Harada, M.
Osaka J. Math.
23 (1986), 509–521

GENERALIZATIONS OF NAKAYAMA RING II

MANABU HARADA

(Received December 18, 1984)

We have defined (right US-3) rings satisfying (**, 3) in [5], which are rings generalized from Nakayama ring (right generalized uni-serial rings). As stated in [5], we shall give, in this note, another generalization of Nakayama rings, which is related to the condition (*, 3), and give a characterization of those rings.

1. Preliminary results. Let R be a ring with identity. We assume always throughout this note that R is a right artinian ring and every module is a right unitary R-module M with finite length, which we denote by $|M|$. We have studied the following conditions in [3] and [5]:

(**, n) *Every (non-zero) maximal submodule of a direct sum $D(n)$ of n non-zero hollow modules contains a non-trivial direct summand of $D(n)$.*

(*, n) *Every (non-zero) maximal submodule of the $D(n)$ is also a direct sum of hollow modules.*

We shall study mainly, in this note, rings satisfying (*, 3) for any direct sum of three hollow modules. We shall use the same notations as given in [3] and [5].

Let e be a primitive idempotent in R.

CONDITION II [3]. $|eJ/eJ^2|\leqslant 2$ *for each e, where J is the Jacobson radical of R.*

In [3] we have given the structure of rings which satisfy Condition II and

CONDITION I. *Every submodule in any direct sum of (three) hollow modules is also a direct of hollow modules.*

However, checking carefully each step, we know that we utilize only (*, 3) for any direct sum of three hollow modules. Thus we have the following theorem.

Theorem 1. *Let R be a right artinian ring. Assume that (*, 3) for any direct sum of three hollow modules and Condition II hold. Then for each primitive idempotent e in R, we have the following properties:*

1) $eJ=A_1\oplus B_1$, *where A_1 and B_1 are uniserial modules. Further, if $A_1/J(A_1)\approx B_1/J(B_1)$, $\alpha A_1=B_1$ for some unit α in eRe.*

2) *For every submodule N in eJ, there exists a trivial submodule $A_i \oplus B_j$ of eJ and a unit γ in eRe such that $N=\gamma(A_i \oplus B_j)$, where $A_i = A_1 J^{i-1} \subset A_1$ and $B_j = B_1 J^{j-1} \subset B_1$.*

3) *If $A_1 \approx B_1$, then $\Delta(A_i \oplus B_i) = \Delta$ and $[\Delta: \Delta(A_i \oplus B_j)] = 2$ provided $i \neq j$; further $\Delta(A_1) = \Delta(A_i) = \Delta(A_i \oplus B_j)$ $(i<j)$ and $\Delta(B_1) = \Delta(B_j) = \Delta(A_i \oplus B_j)$ $(i>j)$. If $A_1 \not\approx B_1$, then $\Delta(N) = \Delta$ for any submodule N in eJ.*

Here we shall recall the notations above. Put $\Delta = \overline{eRe} = eRe/eJe$. For any right ideal A in eR, $\Delta(A) = \{\bar{x} \mid \in \Delta, (x+j)A \subset A$ for some element j in $eJe\}$. Then $\Delta(A)$ is a subdivision ring of Δ and $[\Delta: \Delta(A)]$ means the dimension of Δ over $\Delta(A)$ as a right $\Delta(A)$-vector space.

2. Rings with (*, 3). We shall study, in this section, the converse of Theorem 1. We assume that R has the structure given in Theorem 1, unless otherwise stated.

We have given the following lemma in [3], provided Condition II″ in [3] is satisfied. We shall show in the same manner that the lemma is valid under a weaker condition.

Lemma 1 ([3]). *Let R be a ring whose structure is given as in Theorem 1, and e a primitive idempotent. Let $\{E_i\}_{i=1}^{n}$ be a family of right ideals in eR and $D = \sum_{i=1}^{n} \oplus eR/E_i$. Then, if $\Delta(A_1) = \Delta(B_1) = \Delta$, D satisfies (*, n)*

Proof. We shall quote the same argument as given in the last part of § 3 in [3], and hence use the induction on the nilpotency of J. If $E_i \subset E_j$ for some i, j, every maximal submodule of D contains a direct summand of D by assumption and [3], Lemma 27 (cf. the proof of Lemma 3 below). By induction we may consider the following case:

$$E_0 = A_i,\ E_k = A_{i_k} \oplus B_{j_k};$$
$$i < i_1 < i_2 < \cdots < i_p, \quad j_1 > j_2 > \cdots > j_p \quad \text{and} \quad D = \sum_{i=0}^{p} \oplus eR/E_i\,.$$

Assume $i_t \leqslant j_t$, $i_{t+1} > j_{t+1}$. Let M be a maximal submodule of D. We may assume that $\bar{M} = M/J(D)$ $(\subset \bar{D} = D/J(D))$ has a basis $\{(0, \cdots, 0, \overset{i}{\check{\bar{e}}}, \bar{k}_i, \cdots, 0)\}_1^p$. Since $\Delta(A_1) = \Delta(B_1) = \Delta$, we can take k_s with $k_s A_1 = A_1$ for $s \leqslant t$ and $k_r B_1 = B_1$ for $r > t$. Set $M^* = A_1/A_i \oplus \sum_{s=1}^{p} \oplus eR/(A_{i_s} \oplus B_{j_{s-1}}) \oplus B_1/B_{j_p}$, $(B_{j_0} = 0)$, then $|M^*| = |D| - 1$. Define a homomorphism f of M^* to D by setting

$$\begin{aligned} &f((x+A_i) + \sum_{s=1}^{p} (y_s + (A_{i_s} \oplus B_{j_{s-1}})) + (z + B_{j_p})) \\ &\quad = (x + y_1 + A_i) + (ek_1 y_1 + y_2 + (A_{i_1} \oplus B_{j_1}) \\ &\qquad + (ek_2 y_2 + y_3 + (A_{i_2} \oplus B_{j_2})) + \cdots + (ek_p y_p + z + (A_{i_p} \oplus B_{j_p}))\,, \end{aligned}$$

where $x \in A_1$, $y_s \in eR$ and $z \in B_1$. $A_{i_a} \oplus B_{j_{a-1}} \subset A_{i_{a-1}} \oplus B_{j_{a-1}}$, $k_a(A_{i_a} \oplus B_{j_{a-1}})$

