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COMMUTATIVE ALGEBRAS ASSOCIATED WITH A DOUBLY TRANSITIVE GROUP

Dedicated to Professor Hirosi Nagao on his 60th birthday

Hiroshi SUZUKI

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1. Introduction

This paper is devoted to the study of the automorphism groups of commutative (nonassociative) algebras related to a certain family of doubly transitive groups.

Since R. Griess succeeded in demonstrating the existence of the Friendly Giant in [3] using a 196884-dimensional commutative algebra, several studies have been made to present some finite groups as automorphism groups of commutative algebras.

S. Norton constructed commutative algebras, so called Norton algebras, whose automorphism groups contain finite groups generated by 3-transpositions. In [1], P. Cameron, J. Goethals and J. Seidel generalized his method and showed that 'Norton algebra' can be defined for a large class of transitive groups. It seems very natural to ask how close the full automorphism group of a Norton algebra is to the original permutation group. S.D. Smith studied this type of problem in [6] but it seems very hard to answer this question in general at this point.

In [4], K. Harada defined an n-dimensional commutative algebra on the nontrivial irreducible factor of the permutation module of a doubly transitive group G and showed that the full automorphism group of it is isomorphic to Σ_{n+1}, the symmetric group of degree $n+1$. He also showed that such a G-invariant algebra structure is uniquely determined up to a scalar multiple if G is 3-ply transitive.

So the next question is what happens if G is required to be just doubly transitive. Then even in this case, it is not very hard to compute the structure constants of G-invariant commutative algebras. (See Section 3.) But the determination of the full automorphism group seems to be more demanding. The first development in this direction was made in [5] by K. Narang. He took the natural doubly transitive action of a group G satisfying $PSL(m, q)\leqq G\leqq P\Gamma L(m, q)$ of degree $n=(q^m-1)/(q-1)$ and showed that there exists an

$n-1$ dimensional algebra whose full automorphism group is isomorphic to $P\Gamma L(m, q)$ if $m \geqq 3$.

Let G be a doubly transitive group on the set Ω of degree n. Suppose that the global stabilizer of two points a, b of Ω in G, i.e., $G_{\{a,b\}}$ has r-orbits on $\Omega-\{a, b\}$. Let $\boldsymbol{C}[\Omega]$ be the permutation module over the complex number field with the natural basis $\{x_1, \cdots, x_n\}$. Let $V_0=\langle x_1+\cdots+x_n\rangle$, and

$$V_1=\langle\sum_{i=1}^{n} \lambda_i x_i\colon \sum_{i=1}^{n} \lambda_i=0\rangle .$$

Then $\boldsymbol{C}[\Omega]=V_0 \oplus V_1$. After we determine the structure constants of the G-invariant algebra A_1 on V_2, with r parameters, we extend the multiplication to $\boldsymbol{C}[\Omega]$ so that the automorphism group of this new algebra A on $\boldsymbol{C}[\Omega]$ is isomorphic to that of A_1. Now we can show that almost always the automorphism group of A, i.e., $\operatorname{Aut} A$, does not grow a lot from G under a certain assumption on G; roughly speaking $r=2$. Our method is as follows. Firstly using the nonassociativity of the algebra A, we obtain two $\operatorname{Aut} A$-invariant multilinear mappings of degree 4. Now we apply a result in [7] (see also Section 2) to get two symmetric trilinear forms θ_0 and θ_1 which are also invariant under the action of $\operatorname{Aut} A$. There are two cases:

(1) $\operatorname{Aut} A$ stabilizes a symmetric trilinear form which is also Σ_n-invariant.

(2) The restrictions of θ_0 and θ_1 are similar on A_1.

If the case (2) occurs it follows that the parameters related to the structure constants of the algebra A must satisfy a polynomial equation of degree 7. So unless the parameters satisfy the equation the case (1) holds. Now it follows from the main result in [2] that $\operatorname{Aut} A$ must be contained in the group isomorphic to $\boldsymbol{Z}_3 \times \Sigma_n$. Thus we have $\operatorname{Aut} A=G$ provided that G is maximal among the doubly transitive groups satisfying the conditions on G.

$P\Gamma L(m, q)$, $Sp(2m, 2)$ (two types), $PSL(2, 11)$ $(n=11)$ and Co. 3 are in the list of the groups satisfying our hypothesis. So in particular our theorem includes K. Narang's.

Recently, in [8], J. Tits showed that the irreducible part of the Griess' algebra has the Friendly Giant as its full automorphism group and also the author was informed that M. Kitazume obtained corresponding results for some of the Conway-Norton algebras in [6] using the similar methods as Tits'.

2. Definitions, notations and preliminary lemmas

Let W be a vector space over the complex number field $\boldsymbol{C}$. We define the following:

$\mathcal{L}(W^r; W)$: the set of multilinear mappings θ of degree r, i.e., $\theta\colon \overbrace{W \times \cdots \times W}^{r} \to W$.

$\mathcal{L}(W^r; \boldsymbol{C})$: the set of multilinear forms θ of degree r, i.e., $\theta\colon \overbrace{W \times \cdots \times W}^{r} \to \boldsymbol{C}$.

$$\mathcal{L}^s(W^r;\, W) = \{\theta\in\mathcal{L}(W^r;\, W)\colon \theta(u_1, \cdots, u_r) = \theta(u_1\sigma, \cdots, u_r\sigma) \\ \text{for all}\quad \sigma\in\Sigma_r\}\,,$$

i.e., the set of symmetric multilinear mappings of degree r.

$$\mathcal{L}^s(W^r;\, \boldsymbol{C}) = \{\theta\in\mathcal{L}(W^r;\, \boldsymbol{C})\colon \theta(u_1, \cdots, u_r) = \theta(u_1\sigma, \cdots, u_r\sigma) \\ \text{for all}\quad \sigma\in\Sigma_r\}\,,$$

i.e., the set of symmetric multilinear forms of degree r.

It is easy to see that these four sets become vector spaces with natural additions and scalar multiples.

For an element θ of $\mathcal{L}(W^r;\, W)$ and $\mathcal{L}(W^r;\, \boldsymbol{C})$, we define the automorphism group of θ as follows:

$$\begin{aligned}\operatorname{Aut}\theta &= \{g\in GL(W)\colon \theta(u_1{}^g, \cdots, u_r{}^g) = \theta(u_1, \cdots, u_r)^g \\ &\quad\text{for all}\quad u_1, \cdots, u_r\in W\},\ \text{if}\ \theta\in\mathcal{L}(W^r;\, W)\,. \\ \operatorname{Aut}\theta &= \{g\in GL(W)\colon \theta(u_1{}^g, \cdots, u_r{}^g) = \theta(u_1, \cdots, u) \\ &\quad\text{for all}\quad u_1, \cdots, u_r\in W\},\ \text{if}\ \theta\in\mathcal{L}(W^r;\, \boldsymbol{C})\,.\end{aligned}$$

Let G be a subgroup of $GL(W)$, and θ an element of $\mathcal{L}(W^r;\, W)$ or $\mathcal{L}(W^r;\, \boldsymbol{C})$. Then θ is said to be G-invariant if G is contained in the automorphism group of θ, i.e., $G\leqq\operatorname{Aut}\theta$.

Let $\mathcal{L}$ be $\mathcal{L}(W^r;\, W)$, $\mathcal{L}(W^r;\, \boldsymbol{C})$, $\mathcal{L}^s(W^r;\, W)$ or $\mathcal{L}^s(W^r;\, \boldsymbol{C})$. Then $\mathcal{L}_G=\{\theta\in\mathcal{L}\colon G\leqq\operatorname{Aut}\theta\}$, the set of G-invariant elements of $\mathcal{L}$.

We note that if θ is an element of $\mathcal{L}(W^2;\, W)$, we identify θ with the algebra A_θ on W whose product is defined by θ. So in particular, if A is an algebra on a vector space W,

$$\operatorname{Aut}A = \{g\in GL(W)\colon u^g v^g = (uv)^g \ \text{for all}\ u,\, v\in W\}\,.$$

Next we define a mapping δ from $\mathcal{L}(W^{r+1};\, W)$ to $\mathcal{L}(W^r;\, \boldsymbol{C})$ introduced in [7]. Let $\{x_1, \cdots, x_n\}$ be a basis of W and B be an element of $\mathcal{L}(W^2;\, \boldsymbol{C})$ satisfying $B(x_i,\, x_j)=\delta_{ij}$, i.e., a nondegenerate symmetric bilinear form with an orthonormal basis $\{x_1, \cdots, x_n\}$. Let θ be an element of $\mathcal{L}(W^{r+1};\, W)$ and $u_1, \cdots, u_r, w\in W$. Let $\theta(u_1, \cdots, u_r, *)$ denote a linear mapping defined by $\theta(u_1, \cdots, u_r, *)(w)=\theta(u_1, \cdots, u_r, w)$. Then $\delta(\theta)$ is a mapping from $W\times\overset{r}{\cdots}\times W$ to $\boldsymbol{C}$ defined by

$$\delta(\theta)(u_1, \cdots, u_r) = \operatorname{Tr}(\theta(u_1, \cdots, u_r, *))\,.$$

Then the following hold.

Proposition 2.1. (1) $\theta\in\mathcal{L}(W^{r+1};\, W)$ *implies* $\delta(\theta)\in\mathcal{L}(W^r;\, \boldsymbol{C})$.
(2) $\theta\in\mathcal{L}^s(W^{r+1};\, W)$ *implies* $\delta(\theta)\in\mathcal{L}^s(W^r;\, \boldsymbol{C})$.

