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p-RADICAL GROUPS ARE p-SOLVABLE

Dedicated to Professor Hirosi Nagao for his 60th birthday

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Let k be an algebraically closed field of characteristic $p>0$. Let G be a finite group with Sylow p-subgroup P. Following Motose and Ninomiya [3] we call G p-radical if $k_P{}^G$ is completely reducible, where k_P is the trivial kP-module. Our aim in this paper is to prove the following theorems.

Theorem 1. *If G is p-radical, then G is p-solvable.*

Theorem 2. *Let G be a p-radical group with Sylow p-subgroup P. Then the following hold;*

(1) *If $D=P\cap P^x$ for some x in G, then D is a vertex of some simple kG-module.*

(2) *If $D=P\cap P^x$ for some x in $C_G(D)$, then D is a defect group of some p-block of G.*

We will write $V\mid W$ if a kG-module V is isomorphic to a direct summand of a kG-module W. For kG-modules V and W and a subgroup H of G, let $(V, W)^G=\mathrm{Hom}_{kG}(V, W)$ and $(V, W)_H^G=T_{H,G}(V, W)^H$, where $T_{H,G}$ is the trace map from $(V, W)^H$ to $(V, W)^G$.

1. Preliminaries

Throughout this paper we let G be a p-radical group with Sylow p-subgroup P and put $Y=k_P{}^G$. In this section we shall prove two lemmas which will be used to prove the theorems stated in the introduction.

Lemma 1. *If S is a simple kG-module with vertex Q, then every indecomposable direct summand of S_P is isomorphic to $k_A{}^P$ for some $A\subset P$ which is conjugate to Q.*

Proof. Since Y is completely reducible, $(Y, S)^G=(Y, S)_Q^G$ and $(Y, S)_R^G=0$ if R does not contain any conjugate of Q. Let X be an indecomposable direct summand of S_P. Then by Mackey decomposition theorem $X\cong k_A{}^P$ for some $A\subset P$ such that A is contained in some conjugate of Q. By the isomorphism

$\mathrm{Hom}_k(Y, S)\cong(\mathrm{Hom}_k(k_P, S_P))^G$ we have $T_{G,A}((Y, S)^A)\neq 0$ (see Lemma 3.5, II in [1]). Thus by the above remark A contains some conjugate of Q and therefore A is conjugate to Q.

Lemma 2. *Let Q be a subgroup of P and put $N=N_G(Q)$ and $R=P\cap N$. Let V be an indecomposable direct summand of $k_R{}^N$. If V_R has an indecomposable direct summand with vertex Q, then V also has Q as a vertex.*

Proof. For our proof of the lemma Scott's study in [5] is very useful. Let Ω be the set of right cosets of P in G. Then Y is the permutation module $k\Omega$. If Ω_Q is the set of fixed points of Q in Ω, then $X=k\Omega_Q$ is a kN-module, $X=\oplus\sum k_{P^x\cap N}{}^N$ where x ranges over the set of representatives of (P, N)-double cosets in G with $P^x\supset Q$ and $k_R{}^N\,|\,X$. Note that every indecomposable direct summand of X has a vertex containing Q as $Q\lhd N$. By Scott's investigation in (section 3, [5]) there exists a k-algebra homomorphism f from $(Y, Y)^G$ to $(X, X)^N$. The map f induces the epimorphism from $(Y, Y)^G_Q$ to $(X, X)^N_Q$ (see Proof of Theorem 3(b), [5]). Since $(Y, Y)^G$ is semisimple we conclude that $(X, X)^N_Q$ is an ideal of $(X, X)^N$ with $J((X, X)^N)\cap(X, X)^N_Q=0$, where $J((X, X)^N)$ denotes the Jacobson radical of $(X, X)^N$. Then it follows that $(X, X)^N_Q$ is a direct summand of $(X, X)^N$ as algebras. Let V be an indecomposable direct summand of $k_R{}^N$ and assume that V_R has an indecomposable direct summand with vertex Q. Then $k_Q{}^R$ is a direct summand of V_R by Mackey decomposition theorem and therefore $(V, k_R{}^N)^N_Q\neq 0$. Thus $(V, X)^N_Q\neq 0$ and this is implies that an idempotent in $(X, X)^N$ corresponding to V is in $(X, X)^N_Q$ as $(X, X)^N_Q$ is an algebra direct summand of $(X, X)^N$. So V also has a vertex Q and the result follows.

2. Proof of Theorem 1

Let S be a simple kG-module in the principal p-block of G and Q be its vertex. By Lemma 1 $S_P=\oplus\sum k_{Q^x}{}^P$ for some x's with $Q^x\subset P$. By the result of Knörr (Corollary 3.6, [2]) Q is a defect group of the principal p-block of $QC_G(Q)$ and therefore is a Sylow p-subgroup of $QC_G(Q)$. Thus $Z(P)\subset Q^x$ for every x with $Q^x\subset P$. So Ker $S\supset Z(P)$ and it follows that $O_{p'p}(G)\supset Z(P)$. As $G/O_{p'p}(G)$ is also p-radical, the theorem follows by induction on the order of G.

3. Proof of Theorem 2

First we shall prove the statement (1) in Theorem 2. For any Sylow intersection $D=P^x\cap P$, $k_D{}^P$ is a direct summand of Y_P by Mackey decomposition theorem and therefore is a direct summand of S_P for some simple kG-module S. Then the result follows from Lemma 1.

Next we shall show the statement (2) in Theorem 2. Let $D=P^x\cap P$ where x is in $C_G(D)$ and put $N=N_G(D)$, $H=DC_G(D)$ and $R=P\cap N$. Since x is in $C_G(D)$ and $D=R^x\cap R$ it follows that $k_D{}^R\,|\,k_R{}^{RH}{}_R$. Then by Lemma 2 $k_R{}^{RH}$

has an indecomposable direct summand V with vertex D as $k_R^{RH} | k_R{}^N{}_{RH}$. Let W be an indecomposable kH-module such that $W^{RH}=V$ and let b be a p-block of H which contains W. We claim that W^N is completely reducible. By the result of Scott (Theoremc, [5]) every indecomposable direct summand of $k_R{}^N$ with vertex D is the Green correspondent of an indecomposable direct summand of Y and is therefore a simple kN-module by (Lemma 2.2, [4]). Since $W^N | V^N | (k_R{}^{RH})^N = k_R{}^N$ and every indecomposable direct summand of W^N has a vertex D we have that W^N is completely reducible by the above remark and our claim follows. Since W^N is completely reducible, W is simple and is a unique simple kH-module in b as $H=DC_G(D)$. Let B be a p-block of N which covers b. Then every simple kN-module in B is a submodule of W^N and therefore direct summand of W^N. This implies that B has a defect group D. So G has a p-block with defect group D by Brauer's First Main Theorem.

References

[1] W. Feit: The representation theory of finite groups, North-Holland, Amsterdam, New York, Oxford, 1982.

[2] R. Knörr: *On the vertices of irreducible modules,* Ann. of Math. (2) **110** (1979), 487–499.

[3] K. Motose and Y. Ninomiya: *On the subgroups H of a group G such that $J(KH)KG \supset J(KG)$*, Math. J. Okayama Univ. **17** (1975), 171–176.

[4] T. Okuyama: *Module correspondence in finite groups*, Hokkaido Math. J. **10** (1981), 299–318.

[5] L.L. Scott: *Modular permutation representations*, Trans. Amer. Math. Soc. **175** (1973), 101–121.

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