$=A_{i_a}\oplus B_{j_{a-1}}\subset A_{i_a}\oplus B_{j_a}$ for $a\neq t+1$, and $k_{t+1}(A_{i_{t+1}}\oplus B_{j_t})\subset k_{t+1}(A_{i_{t+1}}\oplus B_{j_{t+1}})=A_{i_{t+1}}\oplus B_{j_{t+1}}$. Hence f is well defined. Assume that the right hand side of the above is zero. Since $x\in A_1$ and $z\in B_1$, $y_s\in eJ$ for all s. Put $y_r=y_{r1}+y_{r2}$, where y_{r1} is in A_1 and y_{r2} in B_1. Now $x+y_1=x+y_{11}+y_{12}\in A_i$, $x\in A_1$ and so $y_{12}=0$. $ek_1(y_{11}+y_{12})+(y_{21}+y_{22})\in A_{i_1}\oplus B_{j_1}$. Since $k_1A_1\subset A_1$, $y_{22}\in B_{j_1}$. Repeating those arguments, we assume by induction that $y_{l+12}\in B_{j_l}$ for $l<t'\leqslant t$. Put $w=ek_{t'}(y_{t'1}+y_{t'2})+(y_{t'+11}+y_{t'+12})\in A_{i_{t'}}\oplus B_{j_{t'}}$. Since $ek_{t'}$ is an isomorphism of eJ, $ek_{t'}B_{j_{t'-1}}\subset A_{j_{t'-1}}\oplus B_{j_{t'-1}}$. Now $B_{j_{t'-1}}\subset B_{j_{t'}}$ and $k_{t'}A_1\subset A_1$. Let π_2: $eJ\rightarrow B_1$ be the projection. Then $B_{j_{t'}}\in\pi_2(w)=\pi_2(ek_{t'}y_{t'2}+y_{t'+12})=\pi_2(ek_{t'}y_{t'2})+y_{t'+12}$. Since $\pi_2(ek_{t'}y_{t'2})\in B_{j_{t'}}$, $y_{t'+12}\in B_{j_{t'}}$. Consider next from the bottom side. $ek_p(y_{p1}+y_{p2})+z\in A_{i_p}\oplus B_{j_p}$. Since $k_pB_1\subset B_1$ and $x\in B_1$, $y_{p1}\in A_{i_p}$ from the same argument above (take π_1: $eJ\rightarrow A_1$). Repeating those arguments inductively, we obtain $y_{s1}\in A_{i_s}$ for $s\geqslant t+1$. Consider $ek_{t+1}(y_{t+11}+y_{t+12})+(y_{t+21}+y_{t+22})\in A_{i_{t+1}}\oplus B_{j_{t+1}}$. Since $y_{t+12}\in B_{j_t}$, $y_{t+11}\in A_{i_{t+1}}$ and $i_{t+1}>j_{t+1}$, $j_t>j_{t+1}$, $y_{t+22}\in B_{j_{t+1}}$. Similarly, from $ek_t(y_{t1}+y_{t2})+(y_{t+11}+y_{t+12})\in A_{i_t}\oplus B_{j_t}$, $y_{t1}\in A_{i_t}$. Combining the above two steps, we know that f is a monomorphism. Hence $M\approx M^*$.

It is remained for us, from Lemma 1, to study a case of $\Delta(A_1)\neq\Delta$, i.e., $\bar{A}_1\approx\bar{B}_1$. We have shown in [3] that if a right artinian ring R has the structure in Theorem 1, then (*, n) is satisfied for any $D(n)$, provided $J^3=0$. We shall show that (*, 3) is satisfied without the assumption $J^3=0$.

Lemma 2 ([3], Lemma 24). *We assume the above situation. Let δ be an element in eRe such that $\bar{\delta}\notin\Delta(A_1)$. Then $\pi_2\delta A_i=B_i$, where π_2: $eJ\rightarrow B_1$ is the projection.*

Proof. Since $[\Delta\colon\Delta(A_1)]=2$, $\bar{\delta}=\bar{a}_1+\bar{\alpha}\bar{a}_2$; the $\bar{a}_i\in\Delta(A_1)$ and $\alpha A_1=B_1$. Set $\delta=a_1+\alpha_2a_2+j$; $a_iA_1\subset A_1$, $j\in eJe$. Since $jA_i\subset A_{i+1}\oplus B_{i+1}$, $\pi_2\delta A_i=B_i$.

Lemma 3. *Assume that R has the structure* 1), 2) *and* 3) *given in Theorem* 1. *Then* (*, 2) *is fulfiled for any* $D(2)$.

Proof. The assumption 2) in Theorem 1 gives us a guarantee of (*, 1) for any hollow module. Let $eJ=A_1\oplus B_1$. If $A_1\not\approx B_1$, $\Delta(C)=\Delta$ for any submodule C of eR by assumption. Then we have shown by Lemma 1 that (*, 2) is fulfiled for any $D(2)$. Assume that $A_1\approx B_1$. Then $\Delta=\Delta(A_1)\oplus\bar{\alpha}\Delta(A_1)$, where α is the element given in 1). Set $D=eR/N_1\oplus eR/N_2$, where the N_i are submodules of eR. We shall show the lemma by induction on the nilpotency of J. If $J^3=0$, we are done in [3], §4. Assume $eJ^n\neq 0$ and $eJ^{n+1}=0$. If $N_i\supset eJ^n$ for $i=1, 2$, eR/N_i is a hollow R/J^n-module. Hence we may assume that $N_1=A_i=A_1J^{i-1}$ by induction. Let M be a maximal submodule of D, and put $\bar{D}=D/\mathrm{J}(D)\supset\bar{M}=M/\mathrm{J}(D)$. We may assume that $\bar{M}$ has a basis $\{(e+\mathrm{J}(D),\ \delta+\mathrm{J}(D))\}$, where δ is a unit element in eRe (it is sufficient to show the lemma in case R is basic; see [2] and [3]).

i) $N_2=A_k\oplus B_j$, (we may assume $k\leqslant i$ [3]) a) $i\leqslant k\leqslant i$. $F=A_i\cap\delta^{-1}(A_k\oplus B_j)=\delta^{-1}(\delta A_i\cap(A_k\oplus B_j))$. a)–i) If $\bar{\delta}\in\Delta_1=\Delta(A_1)$, $F=A_k$ (we may assume $\delta A_1\subset A_1$). a)–ii). If $\bar{\delta}\notin\Delta_1$, $F=A_j$ by Lemma 2. Put $M^*=eR/A_k\oplus A_1/A_i\oplus B_1/B_j$ for the case a–i). Define a homomorphism f of M^* to D by setting

$$f((x+A_k)+(y+A_i)+(z+B_j))=(x+y+A_i)+(\delta x+z+(A_k\oplus B_j)),$$

where x is in eR, y in A_1 and z in B_1. Then f is well defined. It is easy to check that f is a monomorphism, since $\delta A_1=A_1$. Put $M^*=eR/A_j\oplus A_1/A_i\oplus A_1/A_k$ for the case a)–ii). Define a homomorphism f of M^* into D by setting

$$f((x+A_j)+(y+A_i)+(z+A_k))=(x+y+A_i)+(\delta x+z+A_k\oplus B_j),$$

where x is in eR and y, z are in A_1. We can show from the fact: $\bar{\delta}\notin\Delta_1$, that f is a monomorphism (cf. the proof of Lemma 4 below). Hence $M\approx M^*$, since $|M|=|M^*|$ and $f(M^*)\subset M$.