(3) $\delta(\theta)(u_1, \cdots, u_r)=\sum_{i=1}^{n} B(\theta(u_1, \cdots, u_r, x_i), x_i)$.

(4) Aut $\theta \leqq$ Aut $\delta(\theta)$.

Proof. (1), (2) and (3) are clear from the definitions. (4) is also easily verified. See for example [7].

We note that Proposition 2.1 is one of the keys to our paper. See also [8].

Next we consider multilinear mappings and forms defined on a space related to the permutation module of a doubly transitive group.

Let G be a doubly transitive group on a set $\Omega=\{1, 2, \cdots, n\}$. Let $\Omega_{12}^1, \cdots, \Omega_{12}^r$ be orbits of the global stabilizer of the set $\{1, 2\}$ in G, denoted by $G_{\{1,2\}}$. For $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$ define Ω_{ij}^k by $\Omega_{ij}^k=(\Omega_{12}^k)^g$ by an element g such that $\{1^g, 2^g\}=\{i, j\}$. Then it is easy to see that Ω_{ij}^k does not depend on the choice of g, and $\Omega_{ij}^k=\Omega_{ji}^k$.

Let $V=\boldsymbol{C}[\Omega]$ denote the permutation module of G over the complex number field $\boldsymbol{C}$ with the standard basis $\{x_1, \cdots, x_n\}$ such that $x_i^g=x_{ig}$, for $g \in G$. Let $\delta=x_1+\cdots+x_n$, $V_0=\langle\delta\rangle$ and

$$V_1=\{\sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i=0\}.$$

Then V_0 and V_1 are submodules of V, moreover V_1 is an irreducible module as G is doubly transitive on Ω. Let $e_i=x_i-\frac{1}{n}\delta$. Then $\langle e_1, \cdots, e_n\rangle=V_1$ and $e_1+\cdots+e_n=0$.

Let $S^3(\Omega)$ denote the set of unordered 3-tuples of Ω and r' be the number of orbits of G on $S^3(\Omega)$. Then we have the next proposition.

Proposition 2.2. *The following hold*:

(1) $\dim \mathcal{L}^s(V_1^3; \boldsymbol{C})_G=r'$.

(2) $\dim \mathcal{L}^s(V_1^2; V_1)_G=r \geqq r'$.

Proof. This is well-known and easy to prove.

Let θ_s be an element of $\mathcal{L}^s(V_1^3; \boldsymbol{C})$ defined by

$$\begin{aligned}
&\theta_s(e_i, e_i, e_i)=(n-1)(n-2), && i=1, \cdots, n-1, \\
&\theta_s(e_i, e_i, e_j)=-(n-2), && i \neq j, \quad i, j=1, \cdots, n-1, \\
&\theta_s(e_i, e_j, e_k)=2, && i \neq j \neq k \neq i, \quad i, j, k=1, \cdots, n-1.
\end{aligned}$$

Note that $\{e_1, \cdots, e_{n-1}\}$ defined above is a basis of V_1.

Proposition 2.3. *Suppose $r'=1$. Then the following hold:*

(1) $\mathcal{L}^s(V_1^3; \boldsymbol{C})_G=\langle\theta_s\rangle$.

(2) Aut $\theta_s \cong \boldsymbol{Z}_3 \times \Sigma_n$.

Proof. See Egawa-Suzuki [2].

Lemma 2.4. *The following holds:*
$\#\{j: j=1, \cdots, n, k\in\Omega_{1j}^t\}=r_t$, *for* $1\leqq t\leqq r$, $1\leqq k\leqq n$, *where* $r_t=\#\Omega_{12}^t$.

Proof. Counting the number of pairs (j, k) such that $k\in\Omega_{1j}^t$, we have the equality above.

3. Structure of algebra

In this section we shall study G-invariant algebras, where G is a doubly transitive group for which we set some notations and definitions in the previous section. We note that throughout this paper algebras may not be associative, in fact most of them are nonassociative. We shall define algebra structures on two spaces, namely the permutation module V and the nontrivial irreducible factor V_1 of V, and discuss the correspondence between the automorphism groups of these two algebras. In this section we investigate the structure of the algebra defined on V under some condition on (G, Ω).

Now we begin with the determination of G-invariant algebras defined on V_1. We should note that the following theorem has been known to a lot of mathematicians who are interested in automorphism groups of commutative nonassociative algebras.

Theorem 3.1. *Let* A_1 *be a commutative algebra on* V_1 *satisfying the following conditions*:

(1) $e_ie_i=ae_i$, *for* $i=1, 2, \cdots, n-1$,

(2) $e_ie_j=\sum_{t=1}^{r} c_t \sum_{k\in\Omega_{ij}^t} e_k$, *for* $i, j=1, 2, \cdots, n-1$ $i\neq j$; *and*

(3) $a=\sum_{t=1}^{r} c_tr_t$, *where* $r_t=\#\Omega_{12}^t$.

Here $c_1, \cdots, c_r$ *and* a *are constants in the complex number field* $\mathbf{C}$.

Then A_1 *is G-invariant. Moreover if* A_1 *is a G-invariant commutative algebra defined on* V_1, *then* A_1 *must be the one defined above with constants* $c_1, \cdots, c_r$ *and* a.

Proof. Suppose a commutative algebra A_1 on V_1 is G-invariant. Since $\{e_1, \cdots, e_n\}$ is a generator of an $n-1$ dimensional space V_1 with an equation $e_1+\cdots+e_n=0$, e_ie_i can be written as a scalar multiple of e_i because G is doubly transitive. Let $e_ie_i=ae_i$, for all $i=1, 2, \cdots, n$. Similarly e_ie_j has an expression as in (2) for all $i, j=1, 2, \cdots, n$, $i\neq j$. Note that as G is doubly transitive and $(\Omega_{ij}^t)^g=\Omega_{i^gj^g}^t$ for all $g\in G$, $c_1, \cdots, c_r$ and a do not depend on the choice of i and j. Using Lemma 2.4, we have

$$\begin{aligned} a(e_2+\cdots+e_n) &= -ae_1 = -e_1e_1 = e_1(e_2+\cdots+e_n) \\ &= \sum_{i=2}^{n}\sum_{t=1}^{r} c_t \textstyle\sum_{k\in\Omega_{i1}^t} e_k \end{aligned}$$

$$= \sum_{k=1}^{n} (\sum_{t=1}^{r} c_t(\#\{j: k \in \Omega_{1j}^t\}))e_k$$
$$= \sum_{k=2}^{n} (\sum_{t=1}^{r} c_t r_t)e_k .$$

Hence we have

$$a = \sum_{t=1}^{r} c_t r_t .$$

This in turn implies that there are at most r linearly independent G-invariant symmetric multilinear mappings of degree 2, i.e., bilinear mappings on V_1. Since $\dim \mathcal{L}^s(V_1^s; V_1)=r$ by Proposition 2.2. (2), it follows that these r linearly independent symmetric bilinear mappings on V_1 are all G-invariant. Thus we have the assertions.

Let A be an algebra on V with parameters $c_1, \cdots, c_r$ and a in $\boldsymbol{C}$ defined by the following:

(1) $x_i x_i = a x_i$, for $i=1, 2, \cdots, n$,

(2) $x_i x_j = \sum_{t=1}^{r} c_t \sum_{k \in \Omega_{ij}^t} x_k$, for $i, j=1, 2, \cdots, n$, $i \neq j$; and

(3) $\sum_{t=1}^{r} c_t r_t = a$.

The next main objective is to show that if $a=1$, Aut A is isomorphic to Aut A_1, where A_1 is an algebra on V_1 defined in Theorem 3.1 with parameters $c_1, \cdots, c_r$ and 1 for a. So let A_1 be the algebra on V_1 defined above with parameters $c_1, \cdots, c_r$ and 1.

Lemma 3.2. *The following hold*:

(1) $x_i\delta=\delta$, $\delta^2=n\delta$, *for* $i=1, \cdots, n$.

(2) V_0 *and* V_1 *are ideals in* A.

(3) *The restriction of* A *to* V_1 *is* A_1.

Proof. Since the definition of A is symmetric on the i's, to show $x_i\delta=\delta$ we may assume $i=1$.

$$\begin{aligned} x_1\delta &= x_1(x_1+\cdots+x_n) \\ &= x_1 + \sum_{i=2}^{n} \sum_{t=1}^{r} c_t \sum_{j \in \Omega_{1i}^t} x_j \\ &= x_1 + \sum_{i=2}^{n} (\sum_{t=1}^{r} c_t(\#\{j: i \in \Omega_{1j}^t\}))x_i \\ &= x_1 + \sum_{i=2}^{n} (\sum_{t=1}^{r} c_t r_t)x_i \\ &= \delta . \end{aligned}$$

So $x_1\delta=\delta$. As $\delta=x_1+\cdots+x_n$, $\delta^2=n\delta$ follows immediately. Since $e_i=x_i-\frac{1}{n}\delta$, we have the following.

$$e_i\delta = 0\,.$$
$$e_ie_i = e_i\,.$$

If $i \neq j$, then

$$\begin{aligned} e_ie_j &= \left(x_i - \frac{1}{n}\delta\right)\left(x_j - \frac{1}{n}\delta\right) \\ &= x_ix_j - \frac{1}{n}\delta \\ &= \sum_{t=1}^{r} c_t \textstyle\sum_{k\in\Omega_{ij}^t} x_k - \left(\sum_{t=1}^{r} c_tr_t\right)\frac{1}{n}\delta \\ &= \sum_{t=1}^{r} c_t \textstyle\sum_{k\in\Omega_{ij}^t} e_k\,. \end{aligned}$$

Hence (2) and (3) follow.

Let s be a mapping from V to $\boldsymbol{C}$ defined by

$$s\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i\,.$$

Then $s \in \mathcal{L}(V^1; \boldsymbol{C})$, and $V_1 = \mathrm{Ker}(s)$. Let B be the natural symmetric bilinear form on V according to the basis $\{x_1, \cdots, x_n\}$ i.e., $B \in \mathcal{L}^s(V^2; \boldsymbol{C})$ satisfying $B(x_i, x_j) = \delta_{ij}$.

Proposition 3.3. *The following hold if $a=1$.*

(1) Aut $A \leqq$ Aut s.

(2) *V_0 and V_1 are Aut A-invariant.*

(3) *The restriction of elements of Aut A to V_1 induces an isomorphism Aut $A \cong$ Aut A_1.*

Proof. Let θ be an element of $\mathcal{L}^s(V^2; V)$ associated with the algebra structure A, i.e., $\theta(x, y) = xy$. Let δ be the mapping defined in Section 2 and $\delta(\theta) = \theta^* \in \mathcal{L}(V^1; \boldsymbol{C})$.

$$\theta^*(x_i) = \sum_{i=1}^{n} B(x_ix_j, x_j) = B(x_ix_i, x_i) = 1\,.$$

So $\theta^* = s$. Hence (1) follows from Proposition 2.1. As $V_1 = \mathrm{Ker}(s)$, V_1 is Aut A-invariant. Let

$$V_1^\perp = \{x \in V \colon xy = 0 \text{ for all } y \text{ in } V_1\}\,.$$

Then by Lemma 3.2. (1) and (2) we have

$$V_0 \subseteq V_1^\perp \subsetneqq V\,,$$

and V_1 is Aut A-invariant. Thus in particular $V_1^\perp$ is G-invariant. Since V/V_0 is an irreducible G-module isomorphic to V_1, $V_0 = V_1^\perp$. (2) holds.

Let σ be an element of Aut A. Then (2) implies that the element δ is an eigenvector with a nonzero eigenvalue λ. Applying σ to $\delta^2=n\delta$, we have $\lambda^2\delta^2=n\lambda\delta$, or $n\lambda^2=n\lambda$. Since $\lambda\neq 0$, $\lambda=1$. Now we can define an isomorphism between Aut A and Aut A_1 easily as the restriction of A to V_1 is A_1 by Lemma 3.2.

Lemma 3.4. *For all elements x and y in V $s(xy)=s(x)s(y)$, if $a=1$.*

Proof. Let $\theta(x, y)=s(xy)$ and $\theta'(x, y)=s(x)s(y)$. Then θ and θ' are elements of $\mathcal{L}^s(V^2; \boldsymbol{C})$. So it suffices to check the equality at the basis elements.

$$s(x_ix_i) = s(x_i) = 1 = s(x_i)s(x_i)\,,$$
$$s(x_ix_j) = s(\sum_{t=1} c_t \textstyle\sum_{k\in\Omega^t_{ij}} x_k) = \sum_{t=1} c_t \textstyle\sum_{k\in\Omega^t_{ij}} s(x_k)$$
$$= 1 = s(x_i)s(x_j)\,,$$

for all $i, j=1, \cdots, n$, $i\neq j$. Hence we have the equality.

In Proposition 3.3, we verified that Aut $A_1\simeq$ Aut A under a hypothesis that $a=1$. So we do not know whether or not we can say something about Aut A_1 if $a\neq 1$. However, if $a\neq 0$, it is easy to see that Aut A_1 is isomorphic to the algebra on V_1 defined by the parameters $c_1/a, \cdots, c_r/a$ and 1. Hence $a=0$ is the only case actually excluded. In order to investigate Aut A we require stronger assumption, i.e., the following Hypothesis I on (G, Ω).

HYPOTHESIS I. (1) $r=2$.
(2) $k\in\Omega^t_{ij}$ if and only if $j\in\Omega^t_{ik}$, for all $i, j, k\in\Omega$, $i\neq j\neq k\neq i$ and $t=1, 2$.

For a list of groups which satisfy Hypothesis I, see Section 6, and we also note that (2) automatically holds if a one point stabilizer G_a of G is of even order or $r_1\neq r_2$ by Lemma 2.4.

From now on assume that (G, Ω) satisfies Hypothesis I and A is a G-invariant algebra satisfying the following:

(1) $x_ix_i=x_i$, $i=1, 2, \cdots, n$,
(2) $x_ix_j=c_1\sum_{k\in\Omega^1_{ij}}x_k+c_2\sum_{k\in\Omega^2_{ij}}x_k$, $i, j=1, 2, \cdots, n$, $i\neq j$, and
(3) $c_1r_1+c_2r_2=1$.

We define some constants which we shall need later. Let

$$p^k_{ij} = |\Omega^i_{uv}\cap\Omega^j_{uw}|\,, \qquad \text{if} \quad w\in\Omega^k_{uv},\ i, j, k\in\{1, 2\}.$$

It is well-known that each p^k_{ij} does not depend on the choice of u, v and w. Let

$$b_t = b_t(c_1, c_2) = c_1^2p^t_{11}+2c_1c_2p^t_{12}+c_2^2p^t_{22}\,,$$

$$\begin{aligned} d_t &= d_t(c_1, c_2) \\ &= c_1 b_1 p^t_{11} + (c_1 b_2 + c_2 b_1) p^t_{12} + c_2 b_2 p^t_{22} + c_t(c_1^2 r_1 + c_2^2 r_2) . \end{aligned}$$

By our Hypothesis I, $p^k_{ij} = p^k_{ji}$.

$$\begin{aligned} n^{uv}_{ij} &= n^{uv}_{ij}(c_1, c_2) \\ &= c_1^2 |\Omega^1_{ij} \cap \Omega^1_{uv}| + c_1 c_2 (|\Omega^1_{ij} \cap \Omega^2_{uv}| + |\Omega^2_{ij} \cap \Omega^1_{uv}|) + c_2^2 |\Omega^2_{ij} \cap \Omega^2_{uv}| . \end{aligned}$$

Lemma 3.5. *The following hold if $i \neq 1$.*

(1) $x_1(x_1 x_i) = (c_1^2 r_1 + c_2^2 r_2) + b_1 \sum_{j \in \Omega^1_{1i}} x_j + b_2 \sum_{j \in \Omega^2_{1i}} x_j$.

(2) $x_1(x_1(x_1 x_i)) = (b_1 c_1 r_1 + b_2 c_2 r_2) x_i + d_1 \sum_{j \in \Omega^1_{1i}} x_j + d_2 \sum_{j \in \Omega^2_{1i}} x_j$.

Proof. Let

$$x(\alpha, \beta_1, \beta_2) = \alpha x_i + \beta_1 \sum_{j \in \Omega^1_{1i}} x_j + \beta_2 \sum_{j \in \Omega^2_{1i}} x_j .$$

Then

$$\begin{aligned} & x_1 x(\alpha, \beta_1, \beta_2) \\ &= \alpha x_1 x_i + \beta_1 \sum_{j \in \Omega^1_{1i}} x_1 x_i + \beta_2 \sum_{j \in \Omega^2_{1i}} x_1 x_i \\ &= \alpha c_1 \sum_{j \in \Omega^1_{1i}} x_j + \alpha c_2 \sum_{j \in \Omega^2_{2i}} x_j + \beta_1 c_1 \sum_{j \in \Omega^1_{1i}} \sum_{k \in \Omega^1_{1j}} x_k \\ &\quad + \beta_1 c_2 \sum_{j \in \Omega^1_{1i}} \sum_{k \in \Omega^2_{1j}} x_k + \beta_2 c_1 \sum_{j \in \Omega^2_{1i}} \sum_{k \in ^1_{1j}} x_k \\ &\quad + \beta_2 c_2 \sum_{j \in \Omega^2_{1i}} \sum_{k \in \Omega^2_{1j}} x_k \\ &= (\beta_1 c_1 r_1 + \beta_2 c_2 r_2) x_i \\ &\quad + (\beta_1 c_1 p^1_{11} + (\beta_1 c_2 + \beta_2 c_2 p^1_{12} + \beta_2 c_2 p^1_{22} + \beta c_1) \sum_{j \in \Omega^1_{1i}} x_j \\ &\quad + (\beta_1 c_1 p^2_{11} + (\beta_1 c_2 + \beta_2 c_1) p^2_{12} + \beta_2 c_2 p^2_{22} + \beta c_2) \sum_{j \in \Omega^2_{1i}} x_j . \end{aligned}$$

Since $x_1 x_i = x(0, c_1, c_2)$, (1) follows. Hence by (1), $x_1(x_1 x_i) = x(c_1^2 r_1 + c_2^2 r_2, b_1, b_2)$. So (2) holds.