b) $k\leqslant i\leqslant j$. If $\bar{\delta}\in\Delta_1$ (resp. $\notin\Delta_1$), $F=A_i$ (resp. $F=A_j$). We obtain the same result as in a–ii) for $F=A_j$. If $F=A_i$, put $M^*=eR/A_i\oplus A_1/A_k\oplus B_1/B_j$. Then $M\approx M^*$ as above.

c) $k\leqslant j\leqslant i$. Since $eReA_i\subset A_k\oplus B_j$, M contains a direct summand of D.

ii) $N_2=A_j$, $i\geqslant j$. If $\bar{\delta}\notin\Delta_1$, $\delta A_i\cap A_j=0$ and M is isomorphic to $eR\oplus A_1/A_i\oplus A_1/A_j$. If $\bar{\delta}\in\Delta_1$, $\delta A_i\subset A_j$. Hence we obtain the same situation as in i)–c).

Lemma 4. (*, 3) *is satisfied for any three hollow modules.*

Proof. We may assume $\Delta(A_1)=\Delta_1\neq\Delta$ by Lemma 1. From induction on the nilpotency of $J(R)$, it is sufficient to study the case:

$$E_0=A_i,\ E_1=A_{i_1}\oplus B_{j_1}\quad\text{and}\quad E_2=A_{i_2}\oplus B_{i_2}$$

with $i_k\leqslant j_k$ for $k=1$, 2, and $D=\sum\oplus eR/E_i$. Here B_{j_k} may be equal to zero (cf. [3], § 3).

If $B_{j_1}=B_{j_2}=0$, D satisfies (*, 3) by [4], Corollary 3. Let M be a maximal submodule of D. If M contains a non-zero direct summand D_1 of D, $M=D_1\oplus M_1$ where M_1 is a maximal submodule of $N_1\oplus N_2$; the N_i are isomorphic to some of $\{eR/E_i\}_{i=1}^{3}$. Then M_1 is a direct sum of hollow modules by Lemma 3, and hence so is M. Therefore we consider M not containing a direct summand of D. Put $\bar{D}=D/J(D)\supset\bar{M}=M/J(D)$, and $D=(\bar{e}\Delta,\bar{e}\Delta,\bar{e}\Delta)$. Then the above $\bar{M}$ has a basis $\{(\bar{e},\bar{\delta}_1,0),(0,\bar{e},\bar{\delta}_2)\}$, where $\bar{\delta}_i$ are in Δ and $\bar{\delta}_1\bar{\delta}_2\neq 0$ (cf. [3]). We consider the following situation:

1) $i\leqslant i_1\leqslant i_2\leqslant j_1\leqslant j_2$.

a) $\bar{\delta}_2\in\Delta_1$. Then $\delta_2E_2\subset E_1$. Hence M contains a direct summand of D by [1], Theorem 2. (1)

b) $\bar{\delta}_1 \in \Delta_1$ and $\bar{\delta}_2 \notin \Delta_1$. $M \approx A_1/A_i \oplus eR/A_{i_1} \oplus eR/(A_{j_1} \oplus B_{j_2}) \oplus A_1/A_{i_2}$. (2)

c) $\bar{\delta}_1$ and $\bar{\delta}_2 \notin \Delta_1$. $M \approx A_1/A_i \oplus eR/A_{i_2} \oplus eR/(A_{j_1} \oplus B_{j_2}) \oplus A_1/A_{i_1}$. (3)

2) $i_1 \leqslant i \leqslant i_2 \leqslant j_1 \leqslant j_2$.

a) $\bar{\delta}_1$ or $\bar{\delta}_2 \in \Delta_1$. We obtain (1).

b) $\bar{\delta}_1$ and $\bar{\delta}_2 \notin \Delta_1$. We obtain (3).

3) $i_1 \leqslant i_2 \leqslant i \leqslant j_1 \leqslant j_2$. Since $E_1 \supset E_2 \supset E_3$. We obtain (1) by [4], Corollary 3.

4) $i_1 \leqslant j_1 \leqslant i_2 \leqslant j_2$. Since $eReE_2 \subset E_1$, we obtain (1).

5) $i \leqslant i_1 \leqslant i_2 \leqslant j_2 \leqslant j_1$.

a) $\bar{\delta}_1$ and $\bar{\delta}_2 \in \Delta_1$. We obtain the same situation as in the proof of Lemma 1. i.e., $M \approx A_1/A_i \oplus eR/A_{i_1} \oplus eR/(A_{i_2} \oplus B_{j_1}) \oplus B_1/B_{j_2}$. (4)

b) $\bar{\delta}_1 \in \Delta_1$ and $\bar{\delta}_2 \notin \Delta_1$. $M \approx A_1/A_i \oplus eR/A_{i_1} \oplus eR/(A_{j_2} \oplus B_{j_1}) \oplus A_1/A_{i_2}$. (5)

c) $\bar{\delta}_1 \notin \Delta_1$ and $\bar{\delta}_2 \in \Delta_1$. $M \approx A_1/A_i \oplus eR/A_{j_2} \oplus eR/(A_{i_2} \oplus B_{j_1}) \oplus A_1/A_{i_1}$. (6)

d) $\bar{\delta}_1$ and $\bar{\delta}_2 \notin \Delta_1$. $M \approx A_1/A_i \oplus eR/A_{i_2} \oplus eR/(A_{j_2} \oplus B_{j_1}) \oplus A_1/A_{i_1}$. (7)

6) $i_1 \leqslant i \leqslant i_2 \leqslant j_2 \leqslant j_1$.

a) $\bar{\delta}_1 \in \Delta_1$. We obtain (1).

b) $\bar{\delta}_1 \notin \Delta_1$ and $\bar{\delta}_2 \in \Delta_1$. We obtain (6).

c) $\bar{\delta}_1$ and $\bar{\delta}_2 \notin \Delta_1$. We obtain (7).

7) $i_1 \leqslant i_2 \leqslant i \leqslant j_2 \leqslant j_1$.

a) $\bar{\delta}_1 \in \Delta_1$ or $\bar{\delta}_2 \notin \Delta_1$ and $\bar{\delta}_2 = \bar{\delta}_1 \bar{x}_2$; $\bar{x}_2 \in \Delta_1$. We obtain (1).

b) $\bar{\delta}_1 \notin \Delta_1$ and $\bar{\delta}_2 \in \Delta_1$. We obtain (6).

c) $\bar{\delta}_1$ and $\bar{\delta}_2 \notin \Delta_1$ and $\{\bar{\delta}_1, \bar{\delta}_2\}$ is linearly independent over Δ_1.
$M \approx A_1/A_{i_1} \oplus eR/A_i \oplus eR/(A_{j_2} \oplus B_{j_1}) \oplus A_1/A_{i_2}$. (8)

8) $i_1 \leqslant i_2 \leqslant j_2 \leqslant i \leqslant j_1$ or $i_1 \leqslant i_2 \leqslant j_2 \leqslant j_2 \leqslant i$. Since $eReE_0 \subset E_2$, we obtain (1).

We shall give a sample of proofs.

1)–c). Put $\xi' = (\bar{e}, \bar{\delta}_1, 0)$ and $\eta' = (0, \bar{e}, \bar{\delta}_2)$. Consider $\{\xi', \eta'' = (0, \bar{\delta}_2', \bar{e})\}$, where $\bar{\delta}_2' = \bar{\delta}_2^{-1} \notin \Delta_1$. If $\{\bar{\delta}_1, \bar{\delta}_2'\}$ is lineraly independent, there exist $\bar{a}_1'$ and $\bar{a}_2'$ in Δ_1 such that $\bar{e} = \bar{\delta}_1 \bar{a}_1' + \bar{\delta}_2' \bar{a}_2'$ and $\bar{a}_1' \bar{a}_2' \neq 0$, since $[\Delta : \Delta_1] = 2$. Then $\bar{M}$ has a basis $\{\xi = \xi' + \eta'' \bar{a}_2' \bar{a}_1'^{-1} = (\bar{e}, \bar{a}_1, \bar{a}_2)$ and $\eta = \eta'' = (0, \bar{\delta}_2', \bar{e})\}$, where $\bar{a}_1 = \bar{a}_1'^{-1}$ and $\bar{a}_2 = \bar{a}_2' \bar{a}_1'^{-1}$. On the other hand, if $\bar{\delta}_1 = \bar{\delta}_2' \bar{a}_2''$, $\bar{M}$ has a basis $\{\xi = \xi - \eta'' \bar{a}_2'' = (\bar{e}, 0, \bar{a}_2)$ $(\bar{a}_2'' = \bar{a}_2)$ and $\eta = \eta'' = (0, \bar{\delta}_2', \bar{e})\}$. In either case, $\bar{a}_2 \neq 0$ and define a homomorphism f of $M^* = A_1/A_i \oplus eR/A_{i_2} \oplus eR/(A_{j_1} \oplus B_{j_2}) \oplus A_1/A_{i_1}$ to D by setting

$$f((x + A_i) + (y + A_{i_2}) + (z + (A_{j_1} \oplus B_{j_2})) + (w + A_{i_1}))$$
$$= (x + y + A_i) + (a_1 y + \delta_2' z + w + (A_{i_1} \oplus B_{j_1})) + (a_2 y + z + (A_{i_2} \oplus B_{j_2})),$$

where x is in A_1, y and z in eR and w in A_1.

Since $A_i \cap a_1^{-1}(A_{i_1} \oplus B_{j_1}) \cap a_2^{-1}(A_{i_2} \oplus B_{j_2}) = A_{i_2}$ $(0^{-1}(A_{i_1} \oplus B_{j_1}) = eR)$ and $\delta_2'^{-1}(A_{i1} \oplus B_{j_1}) \cap (A_{i_2} \oplus B_{j_2}) = A_{j_1} \oplus B_{j_2}$ by Lemma 2, f is well defined. Assume that the latter term of the above equation is zero, i.e.,

0) $x, w \in A_1$.

1) $x + y \in A_i$.

2) $a_1 y + \delta_2' z + w \in A_{i_1} \oplus B_{j_1}$.

3) $a_2 y + z \in A_{i_2} \oplus B_{j_2}$.

Since x is in $A_1\subset eJ$, y and z are in eJ by 1) and 3). Put $z=a+b$; $a\in A_1$, $b\in B_1$. Since we may assume $a_iA_1=A_1$, b is in B_{j_2} by 3). $\delta_2'z=\delta_2'a+\delta_2'$ and $\delta_2'b\in A_{j_2}\oplus B_{j_2}\subset A_{i_1}\oplus B_{j_1}$. Hence a is in A_{j_1} by 2) and Lemma 2, and so z is in $A_{j_1}\oplus B_{j_2}\subset A_{i_2}\oplus B_{j_2}$. Therefore y is in A_{i_2} by 3), since $\bar{a}_2\neq 0$, and so x in A_i, w in A_{i_1}. We have shown that f is a monomorphism. On the other hand, $|D|=n+i+i_1+i_2+j_1+j_2-2$ and $|M^*|=n+i+i_1+i_2+j_1+j_2-3=|D|-1$, where $eJ^n\neq 0$, $eJ^{n+1}=0$. Hence $f(M^*)=M$, for $M\supset \mathrm{J}(D)$ and $f(M^*)=M$.

Now let $eJ=A_1\oplus B_1$ be as before and $eJ^n\neq 0$ and $eJ^{n+1}=0$. We consider here together all cases: a) $B_1=0$, b) $\bar{A}\not\approx\bar{B}_1$ and c) $\bar{A}_1\approx\bar{B}_1$. We obtain the following three hollow modules;

1) $S_i(e)=eR/(A_1\oplus B_i)$, 2) $T_i(e)=eR/A_i$ (or eR/B_j) and 3) $U_{ij}(e)=eR/(A_i\oplus B_j)$ (we denote those modules by $H(e)$).

Now S_i and U_{ij} are R/J^t-modules, where $t=i$ and max $\{i, j\}$, respectively. We shall give a weight for each hollow module H as follows; $\mathrm{w}(H)=|\mathrm{J}(H)/\mathrm{J}^2(H)|$, i.e., $\mathrm{w}(S_i)=1$, $\mathrm{w}(T_i)=2$ $(i\neq 1)$, $\mathrm{w}(T_1)=1$ and $\mathrm{w}(U_{ij})=2$ $(i\neq 1$ and $j\neq 1)$.