Lemma 3.6. *The following holds. $B(x_i(x_j x_u), x_v) = n^{iv}_{ju}(c_1, c_2)$, if i, j, u, v are distinct.*

Proof. Since $G \leqq \mathrm{Aut}\, A$ is doubly transitive, we may assume $i = 1$ and $j = 2$. Let 1, 2, i and j be distinct.

$$B(x_1(x_2 x_i), x_j) = c_1 B(\textstyle\sum_{k \in \Omega^1_{2i}} x_1 x_k, x_j) + c_2 B(\textstyle\sum_{k \in \Omega^2_{2i}} x_1 x_k, x_j) .$$

Since $B(x_1 x_1, x_j) = B(x_1, x_j) = 0$, we may assume $1 \neq k$. So $B(x_1(x_2 x_i), x_j) = n^{1j}_{2i}(c_1, c_2)$. Hence we have the assertion.

Lemma 3.7. *The following are equivalent.*

(1) $r_2^2(p^1_{11} - p^2_{11}) - 2 r_1 r_2 (p^1_{12} - p^2_{12}) + r_1^2 (p^1_{22} - p^2_{22}) = 0$.

(2) i) $r_1 = r_2$, *and*

ii) *If* $p^1_{11} = a$, *then* $r_1 = 2(a+1)$, $p^2_{22} = a$ *and* $p^1_{12} = p^1_{22} = p^2_{11} = p^2_{12} = a + 1$.

Proof. It is clear that (2) implies (1).

Assume (1). Since (1) is symmetric, by way of contradiction we may assume $r_1<r_2$. By definition,

$$p_{11}^1+p_{12}^1+1=r_1\,,\quad p_{12}^1+p_{22}^1=r_2\,,$$
$$p_{11}^2+p_{12}^2=r_1\quad\text{and}\quad p_{12}^2+p_{22}^2+1=r_2\,.$$

Hence $(p_{11}^1-p_{11}^2)+1=-(p_{12}^1-p_{12}^2)$, so

$$(p_{11}^1-p_{11}^2)+2=1-(p_{12}^1-p_{12}^2)=p_{22}^1-p_{22}^2\,.$$

So (1) implies

$$r_2^2(p_{11}^1-p_{11}^2)+2r_2r_1(p_{11}^1-p_{11}^2+1)+r_1(p_{11}^1-p_{11}^2+2)=0\,,$$

or

$$(p_{11}^1-p_{11}^2)(r_1+r_2)^2+2r_1(r_1+r_2)$$
$$=(r_1+r_2)((p_{11}^1-p_{11}^2)(r_1+r_2)+2r_1)=0\,.$$

Thus we have

$$p_{11}^2-p_{11}^1=2r_1/(r_1+r_2)\,.$$

Since $r_1<r_2$, $0<p_{11}^2-p_{11}^1<1$. A contradiction. Therefore $r_1=r_2$ and $p_{11}^2-p_{11}^1=1$. Let $p_{11}^1=a$. Counting the number of elements of the set

$$\{(x, y)\colon x\in\Omega_{12}^1,\ y\in\Omega_{12}^2,\ x\in\Omega_{1y}^1\}\,,$$

we have $r_1p_{12}^1=r_2p_{11}^2$. So $r_1=r_2$ implies $p_{12}^1=p_{11}^2$. Now the rest of the assertion in (ii) follows easily from the four equations above.

4. Aut A and Aut A-invariant trilinear forms

In this section we consider Aut A under Hypothesis I, using Aut A-invariant trilinear forms. Our goal of this section is to prove the following theorems.

Theorem 4.1. *Suppose (G, Ω) satisfies Hypothesis I. Let A be the G-invariant commutative algebra on $V=\boldsymbol{C}[\Omega]$ with parameters c_1 and c_2 defined in Section* 3. *Then one of the following holds:*

(i) Aut $A\leqq\Sigma_n$,

(ii) $r_2^2(p_{11}^1-p_{11}^2)-2r_1r_2(p_{12}^1-p_{12}^2)+r_1^2(p_{22}^1-p_{22}^2)=0$,

(iii) $c_1r_1+c_2r_2=0$,

(iv) $c_1^2r_1+c_2^2r_2=0$, *or*

(v) $c_1r_1+c_2r_2=a\neq0$ *and c_1/a is a root of a polynomial $f(X)\in\boldsymbol{Z}[X]$ of degree* 4 *which depends only on (G, Ω).*

Theorem 4.2. *Suppose (G, Ω) satisfies Hypothesis I. Let A_1 be the*

G-invariant commutative algebra on the nontrivial irreducible factor V_1 of V with parameters c_1 and c_2 in Theorem 3.1. *Then one of the following holds:*

(i) $Aut\ A_1 \leqq \Sigma_n$,

(ii) $r_2^2(p_{11}^1 - p_{11}^2) - 2r_1r_2(p_{12}^1 - p_{12}^2) + r_1^2(p_{22}^1 - p_{22}^2) = 0$,

(iii) $c_1r_1 + c_2r_2 = 0$,

(iv) $c_1^2r_1 + c_2^2r_2 = 0$, *or*

(v) $c_1r_1 + c_2r_2 = a \neq 0$ *and c_1/a is a root of a polynomial $f(X) \in \boldsymbol{Z}[X]$ of degree* 4 *which depends only on* (G, Ω).

As we have noted in Section 3, the G-invariant algebra with parameters c_1 and c_2, and the one with parameters αc_1 and αc_2 have isomorphic automorphism groups if α is a nonzero constant. So (iii), (iv) and (v) in Theorem 4.1 and 4.2 can be said as follows:

(vi) $f(c_1, c_2) = 0$, where $f(X_1, X_2) \in \boldsymbol{Z}[X_1, X_2]$ is a homogeneous polynomial of degree 7 which depends only on (G, Ω).

Viewing (c_1, c_2) as a point on the projective line $P^1(\boldsymbol{C})$, we can say that if (ii) does not occur, (i) holds unless (c_1, c_2) is one of the seven points on $P^1(\boldsymbol{C})$ which are determined by (G, Ω).

By Lemma 3.7 the condition (ii) can be replaced by the two conditions in Lemma 3.7. (2). Hence if $r_1 \neq r_2$, Hypothesis I (1) implies (2) and the case (ii) does not occur.

Since $c_1r_1 + c_2r_2 \neq 0$ implies Aut $A \cong$ Aut A_1 by Proposition 3.3, Theorem 4.1 implies Theorem 4.2 and vice versa.

Assume $c_1r_1 + c_2r_2 \neq 0$. Replacing c_i with c_i/a, we may assume $c_1r_1 + c_2r_2 = 1$.

Lemma 4.3. *Let B_1 be a symmetric bilinear form on V satisfying the following*:

(1) $B_1(x_i, x_i) = 1 + (n-1)(c_1^2r_1 + c_2^2r_2)$, $i = 1, \cdots, n$.

(2) $B_1(x_i, x_j) = 1 - (c_1^2r_1 + c_2^2r_2)$, $i, j = 1, \cdots, n$, $i \neq j$.

Then Aut $A \leqq$ AutB_1.

Proof. Let $\theta(x, y, z) = x(yz)$. Then it is easy to see that $\theta \in \mathcal{L}(V^3; V)$ and that Aut $A \leqq$ Aut θ. Let $\delta(\theta) = \theta^*$, where δ is a mapping from $\mathcal{L}(V^3; V)$ to $\mathcal{L}(V^2; \boldsymbol{C})$ defined in Section 2. Then by Proposition 2.1. (4), Aut $A \leqq$ Aut θ $\leqq$ Aut θ^*. So to have the assertion of this lemma, it suffices to show $\theta^* = B_1$. Applying Proposition 2.1. (3) and Lemma 3.5. (1), we have

$$\begin{aligned}\theta^*(x_1, x_1) &= \sum_{i=1}^{n} B(\theta(x_1, x_1, x_i), x_i) \\ &= \sum_{i=1}^{n} B(x_1(x_1x_i), x_i) \\ &= 1 + (n-1)(c_1^2r_1 + c_2^2r_2)\,.\end{aligned}$$

Since $G \leqq$ Aut θ^* and G is doubly transitive, we have (1). Again using the double transitivity, we have

$$\theta^*(x_1, x_i) = \theta^*(x_1, x_2) \qquad \text{for all} \quad i = 2, \cdots, n .$$

Moreover it follows from Lemma 3.2

$$\theta^*(x_1, \delta) = \sum_{i=1}^{n} B(x_1(\delta x_i), x_i) = \sum_{i=1}^{n} B(\delta, x_i) = n$$

Hence $n=\theta^*(x_1, \delta)=\theta^*(x_1, x_1)+(n-1)\theta^*(x_1, x_2)$. Solving the equation above using (1), we have (2).

Lemma 4.4. *Let* $\theta_0(x, y, z)=B(xy, z)$. *Then the following hold*:

(1) $\theta_0 \in \mathcal{L}^s(V^3; \boldsymbol{C})$.

(2) $\theta_0(x_i, x_i, x_i)=1$, $\theta_0(x_i, x_i, x_j)=0$ *and* $\theta_0(x_i, x_j, x_k)=c_t$, *where* $i, j=1, \cdots, n$, $i \neq j$ *and* $k \in \Omega^t_{ij}$.

(3) $\theta_0(e_i, e_i, e_i)=1-\frac{1}{n}$, $\theta_0(e_i, e_j, e_j)=-\frac{1}{n}$ *and* $\theta_0(e_i, e_j, e_k)=c_t-\frac{1}{n}$, *where* $i, j=1, \cdots, n$, $i \neq j$ *and* $k \in \Omega^t_{ij}$.