Lemma 5. *Let $S(e)$, $T(e)$ and $U(e)$ be as above. Then for a maximal submodule M of D below, we obtain the following*:

1) $D=S(e)\oplus S'(e)$. $M\approx S(f_1)\oplus S(e)$, $S(e)\oplus S(f_2)$ *or* $U(e)$.

2) $D=T(e)\oplus S(e)$. $M\approx S(f_1)\oplus S(f_2)\oplus S(e)$ *or* $T'(e)\oplus S(f)$.

3) $D=U(e)\oplus S(e)$. $M\approx S(f_1)\oplus S(f_2)\oplus S(e)$, $U(e)\oplus S(f)$ *or* $U'(e)\oplus S(f)$, *where e and f are primitive idempotents.*

Proof. We can show the lemma from Lemmas 1 and 3 (consider D as R/J^t-modules for 3); $t\leqslant n$).

Assume that

$$C=\sum_{i=1}^{p}\sum_{j=1}^{j_i}\oplus H_j(e_i)\,,$$

where $1=\sum e_i$, $\{e_i\}$ is a set of mutually orthogonal primitive idempotents (and R is basic). Let M be a maximal submodule of C. Since $H_j(e_i)/J(H_j(e_i))\not\approx H_{j'}(e_{i'})/J(H_{j'}(e_{i'}))$ for $i\neq i'$, $M=\sum_i\oplus M_i$, where $M_k=\sum_j\oplus H_j(e_k)$ for all k except some q and M_q is a maximal one in $\sum_j\oplus H_j(e_q)$. Put $\mathrm{w}(C)=\sum_i\sum_j \mathrm{w}(H_j(e_i))$.

Lemma 6. *Every submodule F of $D(q)$ is a direct sum of hollow modules H_i and $w(F)\leqslant 2q$ $(q\leqslant 3)$.*

Proof. We shall show the lemma for a case $q=3$. The remaining parts are same. In order to prove the lemma, we may show that any maximal submodule M of C above with $t=\mathrm{w}(C)\leqslant 6$ has a similar direct decomposition and $\mathrm{w}(M)\leqq t$. Further, from the argument before Lemma 6, we may assume $e_i=e$, and show that

$$M=\sum_{s=1}^{m}\oplus H_s \quad \text{and} \quad \mathrm{w}(M)\leqslant t \qquad (\sharp).$$

We note that if $\mathrm{w}(H_i(e))=2$, $\mathrm{J}(H_i(e))$ is a direct sum of two uniserial modules. If $H_i(e)=eR$ for some i, M contains a direct summand of C by [1], Theorem 2. Hence M satisfies ($\sharp$) by induction on m and the above remark. We shall show ($\sharp$) by induction on n ($J^{n+1}=0$). If $n=0$, then ($\sharp$) is trivial. We assume that every maximal submodule M satisfies ($\sharp$) for $k\leqslant n-1$. Start from

$$D=H(e_1)\oplus H(e_2)\oplus H(e_3)\,.$$

$\mathrm{w}(D)=6$ provided no-one of $\{H(e_i)\}$ is uniserial, and $\mathrm{w}(D)\leqslant 5$ for other cases. Further, if no-one of $\{H(e_i)\}$ is isomorphic to $T_i(e)$, the $H(e_j)$ are R/J^t-modules for some $t\leqslant n$. Then we can show ($\sharp$) by the induction hypothesis. Hence assume $H(e_1)=T_i(e_1)$. We may further assume $e=e_i$ for all i from the remark before Lemma 6. Let M be a maximal submodule of D. Then from Lemma 4 $M=\sum_{i=1}^{4}\oplus H(f_i)$; $f_i=e$ if $H(f_i)\approx T$ or U, and $\mathrm{w}(D)\geqslant \mathrm{w}(M)$. Put $M_0=\sum_{f_i\neq e}\oplus H(f_i)$. First we remark that the M_0 is an R/J^t-module, and hence ($\sharp$) is satisfied for M_0. Further, if no-one of $\{H(e_i)\}$ is isomorphic to $T_1(e)=eR/A_1$, the same for $\{H(f_i)\}$. Now let M be the maximal submodule in $C(\subset D)$ given in the beginning. Remarking the above fact (the case $H(e)=T_1(e)$), we have the following cases:

I) $C=T_{i_1}\oplus T_{i_2}\oplus T_{i_3}$, $T_{i_1}\oplus T_{i_2}\oplus U_{k_1j_1}$, *or* $T_{i_1}\oplus U_{k_1j_1}\oplus U_{k_2j_2}$.
In the first case M contains a direct summand of C, and hence we have ($\sharp$) by Lemmas 1 and 3. For the remaining cases we can use Lemmas 1 and 4.

II) $C=T_{i_1}\oplus T_{i_2}\oplus S_{k_1}\oplus S_{k_2}$.
M contains a direct summand of C by [1], Theorem 2. Repeating this argument, we can reduce M to a case $M=M_1\oplus S_{k_1}\oplus S_{k_2}$ (M_1 is a maximal in $T_{i_1}\oplus T_{i_2}$), $M=M_2\oplus T_{i_2}\oplus S_{k_2}$ (M_2 is maximal in $T_{i_1}\oplus S_{k_1}$) or $M=M_3\oplus T_{i_1}\oplus T_{i_2}$ (M_3 is maximal in $S_{k_1}\oplus S_{k_2}$). Therefore M satisfies ($\sharp$) by Lemma 5.

III) $C=T_{i_1}\oplus U_{k_1j_1}\oplus S_{h_1}\oplus S_{h_2}$, *or* $T_1\oplus T_1\oplus U_{i_1j_1}\oplus S_{k_1}$.
We can make use of the same argument as in I).

IV) T_i does not appear in a direct summand of C, for instance $C=U_{i_1j_1}\oplus U_{i_2j_2}\oplus U_{i_3j_3}$.
We can use the induction hypothesis.

V) Some of T, U and S are equal to zero.
We have the same result as above.

Thus we have

Theorem 2. *Let R be a right artinian ring satisfying Condition II. Then the following conditions are equivalent*:

1) *Every submodule of any $D(3)$ is a direct sum of hollow modules.*
2) (*, 3) *holds for any $D(3)$.*

3) *eR has the structure given in Theorem* 1 *for each primitive idempotent e.*

In this case every submodule of $D(i)$ is a direct sum of at most $2i$ hcllow modules for $i\leqslant 3$.

Remark. If R is an algebra of finite dimension over a field K, then H. Asashiba has shown that (*, 3) implies Condition II. Further, if K is algebraically closed, $\Delta(N)=\Delta=K$ for any submodule N of eR. If $\Delta(N)=\Delta$ for N, (*, 3) implies Condition II by [2], Proposition 10.

Theorem 3. *Let R be as above. Assume that $\Delta(N)=\Delta$ for any submodule N of eR. Then the following statements are equivalent:*

1) *Every submodule of a finite direct sum of any hollow modules is also a direct sum of hollow modules.*

2) *Every submodule of a direct sum of any three hollow modules is also a direct sum of hollow modules.*

3) (*, 3) *holds for any $D(3)$.*

In this case every submodule M of $D(i)$ is a direct sum of at most $2i$ hollow modules.

The author believes that Theorem 3 will be true without assumption $\Delta(N)=\Delta$. However, he can not find a systematic proof. We have studied this problem in [3], § 4, provided $J^3=0$. We shall extend this manner to the case $J^4=0$.

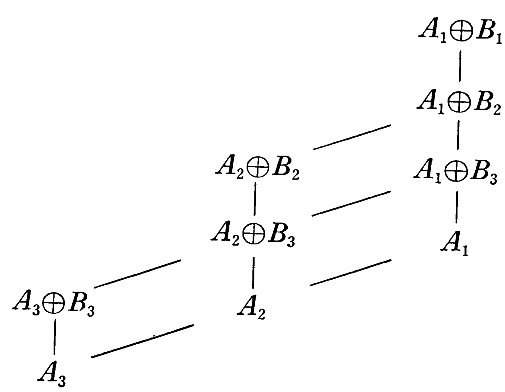
Proposition 4. *Let R be a right artinian ring with $J^4=0$ and assume that Condition* II. *Then the following conditions are equivalent:*

1) *Condition I for any direct sum of hollow modules holds.*

2) *Condition I for any direct sum of three hollow modules holds.*

3) *eR has the structure given in Theorem* 1.

Proof. We may consider the proposition in case of $\Delta(A_1)\neq\Delta$. Under the assumption above, we obtain the diagram of submodules in eJ up to isomorphism:



Let $\{E_i\}_{i=1}^4$ be a family of the modules above. Put $D=\sum_{i=1}^4 \oplus eR/E_i$. Then, since $\Delta(A_2\oplus B_2)=\Delta$, every maximal submodule M of D contains a non-trivial direct summand of D by [1], Theorem 2 and [4], Corollary 3, except $D_1=eR/A_1 \oplus eR/A_1\oplus eR/(A_2\oplus B_3)\oplus eR/(A_2\oplus B_3)$. Let M be the maximal submodule such that $\bar{M}=M/J(D)=\xi\Delta\oplus\eta\Delta\oplus\zeta\Delta$, where $\xi=(\bar{e}, \bar{\delta}_1, 0, 0)$, $\eta=(0, \bar{e}, \bar{\delta}_2, 0)$ and $\zeta=(0, 0, \bar{e}, \bar{\delta}_3)$. If $\bar{\delta}_1$ or $\bar{\delta}_3$ is in Δ_1, M contains a direct summand of D. Assume $\bar{\delta}_1$ and $\bar{\delta}_3 \notin \Delta_1$. If $\bar{\delta}_2\in\Delta_1$, there exist $\bar{a}_2$, $\bar{a}_3$ $(\neq 0)$ in Δ_1 such that $\bar{\delta}_2\bar{a}_2^{-1}-\bar{\delta}_2^{-1}\bar{\delta}_3\bar{a}_3 =-\bar{\delta}_1$, for $[\Delta: \Delta_1]=2$. Then $\bar{M}$ has a basis $\{\xi(-\bar{\delta}_1^{-1}\bar{\delta}_2^{-1}(\bar{a}_2-\bar{\delta}_3^{-1})+\eta\bar{\delta}_2^{-1}(\bar{a}_2-\bar{\delta}_3^{-1}\bar{a}_3)+\zeta\bar{\delta}_3^{-1}\bar{a}_3=(\bar{e}, 0, \bar{a}_2, \bar{a}_3), \eta, \zeta\}$. Then $M\approx eR/A_2\oplus eR/A_2\oplus eR/(A_3\oplus B_3)$ as in the proof of Lemma 4. Next assume $\bar{\delta}_2\notin\Delta_1$. If $\bar{\delta}_1=\bar{\delta}_2^{-1}\bar{a}_2$, $\bar{M}$ has a basis $\{(\bar{e}, 0, \bar{a}_2, 0), \eta, \zeta\}$. If $\{\bar{\delta}_1, \bar{\delta}_2^{-1}\}$ is linearly independent, there exist $\bar{a}_1$, $\bar{a}_2$ in Δ_1 with $\bar{a}_1\bar{a}_2\neq 0$ such that $\bar{e}=\bar{\delta}_1\bar{a}_1+\bar{\delta}_2^{-1}\bar{a}_2$. Then $\bar{M}$ contains a basis $\{(\bar{e}, \bar{a}_1, \bar{a}_2, 0), \eta, \zeta\}$. Repeating this argument for η and ζ, we obtain a basis $\{(\bar{e}, \bar{a}_1, \bar{a}_2, 0), (0, \bar{e}, \bar{b}_1, \bar{b}_2), \zeta\}$, where $\bar{a}_2\bar{b}_2\neq 0$. In this case we obtain also the same result. Therefore every maximal submodule of D is a direct sum of hollow modules. Finally, if D is a direct sum of m hollow modules $(m\geqslant 5)$, M contains a non-trivial direct summand of D by [1], Theorem 2 and [4], Corollary 3. Hence we can prove the proposition by induction on m.

3. Right US-3 rings with (*, n). We have defined right US-3 rings in [5], i.e., rings satisfying (**, 3). In this section we shall study the structure of right US-3 rings with (*, 1) or (*, 2).

Lemma 7. *If a right US-3 ring satisfies (*, 2) for any $D(2)$, then Condition I is satisfied for any $D(n)$.*

Proof. Let $\{N_i\}_{i=1}^n$ be a set of hollow modules, and put $D=\sum_{i=1}^n \oplus N_i$. If $n\geqslant 3$, every maximal submodule M of D is of a form $M_1\oplus\sum_{i=3}^n \oplus N_i'$, where M_1 is a maximal submodule of $N_1'\oplus N_2'$ and the N_i' are isomorphic to some in $\{N_i\}$. Hence M_1 is a direct sum of hollow modules by (*, 2).