(4) $B_1(x, y)=n(c_1^2 r_1+c_2^2 r_2)B(x, y)+(1-(c_1^2 r_1+c_2^2 r_2))s(xy)$.

(5) *If* $c_1^2 r_1+c_2^2 r_2 \neq 0$, *then* $\mathrm{Aut}\, A \leqq \mathrm{Aut}\, B \cap \mathrm{Aut}\, \theta_0$.

Proof. By the definition of θ_0,

$$\begin{aligned} &\theta_0(x_i, x_i, x_i) = B(x_i x_i, x_i) = B(x_i, x_i) = 1 , \\ &\theta_0(x_i, x_j, x_j) = \theta_0(x_j, x_i, x_j) = B(x_i x_j, x_j) \\ =\ & B(c_1 \textstyle\sum_{k \in \Omega^1_{ij}} x_k + c_2 \sum_{k \in \Omega^2_{ij}} x_k, x_j) = 0 , \\ &\theta_0(x_j, x_j, x_i) = B(x_j x_j, x_i) = B(x_j, x_i) = 0. \quad \text{and} \\ &\theta_0(x_i, x_j, x_k) = B(x_i x_j, x_k) \\ =\ & B(c_1 \textstyle\sum_{h \in \Omega^1_{ij}} x_h + c_2 \sum_{h \in \Omega^2_{ij}} x_h, x_k) = c_t, \quad \text{where} \quad k \in \Omega^t_{ij} . \end{aligned}$$

Since $k \in \Omega^t_{ij}$ if and only if $i \in \Omega^t_{jk}$, and $i \in \Omega^t_{jk}$ if and only if $j \in \Omega^t_{ki}$ by Hypothesis I, we have (1) and (2). Moreover using Lemma 3.4, we have

$$\theta_0(x, y, \delta) = B(xy, \delta) = s(xy) = s(x)s(y) .$$

So $\theta_0(e_i, e_j, e_k)=\theta_0\left(x_i-\frac{1}{n}\delta, x_j-\frac{1}{n}\delta, x_k-\frac{1}{n}\delta\right)= \theta_0(x_i, x_j, x_k)-\frac{1}{n}$. Thus (3) follows. Let $i \neq j$. Then

$$\begin{aligned} &n(c_1^2 r_1+c_2^2 r_2)B(x_i, x_i)+(1-(c_1^2 r_1+c_2^2 r_2))s(x_i x_i) \\ =\ & 1-(n+1)(c_1^2 r_1+c_2^2 r_2) . \\ &n(c_1^2 r_1+c_2^2 r_2)B(x_i, x_j)+(1-(c_1^2 r_1+c_2^2 r)_2)s(x_i x_j) \\ =\ & 1-(c_1^2 r_1+c_2^2 r_2) . \end{aligned}$$

This implies (4).

Suppose $c_1^2r_1+c_2^2r_2 \neq 0$. Then $B(x, y)$ can be written as a linear combination of $B_1(x, y)$ and $s(xy)$. Now Proposition 3.3 (1) and Lemma 4.3. imply $\mathrm{Aut}\, A \leqq \mathrm{Aut}\, B$. (5) follows from the definition of θ_0.

Lemma 4.5. *For $x, y, z \in V$ let*

$$\theta_1(x, y, z) = \sum_{i=1}^{n} B(x(y(zx_i)), x_i)\,.$$

Then the following hold:

(1) $\theta_1 \in \mathcal{L}^s(V^3; \boldsymbol{C})$.

(2) $\theta_1(x_i, x_i, x_i)=1+(n-1)(b_1c_1r_1+b_2c_2r_2)$, $\theta_1(x_i, x_j, x_j)=1-(b_1c_1r_1+b_2c_2r_2)$, *where* $i, j=1, \cdots, n$, $i \neq j$.

(3) $\theta_1(e_i, e_i, e_i)=(n-1)(b_1c_1r_1+b_2c_2r_2)$, $\theta_1(e_i, e_j, e_j)=-(b_1c_1r_1+b_2c_2r_2)$, *where* $i, j=1, \cdots, n$, $i \neq j$.

(4) $\mathrm{Aut}\, A \leqq \mathrm{Aut}\, \theta_1$.

Proof. Let $\theta(x, y, z, w)=x(y(zw))$ for x, y, z and $w \in V$. Then it is clear that $\theta \in \mathcal{L}(V^4; V)$. So it follows frcm Proposition 2.1 that $\delta(\theta)=\theta_1 \in \mathcal{L}(V^3; \boldsymbol{C})$. Moreover

$$\mathrm{Aut}\, A \leq \mathrm{Aut}\, \theta \leq \mathrm{Aut}\, \delta(\theta) = \mathrm{Aut}\theta_1\,.$$

Thus we have (4).

Since $B(x_1(x_1(x_1x_i)), x_i)=b_1c_1r_1+b_2c_2r_2$ by Lemma 3.5. (2), unless $i=1$,

$$\theta_1(x_1, x_1, x_1) = 1+(n-1)(b_1c_1r_1+b_2c_2r_2)\,.$$

As $\mathrm{Aut}\, \theta_1$ contains a subgroup G which acts double transitively on the set $\{x_1, \cdots, x_n\}$,

$$\theta_1(x_i, x_i, x_i) = 1+(n-1)(b_1c_1r_1+b_2c_2r_2)$$

holds for all i. Using the definition of θ_1, we have

$$\theta_1(x_i, x_i, \delta) = \theta_1(x_i, \delta, x_i) = \theta_1(\delta, x_i, x_i) = n\,.$$

Again using the double transitivity of a subgroup G of $\mathrm{Aut}\, \theta_1$, we have for all $i \neq j$

$$\begin{aligned}\theta_1(x_i, x_j, x_j) &= \theta_1(x_j, x_i, x_j) = \theta_1(x_j, x_j, x_i) \\ &= 1-(b_1c_1r_1+b_2c_2r_2)\,.\end{aligned}$$

Thus (2) holds.

(3) can be verified easily by the similar method we employed to calculate the values of θ_0 in the previous lemma.

To show the symmetricity of θ_1 it remains to show the following equalities:

$$\theta_1(x_i,\ x_j,\ x_k) = \theta_1(x_k,\ x_j,\ x_i) = \theta_1(x_j,\ x_i,\ x_k)\ .$$

Firstly, using the symmetricity of θ_0 in Lemma 4.4. (1), we have $B(xy,\ z)= B(x,\ yz)$, in general. So

$$\begin{aligned}\theta_1(x_i,\ x_j,\ x_k) &= \sum_{h=1}^{n} B(x_i(x_j(x_k x_h)),\ x_h) \\ &= \sum_{h=1}^{n} B(x_k(x_j(x_i x_h)),\ x_h) \\ &= \theta_1(x_k,\ x_j,\ x_i)\ .\end{aligned}$$

Since Ω^1_{ik} and Ω^2_{ik} are orbits of $G_{\{i,k\}} \leqq G \leqq \mathrm{Aut}\,\theta_1$, $\theta_1(x_i,\ x_j,\ x_k)=\alpha_t$ if $j\in\Omega^t_{ik}$, i.e., $\theta_1(x_i,\ x_j,\ x_k)$ is a constant as x_j varies on an orbit Ω^t_{ik}. Let σ be an element in G_k such that $i^\sigma=j$. Applying the automorphism of V corresponding to σ, we have $\theta_1(x_j,\ x_s,\ x_k)=\alpha_t$, if $s\in\Omega^t_{jk}$, where $s=j^\sigma$. Since $j\in\Omega^t_{ik}$ implies $i\in\Omega^t_{jk}$ by Hypothesis I,

$$\theta_1(x_i,\ x_j,\ x_k) = \theta_1(x_j,\ x_i,\ x_k)\ .$$

Therefore (1) holds.

Lemma 4.6. *The following hold*:
(1) $\theta_1(x_i,\ x_j,\ x_k)-1=\theta_1(e_i,\ e_j,\ e_k)$, *where* $i,\ j,\ k=1,\ \cdots,\ n,\ i\neq j\neq k\neq i$.
(2) *Suppose* $k\in\Omega^t_{ij}$. *Then*

$$\theta_1(x_i,\ x_j,\ x_k) = 3b_t+2d_t-c_t(c_1{}^2r_1+c_2{}^2r_2)+\sum_{u=1}^{2} c_u \sum_{\substack{r\neq s,\ k\in\Omega^u_{rs} \\ \{r,s\}\cap\{i,j\}=\emptyset}} n^{ij}_{rs}(c_1,\ c_2)\ .$$

Proof. (1) follows easily as for all $x,\ y\in V$, $\theta_1\left(x,\ y,\ \dfrac{1}{n}\delta\right)=s(xy)$.

To prove (2), we set $i=1, j=2$ and $k=3$ in order to save symbols. Assume $3\in\Omega^t_{12}$.

$$\begin{aligned}&\theta_1(x_1,\ x_2,\ x_3) \\ &= \sum_{i=1}^{n} B(x_1(x_2(x_3x_i)),\ x_i) \\ &= B(x_1(x_2(x_3x_1)),\ x_1)+B(x_2(x_3(x_2x)),\ x_2)+B(x_1(x_2(x_3x_3)),\ x_3) \\ &\quad +\sum_{i\neq 1,2,3}(c_1\textstyle\sum_{j\in\Omega^1_{3i}} B(x_1(x_2x_j),\ x_i)+c_2\sum_{j\in\Omega^2_{3i}} 2B(x_1(x_2x_j),\ x_i)) \\ &= B(x_1(x_1x_3),\ x_2)+B(x_1(x_2(x_2x_1)),\ x_3)+B(x_3(x_3x_1),\ x_2) \\ &\quad +c_1\sum_{\substack{i\in\Omega^1_{31} \\ \neq 2}} B(x_1(x_1x_2),\ x_i)+c_2\sum_{\substack{i\in\Omega^1_{32} \\ \neq 1}} B(x_1(x_1x_2),\ x_i) \\ &\quad +c_1\sum_{\substack{i\in\Omega^2_{31} \\ \neq 2}} B_1(x_2x,\ x_i)+c_2\sum_{\substack{i\in\Omega^2_{32} \\ \neq 1}} B(x_1x_2,\ x_i) \\ &\quad +c_1\sum_{\substack{i\neq j,\ 3\in\Omega^1_{1j} \\ \{i,j\}\cap\{1,2\}=\emptyset}} B(x_1(x_2x_j),\ x_i)+c_2\sum_{\substack{i\neq j,3\in\Omega^2_{ij} \\ \{i,j\}\cap\{1,2\}=\emptyset}} B(x_1(x_2x_j),\ x_i)\ .\end{aligned}$$

Hence by Lemma 3.5 and Lemma 3.6,

$$\theta_1(x_1, x_2, x_3) = 3b_t + 2d_t + c_t(c_1^2 r_1 + c_2^2 r_2) + \sum_{u=1}^{2} c_u \sum_{\substack{j \neq j,\, 3 \in \Omega^u_{ij} \\ \{i,j\} \cap \{1,2\} = \emptyset}} n^{1i}_{2j}(c_1, c_2)\,,$$

as

$$\begin{aligned} & c_1 \sum_{\substack{i \in \Omega^1_{31} \\ \neq 2}} B(x_1(x_1x_2), x_i) + c_2 \sum_{\substack{i \in \Omega^1_{32} \\ \neq 2}} B(x_1(x_1x_2), x_i) \\ &= B(x_1(x_1x_2), x_1x_3) - c_t B(x_1(x_1x_2), x_2) \\ &= B(x_1(x_1(x_1x_2)), x_3) - c_t B(x_1(x_1x_2), x_2) \\ &= d_t - c_t(c_1^2 r_1 + c_2^2 r_2)\,, \\ & c_1 \sum_{\substack{i \in \Omega^2_{31} \\ \neq 1}} B(x_1x_2, x_i) + c_2 \sum_{\substack{i \in \Omega^2_{32} \\ \neq 1}} B(x_1x_2, x_i) \\ &= B(x_1x_2, x_2x_3) \\ &= b_t\,. \end{aligned}$$

Now assume $c_1^2 r_1 + c_2^2 r_2 \neq 0$. Since the restriction mapping $\operatorname{Aut} A \to \operatorname{Aut} A_1$ sending $\sigma \in \operatorname{Aut} A$ to $\sigma_{|V_1} \in \operatorname{Aut} A_1$ is an isomorphism and $\operatorname{Aut} A$ acts trivially on $V_0 = \langle \delta \rangle$, by Proposition 3.3, we have

$$\operatorname{Aut} A_1 \leqq \operatorname{Aut} \theta_{1|V_1} \cap \operatorname{Aut} \theta_{0|V_1}\,,$$

by Lemma 4.4 and Lemma 4.5. So $\theta_{1|V_1}$, $\theta_{0|V_1}$ are elements of $\mathcal{L}^s(V_1^3; \boldsymbol{C})_G$. Since

$$\dim \mathcal{L}^s(V_1^3 \colon \boldsymbol{C})_G \leqq \dim \mathcal{L}^s(V_1^2; V_1)_G = 2$$

by Proposition 2.2, one of the following holds:

(i) $\theta_{1|V_1}$ is a scalar multiple of $\theta_{0|V_1}$, or

(ii) $\dim\langle \theta_{1|V_1}, \theta_{0|V_1} \rangle = 2$ and $\langle \theta_{1|V_1}, \theta_{0|V_1} \rangle = \mathcal{L}^s(V_1^3; \boldsymbol{C})_G$.

Suppose (ii) holds. Since $G \leqq \Sigma_n$ and

$$\langle \theta_s \rangle = \mathcal{L}^s(V_1^3; \boldsymbol{C})_{\Sigma_n} \leqq \mathcal{L}^s(V_1^3; \boldsymbol{C})_G = \langle \theta_{1|V_1}, \theta_{0|V_1} \rangle\,,$$

by Proposition 2.3, θ_s can be written as a nontrivial linear combination of $\theta_{1|V_1}$ and $\theta_{0|V_1}$. Say

$$\theta_s = \alpha \theta_{1|V_1} + \beta \theta_{0|V_1}\,.$$

As

$$\operatorname{Aut} A_1 \leqq \operatorname{Aut} \theta_{1|V_1} \cap \operatorname{Aut} \theta_{0|V_1} \leqq \operatorname{Aut} \theta_s\,,$$

Proposition 2.3. (2) implies that $\operatorname{Aut} A_1$ is a subgroup of $\boldsymbol{Z}_3 \times \Sigma_n$. Because of the irreducibility of $\operatorname{Aut} A_1$, we can conclude by Schur's lemma that $\boldsymbol{Z}_3$-part acts as scalars on V_1. So if σ is an element of the center of $\operatorname{Aut} A_1$ and $e_i^\sigma = \lambda e_i$, $e_i e_i = e_i$ implies $\lambda^2 = \lambda$. Hence $\lambda = 1$. Thus we have $\operatorname{Aut} A_1 \leqq \Sigma_n$ in this case.

On the other hand, suppose (i) holds. Let

$$\theta_{1|V_1} = \alpha\theta_{0|V_1}\,.$$

Since $\theta_0(e_i, e_j, e_j) = -\dfrac{1}{n}$ and $\theta_1(e_i, e_j, e_j) = -(b_1c_1r_1 + b_2c_2r_2)$ by Lemma 4.4 and Lemma 4.5, $\alpha = n(b_1c_1r_1 + b_2c_2r_2)$. So

$$\theta_1(e_i, e_j, e_k) = n(b_1c_1r_1 + b_2c_2r_2)c_t - (b_1c_1r_1 + b_2c_2r_2)\,.$$

It follows from Lemma 4.6 that $\theta_1(e_i, e_j, e_k)$ can be written as a polynomial g_{ijk} of c_1 of degree at most 3 as $c_1r_1 + c_2r_2 = 1$, where $g_{ijk} \in \boldsymbol{Q}[c_1]$ and g_{ijk} depends only on (G, Ω), namely r_1, r_2, p^w_{uv} and $|\Omega^x_{uv} \cap \Omega^{x'}_{u'v'}|$. So it suffices to have the condition when $n(b_1c_1r_1 + b_2c_2r_2)c_i - (b_1c_1r_1 + b_2c_2r_2)$ is a polynomial of c_1 of degree exactly 4. Since the degree of the second term is at most 3, we need to see the degree of $b_1c_1r_1 + b_2c_2r_2$ in terms of c_1.

$$\begin{aligned}
&b_1c_1r_1 + b_2c_2r_2 \\
&= (c_1^2p^1_{11} + 2c_1c_2p^1_{12} + c_2^2p^1_{22})c_1r_1 + (c_1^2p^2_{21} + 2c_1c_2p^2_{12} + c_2^2p^2_{22})c_2r_2 \\
&= ((c_1^2p^1_{11} + 2c_1c_2p^1_{12} + c_1^2p^1_{22}) - (c_1^2p^2_{11} + 2c_1c_2p^2_{12} + c_1^2p^2_{22}))c_1r_1 \\
&\quad + (c_1^2p^2_{11} + 2c_1c_2p^1_{22} + c_2^2p^2_{22})\,.
\end{aligned}$$

Since

$$\begin{aligned}
&r_2^2((c_1^2p^1_{11} + 2c_1c_2p^1_{12} + c_2^2p^1_{22}) - (c_1^2p^2_{11} + 2c_1c_2p^2_{12} + c_2^2p^2_{22})) \\
&= c_1^2(r_2^2(p^1_{11} - p^2_{11}) - 2r_1r_2(p^1_{12} - p^2_{12}) + r_1^2(p^1_{22} - p^2_{22})) \\
&\quad + 2c_1r_2p^1_{12} + p^1_{22} - 2c_1r_1p^1_{22} - 2c_1r_2p^2_{12} - p^2_{22} + 2c_1r_1p^2_{22}\,,
\end{aligned}$$

$n(b_1c_1r_1 + b_2c_2r_2)c_t - (b_1c_1r_1 + b_2c_2r_2)$ is a polynomial of c_1 of degree exactly 4, if and only if

$$r_2^2(p^1_{11} - p^2_{11}) - 2r_1r_2(p^1_{12} - p^2_{12}) + r_2^2(p^1_{22} - p^2_{22}) \neq 0\,.$$

Thus we have Theorem 4.1 and Theorem 4.2.

As a corollary of our proof, we have the following.