Theorem 5. *Let R be a right artinian ring. Then R is a right US-3 ring and (*, 2) holds for any $D(2)$ if and only if, for each primitive idempotent e, eJ has the following structure:*

I) $eJ^2=0$. 1) $eJ=A_1\oplus B_1$ *with* A_1, B_1 *simple or zero.* 2) *If* $A_1\approx B_1$, $[\Delta: \Delta(A_1)]=2$ *and, for any simple submodule C in eJ, $A_1\sim C$, i.e., there exists a unit x in eRe such that $xC\subset A_1$.*

II) $eJ^2\neq 0$. 1) $eJ=A_1\oplus B_1$ *with* A_1 *uniserial and* B_1 *simple or zero.* 2) $\Delta=\Delta(E)$ *and* 3) $xE=A_i$ *or* $xE=A_i\oplus B_1$, *where E is a submodule of eJ, A_i is a submodule of A_1 and x is a unit in eRe.*

Proof. If (*, 2) and (**, 3) hold, $|eJ/eJ^2|\leqslant 2$ by [5], Proposition 1, and

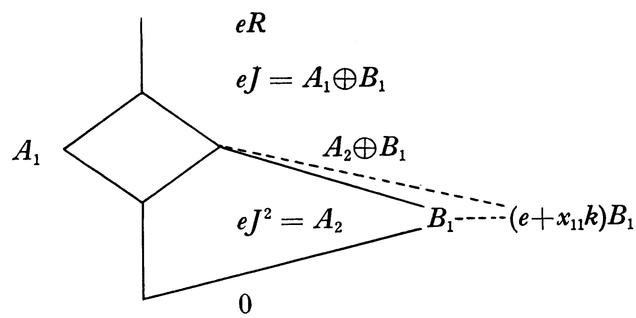
Condition I holds for any D(n) by Lemma 7. Hence eR has the structure in Theorem 1. If $eJ^2=0$, we are done. Assume that $eJ^2\neq 0$, and $eJ=A_1\oplus B_1$ with A_1, B_1 uniserial. Put $A_i=A_1J^{i-1}$ and $B_j=B_1J^{j-1}$. If $A_1\approx B_1$, $[\Delta\colon \Delta(A_1)]=2$ by [3], Theorem 2. Since $A_2\neq 0$ and hence $B_2\neq 0$, $eR/A_1\oplus eR/A_1\oplus eR/(A_2\oplus B_2)$ does not satisfy (*, 3) from [4], Corollary 2. Hence $A_1\not\approx B_1$. If $A_1\neq 0$ and $B_2\neq 0$, any two modules of $\{A_1, A_2\oplus B_2, B_1\}$ are not related by $\sim$, which contradicts [5], Lemma 1 (note that $\Delta=\Delta(E)$ and $eJ=A_1\oplus B_1$). Hence B_1 (or A_1) is simple or zero. The remaining parts are clear from [3], Theorem 1. Conversely, if the case I) occurs, Condition I and (**, 3) hold by [2], Theorem 12 and [3], Theorem 2 (note that $\Delta=\Delta(A)$ provided $A_1\not\approx B_1$). Assume the case II). Then (*, 2) holds for any $D(2)$ by Lemma 3. Further, since $\Delta=\Delta(E)$, $N_1\oplus N_2$ satisfies (**, 2) provided $N_i=eR/C_i$ and $C_1\sim C_2$ by [4], Corollary 1. If $\{E_i\}_{i=1}^3$ is a family of submodules in eJ, then $E_{i_1}\sim E_{i_2}$ by the assumption 3). Hence (**, 3) holds for any $D(3)$.

Theorem 6. *Let R be a right US-3 ring. Then (*, 1) holds for any hollow module if and only if eR has one of the following structure for each primitive idempotent e:*

1) $|eJ/eJ^2|\leqslant 1$.
2) $|eJ/eJ^2|=2$
 i) $eJ^2=0$
 ii) $eJ^2\neq 0$, $eJ=A_1\oplus B_1$ *has the structure as in Theorem* 1, *where A_1 is uniserial and B_1 is simple* ($A_1\not\approx B_2$).

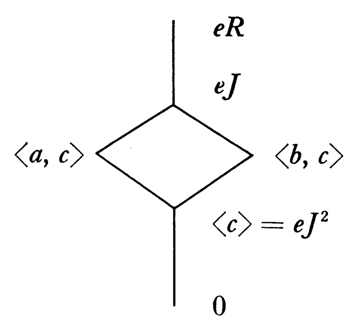
Proof. Since R is a right US-3 ring, $|eJ/eJ^2|\leqslant 2$ by [5], Theorem 2. Assume that (*, 1) holds and $|eJ/eJ^2|=2$. Then $eJ=A_1\oplus B_1$ by assumption, where A_1 and B_1 are hollow. If $\bar{A}_1=A_1/A_1J\approx\bar{B}_1$, eJ^2 is a waist and $A_1\approx B_1$ by [5], Theorem 2. Hence, if $eJ^2\neq 0$, $A_1J\subsetneqq eJ^2$. Then eR/A_1J contains a non-trivial waist eJ^2/A_1J and eJ/A_1J is not hollow. Accordingly, eJ/A_1J is not a direct sum of hollow modules. Therefore $eJ^2=0$. Next assume $eJ^2\neq 0$, and hence $A_1\not\approx B_1$. Then $\Delta(A_1)=\Delta(B_1)=\Delta$ and $A_1\not\sim B_1$. From the proof of Theorem 5, we can show that either A_1 or B_1 is simple (note $|eJ/eJ^2|=2$), say B_1. We shall show that A_1 is uniserial. We know from the proof of [5], Theorem 2 that if $\Delta(C)\neq\Delta$ for some submodule C of eJ, then eJ contains a non-trivial waist module eJ^i with $|eJ^i/eJ^{i+1}|=2$. Then (*, 1) does not hold from the observation of the case $eJ^2=0$. Hence $\Delta(C)=\Delta$ for all C in eR. Now $J(A_1)=A_2\oplus A_2'\oplus A_2''\oplus\cdots$ from (*, 1), where A_2, A_2', $\cdots$ are hollow (actually $A_2''=\cdots=0$ from [5], Theorem 2). Being $A_2\not\sim B_1$ and $A_2'\not\sim B_1$, we know that $A_2\sim A_2'$. Let a_2 be in A_2-A_2J. Since $\Delta(A_2)=\Delta$ and $A_2\sim A_2'$, there exist a unit x in eRe and j in eJe such that $xA_2=A_2$ and $(x+j)A_2=A_2'$. Put $a_2'=(x+j)a_2\in A_2'$. Since x is an isomorphism of A_2, $xa_2\notin A_2J$, $ja_2\in eJeeJ\subset eJ^2=A_2J\oplus A_2'\oplus\cdots$, which is a contradiction. Hence $A_2=A_2'$. Repeating this procedure, we know that A_1 is uniserial. Therefore every submodule of eJ is one of the following: 1) A_i, 2) $A_i\oplus B_1$, and 3) $A_i(f)$, where $A_i=A_1J^{i-1}$ and $A_i(f)=\{a_i+f(a_i)\mid a_i\in A_i, f\in \mathrm{Hom}_R(A_i, B_1)\}$. Assume $A_n\neq 0$ and $A_{n+1}=0$. Then considering $\{A_i, A_i(f), B_1\}$ $(i<n)$, $A_i\sim A_i(f)$. It is clear from [5], Lemma 1 that $A_n\sim A_n(f)$ or $A_n(f)\sim B_1$ (if $A_n\sim A_n(f)$, $A_n=A_n(f)$ for $eJeA_n=0$). Therefore eJ has the structure in Theorem 5. Conversely, assume that eR has the structure of the theorem. If $|eJ/eJ^2|\leqslant 1$, eJ^2 is a waist, and hence, for any submodule $C\subset eJ^2$, $J(eR/C)=eJ/C$ contains a unique maximal submodule eJ^2/C. If $eJ^2=0$, (*, 2) holds for any two hollow modules by [3], Proposition 3. It is clear for the last case to show that (*, 1) holds.