Corollary 4.7. *Suppose (G, Ω) satisfies Hypothesis I, and $(c_1r_1 + c_2r_2) \times (c_1^2r_1 + c_2^2r_2) \neq 0$. Moreover assume*

$$\theta_1(x_i, x_j, x_k) \neq n(b_1c_1r_1 + b_2c_2r_2)c_t - (b_1c_1r_1 + b_2c_2r_2) + 1$$

or

$$\theta_1(e_i, e_j, e_k) \neq n(b_1c_1r_1 + b_2c_2r_2)c_t - (b_1c_1r_1 + b_2c_2r_2)\,,$$

for a set of three numbers i, j, k $(\neq)$, where $k \in \Omega^t_{ij}$. Then $\operatorname{Aut} A_1 \leqq \Sigma_n$ *and* $\operatorname{Aut} A \leqq \Sigma_n$.

REMARK. Since we have Proposition 2.2, if G is a doubly transitive group which is maximal among the ones satisfying Hypothesis I, $\mathrm{Aut}\,A \cong \mathrm{Aut}\,A_1 \cong G$, unless $c_1=c_2$ in which case $\mathrm{Aut}\,A \cong \mathrm{Aut}\,A_1 \cong \Sigma_n$, whenever we have the case (i) in our theorems.

5. $\theta_1(x_i, x_j, x_k)$

In this section we shall determine the value $\theta_1(x_i, x_j, x_k)$ under a stronger hypothesis in order to simplify the condition in Corollary 4.7 and the case (v) of the theorems in the previous section.

HYPOTHESIS II. Let (G, Ω) satisfy Hypothesis I. Moreover

$$|\Omega_{12}^t \cap \Omega_{ij}^t| = |\Omega_{1i}^t \cap \Omega_{2j}^t|,$$

for all $1 \leqq i \neq j \leqq n$, with $t=1$ or 2.

We begin with an introduction of another trilinear form invariant under the action of Aut A.

Lemma 5.1. *For all $x, y, z \in V$, let*

$$\theta_0'(x, y, z) = \sum_{i=1}^{n} B((xy)(zx_i), x_i)\,.$$

Then the following hold.

(1) $\theta_0' \in \mathcal{L}^s(V^3; \boldsymbol{C})$.

(2) $\mathrm{Aut}\,A \leqq \mathrm{Aut}\,\theta_0'$.

(3) $\theta_0'(x, y, z) = B_1(xy, z) = n(c_1^2 r_1 + c_2^2 r_2)\theta_0(x, y, z) + (1-(c_1^2 r_1 + c_2^2 r_2))s(x) \times s(y)s(z)$.

(4) $\theta_0'(x_i, x_i, x_i) = (n-1)(c_1^2 r_1 + c_2^2 r_2) + 1$.

$$\theta_0'(x_i, x_j, x_j) = 1-(c_1^2 r_1 + c_2^2 r_2)$$
$$\theta_0'(x_i, x_j, x_k) = n(c_1^2 r_1 + c_2^2 r_2)c_t + 1 - (c_1^2 r_1 + c_2^2 r_2)\,,$$

where $1 \leqq i \neq j \leqq n$ and $k \in \Omega_{ij}^t$, $t=1, 2$.

(5)
$$\theta_0'(e_i, e_i, e_i) = (n-1)(c_1^2 r_1 + c_2^2 r_2)$$
$$\theta_0'(e_i, e_j, e_j) = -(c_1^2 r_1 + c_2^2 r_2)$$
$$\theta_0'(e_i, e_j, e_k) = n(c_1^2 r_1 + c_2^2 r_2)c_t - (c_1^2 r_1 + c_2^2 r_2)\,,$$

where $1 \leqq i \neq j \leqq n$ and $k \in \Omega_{ij}^t$, $t=1, 2$.

Proof. Let $\theta(x, y, z, w) = (xy)(zw)$. Then $\theta \in \mathcal{L}(V^4; V)$ and $\mathrm{Aut}\,A \leqq \mathrm{Aut}\,\theta$. Hence Proposition 2.1. (1) implies $\delta(\theta) \in \mathcal{L}(V^3; \boldsymbol{C})$, (3) implies $\delta(\theta) = \theta_0'$ and (4) implies $\mathrm{Aut}\,\theta \leqq \mathrm{Aut}\,\delta(\theta)$. Thus we have (1) and (2). By the definition of B_1, (see the proof of Lemma 4.3), we have

$$B_1(xy, z) = \sum_{i=1}^{n} B((xy)(zx_i), x_i) = \theta_0'(x, y, z)\,.$$

Now (3) follows from Lemma 4.4. Since $s(xy)=s(x)s(y)$ by Lemma 3.4, (4) and (5) follow from the value of θ_0 calculated in Lemma 4.4.

Lemma 5.2. *Suppose* $1\leqq i\neq j\leqq n$ *and* $k\in\Omega^t_{ij}$. *Then*

$$\theta_0'(x_i, x_j, x_k)=4d_t+c_t-2c_t(c_1{}^2r_1+c_2{}^2r_2)+\sum_{u=1}^{2} c_u \sum_{\substack{r\neq s, k\in\Omega^n_{rs}\\ \{r,s\}\cap\{i,j\}=\emptyset}} n^{rs}_{ij}(c_1, c_2)\,.$$

Proof. To save symbols let $i=1$, $j=2$ and $k=3$.

$$\begin{aligned}
\theta_0'(x_1, x_2, x_3) &= \sum_{i=1}^{n} B((x_1x_2)(x_3x_i), x_i)\\
&= B((x_1x_2)(x_3x_1), x_1)+B((x_1x_2)(x_3x_2), x_2)+B((x_1x_2)(x_3x_3), x_3)\\
&\quad +\sum_{i\neq 1,2,3} \Big(c_1\sum_{j\in\Omega^1_{3i}} B((x_1x_2)x_j, x_i)+c_2\sum_{j\in\Omega^2_{2i}} B((x_1x_2)x_j, x_i)\Big)\\
&= B(x_1(x_1(x_1x_2)), x_3)+B(x_2(x_2(x_2x_1)), x_3)+B(x_1x_2, x_3)\\
&\quad +c_1\sum_{\substack{i\in\Omega^1_{31}\\ \neq 2}} B((x_1x_2)x_1, x_i)+c_2\sum_{\substack{i\in\Omega^2_{31}\\ \neq 2}} B((x_1x_2)x_1, x_i)\\
&\quad +c_1\sum_{\substack{i\in\Omega^1_{32}\\ \neq 1}} B((x_1x_2)x_2, x_i)+c_2\sum_{\substack{i\in\Omega^2_{32}\\ \neq 1}} B((x_1x_2)x_2, x_i)+\Phi\\
&= 2d_t+c_t-2c_t(c_1{}^2r_1+c^2{}_2r_1)+B(x_1(x_1x_2), x_1x_3)+B(x_2(x_2x_1), x_2x_3)+\Phi\\
&= 4d_t+c_t-2c_t(c_1{}^2r_1+c_2{}^2r_2)+\Phi\,,
\end{aligned}$$

where

$$\Phi = \sum_{u=1}^{2} c_u \sum_{\substack{i\neq j, 3\in\Phi^u_{ij}\\ \{i,j\}\cap\{1,2\}=\emptyset}} B((x_1x_2)x_j, x_i) = \sum_{u=1} c_u \sum_{\substack{i\neq j, 3\in\Omega^u_{ij}\\ \{i,j\}\cap\{1,2\}=\emptyset}}^{2} n^{ij}_{12}(c_1, c_2)\,.$$

Hence we have the formula as desired.

Lemma 5.3. *Suppose Hypothesis II holds. Let* $u\in\{1, 2\}-\{t\}$. *Then the following hold.*

(1) $|\Omega^t_{12}\cap\Omega^u_{ji}|+|\Omega^u_{12}\cap\Omega^t_{ij}|=|\Omega^t_{1i}\cap\Omega^u_{2j}|+\Omega^u_{1i}\cap\Omega^t_{2j}|$ *for all* $1\leqq i\neq j\leqq n$.

(2) $|\Omega^u_{12}\cap\Omega^u_{ij}|=|\Omega^u_{1i}\cap\Omega^u_{2j}|$ *for all* $1\leqq i\neq j\leqq n$.

Proof. To show (1) it suffices to show the following.

$$\begin{aligned}
&2|\Omega^t_{12}\cap\Omega^t_{ij}|+|\Omega^t_{12}\cap\Omega^u_{ij}|+|\Omega^u_{12}\cap\Omega^t_{ij}|\\
&= 2|\Omega^t_{1i}\cap\Omega^t_{2j}|+|\Omega^t_{1i}\cap\Omega^t_{2j}|+|\Omega^u_{1i}\cap\Omega^t_{2j}|.
\end{aligned} \tag{*}$$

Since $\Omega=\{v, w\}\cup\Omega^1_{vw}\cup\Omega^2_{vw}$, we have

$$\begin{aligned}
&2|\Omega^t_{12}\cap\Omega^t_{ij}|+|\Omega^t_{12}\cap\Omega^u_{ij}|+|\Omega^u_{12}\cap\Omega^t_{ij}|\\
&= |\Omega^t_{12}\cap\Omega^t_{ij}|+|\Omega^t_{12}\cap\Omega^u_{ij}|+|\Omega^t_{21}\cap\Omega^t_{ij}|+|\Omega^u_{i2}\cap\Omega^t_{ij}|
\end{aligned}$$

$$
\begin{aligned}
&= r_t - |\Omega_{12}^t \cap \{i\}| - |\Omega_{12}^t \cap \{j\}| + r_t - |\Omega_{ij}^t \cap \{1\}| - |\Omega_{ij}^t \cap \{2\}| \\
&= 2r_t - |\Omega_{1i}^t \cap \{2\}| - |\Omega_{2j}^t \cap \{1\}| - |\Omega_{1i}^t \cap \{j\}| - |\Omega_{2j}^t \cap \{i\}| \\
&= |\Omega_{1i}^t \cap \Omega_{2j}^t| + |\Omega_{1i}^t \cap \Omega_{2j}^u| + |\Omega_{1i}^t \cap \Omega_{2j}^t| + |\Omega_{1i}^u \cap \Omega_{2j}^t| \\
&= 2|\Omega_{1i}^t \cap \Omega_{2j}^t| + |\Omega_{1j}^t \cap \Omega_{2j}^u| + |\Omega_{1i}^u \cap \Omega_{2j}^t| .
\end{aligned}
$$

Hence (*) holds. Moreover since (*) is symmetric for t and u, (1) implies (2).