4. Examples. 1. Let R be the algebra over a field K given in [3], Example 2. Then the lattice of submodules of eR is the following:



where k are in K. Hence (**, 3) and (*, 2) are satisfied by Theorem 5.

2. Let R be a vector space over K with basis $\{e, f, a, b, c, d\}$. Define the multiplication among these elements as follows: $e^2=e$, $f^2=f$, $ef=fe=0$, $ea=ae=a$, $eb=bf=b$, $ec=cf=c$, $fd=df=d$, $ab=bd=c$ and other products are equal to zero. Then the lattice of submodules of eR is the following:



Then R is a right US-3 ring with Condition II′. However, eJ is indecomposable, but not hollow. Hence (*, 1) is not satisfied.

3. Let L, K be fields with $[L: K]=2$. Put

$$R=\begin{pmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K \end{pmatrix}.$$

Then R is a right US-3 ring with (*, 1), but without (*, 2) (note that $\Delta(e_{13}K) = K \neq L = \Delta$).

4. Assume that a right artinian ring R has a decomposition $R=eR\oplus fR$ and $J^2=0$, where $\{e, f\}$ is a set of mutually orthogonal primitive idempotents. Then (*, 2) holds for any $D(2)$ by [3], Proposition 3. We shall give the complete list of such rings with (**, 3) and Condition II. If R is the ring mentioned above, $eJ=A_1\oplus A_2$ and $fJ=B_1\oplus B_2$, where the A_i and the B_i are simple or zero. We always assume, in the following observation, that

α) $$\begin{pmatrix} T_1 & A_1\oplus A_2 \\ B_1\oplus B_2 & T_2 \end{pmatrix}$$

means that T_1 and T_2 are local right artinian rings, the A_i (resp. the B_i) are right T_2 and left T_1 (resp. right T_1 and left T_2) simple module, $(A_1\oplus A_2)J(T_2) = J(T_1)(A_1\oplus A_2)=0$ (the same for $B_1\oplus B_2$), and $(A_1\oplus A_2)(B_1\oplus B_2)=(B_1\oplus B_2)(A_1\oplus A_2)=0$.

β) Δ means a division ring.

γ) S means a local serial ring.

δ) L means the following local ring:

$J(L) = A_1\oplus A_2$, $A_1\approx A_2$ as right L-modules,

ξ) $[L/J(L): L/J(L)(A_1)]=2$, and for any simple L-module A_1' in $J(L)$, there exists a unit α in L such that $A_1'=\alpha A_1$ (see [1] for such a ring).

i) $A_1\approx A_2\approx \overline{eR}$ and $B_1\approx B_2\approx \overline{fR}$. Then $eJf=fJe=0$. Hence

$$R=\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

ii) $A_1\approx A_2\approx \overline{fR}$, $B_1\approx B_2\approx \overline{eR}$. Then

$$R=\begin{pmatrix} \Delta_1 & A_1\oplus A_2 \\ B_1\oplus B_2 & \Delta_2 \end{pmatrix},$$

where the A_i (resp. B_i) satisfy ξ) as $\Delta_1-\Delta_2$ (resp. $\Delta_2-\Delta_1$) bimodules.

iii) $A_1\approx A_2\approx B_1\approx B_2\approx \overline{eR}$ (resp. $\approx\overline{fR}$). Then

$$R=\begin{pmatrix} L_1 & 0 \\ B_1\oplus B_2 & \Delta_2 \end{pmatrix} \quad \left(\text{resp. } R=\begin{pmatrix} \Delta_1 & A_1\oplus A_2 \\ 0 & L_2 \end{pmatrix}\right),$$

where the B_i (resp. A_i) satisfy ξ) as $\Delta_2-L_1/J(L_1)$ (resp. $\Delta_1-L_2/J(L_2)$) bimodules.

iv) $A_1 \approx A_2 \approx B_1 \approx \overline{eR}$ and $B_2 \approx \overline{fR}$.
Then

$$R = \begin{pmatrix} \Delta_1 & A_1 \oplus A_2 \\ B_2 & S_2 \end{pmatrix},$$

where the A_i are similar to iii).

v) $A_1 \not\approx A_2$ and $B_1 \not\approx B_2$. Then

$$R = \begin{pmatrix} S_1 & A_2 \\ B_2 & S_2 \end{pmatrix}.$$

vi) Other cases. We may put $A_i=0$ or $B_i=0$ in the above. The right serial rings appear in v) by setting $S_2=\Delta_2$ or $S_i=\Delta_i$ and $B_2=0$ (or $A_2=0$).

References

[1] M. Harada: *On lifting property of direct sums of hollow modules*, Osaka J. Math. **18** (1980), 783–791.

[2] ———: *On maximal submodules of a direct sum of hollow modules* I, Osaka J. Math. **21** (1984), 671–677.

[3] ———: *On maximal submodules of a direct sum of hollow modules* III, Osaka J. Math. **22** (1985), 81–98.

[4] M. Harada and Y. Yukimoto: *On maximal submodules of a direct sum of hollow modules* IV, Osaka J. Math. **22** (1985), 321–326.

[5] M. Harada: *Generalizations of Nakayama ring* I, Osaka J. Math. **23** (1986), 181–200.

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558
Japan