Theorem 5.4. *If (G, Ω) satisfies Hypothesis II, the following hold.*
(1) $n_{ij}^{uv}(c_1, c_2) = n_{iu}^{jv}(c_1, c_2)$ *for all c_1 and c_2, where i, j, u, v are distinct.*

(2)
$$
\begin{aligned}
&\theta_1(x_i, x_j, x_k) \\
&= \theta_0'(x_i, x_j, x_k) + 3b_t - 2d_t - c_t + c_t(c_1^2 r_1 + c_2^2 r_2) \\
&= (n+1)c_t(c_1^2 r_1 + c_2^2 r_2) + 3b_t - 2d_t - c_t + 1 - (c_1^2 r_1 + c_2^2 r_2) ,
\end{aligned}
$$

where $1 \leqq i \neq j \leqq n$ and $k \in \Omega_{ij}^t$.
(3) *If $(c_1 r_1 + c_2 r_2)(c_1^2 r_1 + c_2^2 r_2) \neq 0$ and*

$$
(nc_t - 1)((b_1 - c_1)c_1 r_1 + (b_2 - c_2)c_2 r_2) \neq c_t(c_1^2 r_1 + c_2^2 r_2) + 3b_t - 2d_t - c_t ,
$$

then Aut $A_1 \leqq \Sigma_n$ *and* Aut $A \leqq \Sigma_n$.

Proof. Since

$$
\begin{aligned}
n_{ij}^{uv}(c_1, c_2) &= c_1^2 |\Omega_{ij}^1 \cap \Omega_{uv}^1| + c_2^2 |\Omega_{ij}^2 \cap \Omega_{uv}^2| \\
&\quad + c_1 c_2 (|\Omega_{ij}^1 \cap \Omega_{uv}^2| + |\Omega_{ij}^2 \cap \Omega_{uv}^1|) ,
\end{aligned}
$$

(1) is a consequence of Lemma 5.3.

It follows from (1) that the last term of $\theta_1(x_i, x_j, x_k)$ in Lemma 4.6 and that of $\theta_0(x_i, x_j, x_k)$ in Lemma 5.2 coincide. Hence (2) follows from Lemma 4.6, Lemma 5.2 and Lemma 5.1. (4). Now using the formula in (2), we have (3) by Corollary 4.7.

6. Examples

In this section we study examples of doubly transitive groups satisfying Hypothesis I and show which one satisfies Hypothesis II and which one does not satisfy the condition (ii) in Theorem 4.1 and Theorem 4.2.

EXAMPLE 1. $PSL(m, q) \leqq G \leqq P\Gamma L(m, q)$, $m \geqq 3$ and $n = (q^m - 1)/(q-1)$. In this case $\Omega = P^{m-1}(q)$ or the set of one dimensional subspaces of an m-dimensional vector space over a field of q elements.

$$
\begin{aligned}
&r_1 = q-1 , \quad r_2 = q^{m-1} + \cdots + q^2 , \\
&p_{11}^1 = q-2 , \quad p_{12}^1 = 0 , \quad p_{22}^1 = q^{m-1} + \cdots + q^2 , \\
&p_{12}^2 = 0 , \quad p_{12}^1 = q-1 , \quad p_{22}^2 = q^{m-1} + \cdots + q^2 - q .
\end{aligned}
$$

Since $r_1 \neq r_2$, it follows from Lemma 2.4 that (G, Ω) satisfies Hypothesis I. Moreover it is an easy calculation to show that (G, Ω) satisfies Hypothesis II as well, for $|\Omega_{ij}^1 \cap \Omega_{st}^1|$ is determined according to the following four cases, where v_i, v_j, v_s, and v_t are nonzero vectors of the corresponding one dimensional space.

(1) $\dim\langle v_i, v_j, v_s, v_t\rangle = 2$.

(2) $\dim\langle v_i, v_j, v_s, v_t\rangle = 3$ and there is a 2-dimensional subspace containing three vectors of the four.

(3) $\dim\langle v_i, v_j, v_s, v_t\rangle = 3$ and there is no 2-dimensional subspace containing three vectors of the four.

(4) $\dim\langle v_i, v_j, v_s, v_t\rangle = 4$.

Since $r_1 \neq r_2$, the case (ii) in Theorem 4.1 and Theorem 4.2 does not occur by Lemma 3.7.

Suppose $c_2=0$. Then $c_1=1-r_1$. Hence $b_1=(1-1/r_1)/r_1$, $b_2=0$ and $d_2=0$. So the assumption of Theorem 5.4. (3) is satisfied. Thus $\mathrm{Aut}\, A \simeq \mathrm{Aut}\, A_1$ is a subgroup of Σ_n, which is the result of K. Narang in [5].

EXAMPLE 2. $G=PSL(2, 11)$ and $n=11$. Let $\alpha=(0123456789X)$, $\beta=(0)(13954)(267X8)$ and $\gamma=(0)(19)(26)(3)(45)(78)(X)$. Then $G=\langle\alpha, \beta, \gamma\rangle$ and the following hold.

$$r_1 = 3\,, \quad r_2 = 6\,,$$
$$p_{11}^1 = 0\,, \quad p_{12}^1 = 2\,, \quad p_{22}^1 = 4\,, \quad p_{11}^2 = 1\,, \quad p_{12}^2 = 2\,, \quad p_{22}^2 = 3\,.$$

Since $r_1 \neq r_2$, (G, Ω) satisfies Hypothesis I, and the case (ii) in Theorem 4.1 and Theorem 4.2 does not occur by Lemma 3.7. Further calculation shows that (G, Ω) satisfies Hypothesis II, too. Using the parameters above we can compute the explicit expression of the equation in Theorem 5.4. (3). It yields as follows.

$$f(c) = \frac{1}{108}(-2673c^4-5292c^3+2160c^2-108c-7)\,,$$

where $c=c_1/a$. Hence Theorem 5.4. (3) reads if $(c_1r_1+c_2r_2)(c_1^2r_1+c_2^2r_2) \neq 0$ and $f(c_1/a) \neq 0$, then $\mathrm{Aut}\, A_1 \simeq \mathrm{Aut}\, A$ is a subgroup of Σ_n.

EXAMPLE 3. $G=\mathrm{Co.}\ 3$ and $n=276$. Then we have the following.

$$r_1 = 112\,, \quad r_2 = 162\,,$$
$$p_{11}^1 = 30\,, \quad p_{12}^1 = 81\,, \quad p_{22}^1 = 81\,,$$
$$p_{11}^2 = 56\,, \quad p_{12}^2 = 56\,, \quad p_{22}^2 = 105\,.$$

Since $r_1 \neq r_2$, (G, Ω) satisfies Hypothesis I, and the case (ii) of Theorem 4.1 and Theorem 4.2 does not occur by Lemma 3.7.

EXAMPLE 4. $G=Sp(2m, 2)$. Then there are 2 types of doubly transitive action of G. One point stabilizers of G corresponding to these two actions are $O^{\varepsilon}(2m, 2)$, where $\varepsilon=\pm 1$. And the parameters are as follows.

$$n = 2^{m-1}(2^m+\varepsilon)\,,\quad r_1 = 2(2^{m-1}-\varepsilon)(2^{m-2}+\varepsilon)\,,\quad r_2 = 2^{2m-2}\,,$$
$$p_{11}^1 = 2^{2m-3}+\varepsilon 2^{2m-1}-3\,,\quad p_{12}^1 = 2^{2m-3}\,,\quad p_{22}^1 = 2^{2m-3}\,,$$
$$p_{11}^2 = (2^{m-1}-\varepsilon)(2^{m-2}+\varepsilon)\,,\quad p_{12}^2 = 2^{2m-3}+\varepsilon 2^{m-2}-1\,,$$
$$p_{22}^2 = 2^{m-2}(2^{m-1}-\varepsilon)\,.$$

Unless $\varepsilon=1$ and $m=2$, $r_1 \neq r_2$. And if $\delta=1$ and $m=2$, the case (ii) is satisfied. Hence if $\varepsilon=-1$ or $\varepsilon=1$ and $m\geqq 3$, Hypothesis I is satisfied and the case (ii) in Theorem 4.1 and Theorem 4.2 does not occur.

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Osaka Kyoiku University
4-88, Minami-Kawahori-cho
Tennoji-ku, Osaka 543
Japan