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ON MODULES THAT COMPLEMENT DIRECT SUMMANDS

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A module M is said to *complement direct summands* if every direct summand of M has the exchange property with respect to completely indecomposable modules, or in other words if for each direct summand B of M and for each decomposition $M=\bigoplus_I A_i$, where every A_i is completely indecomposable (i.e. has local endomorphism ring), there exists a subset K of I with $M=B\oplus\bigoplus_K A_k$.

There are several characterisations by a theorem of Harada [3, 3.1.2].

Theorem. *Let $M=\bigoplus_I A_i$ be a c. indec. decomposition. Equivalent are*

(1) *M satisfies the take-out property.*
(2) *Every direct summand of M has the exchange property in M.*
(3) *M complements direct summands.*
(4) *$(A_i: I)$ is a locally-semi-T-nilpotent family.*
(5) *$J'\cap End(M)$ is equal to the Jacobson radical of $End(M)$.*

One step of the proof, "(4)$\Rightarrow$(5)", does merit a certain attention. In an earlier version of the theorem by Harada and Sai [2, Thm 9], the proof of that step uses assumptions stronger than at hand [2, Lemma 12]. We would like to present an alternative and elementary proof of that step. In particular one does not need transfinite induction as in [3, Lemma 2.2.3]. All notation may be found in [3]. For the proofs let perpetually be $M=\bigoplus_I A_i$ a completely indec. decomposition and let $(e_i: I)$ be a related set of orthogonal idempotents (i.e. $e_i(M)=A_i$).

By definition, for an element f of End(M) not contained in J', there exist some elements $i, j\in I$ and $g\in$End(M) with $ge_jfe_i=e_i$. Thus the Jacobson radical of End(M) is always contained in $J'\cap$End(M), otherwise it would contain a nonzero idempotent.

Lemma 1. *For all $t\in J'\cap End(M)$ and for all $i\in I$, e_it and te_i are elements of the Jacobson radical.*

Proof. Write $e_i = vp$, where v is the inclusion of A_i in M and p is the projection onto A_i induced by e_i. $J' \cap \mathrm{End}(M)$ is an ideal, thus the composition $pstv$ is not an isomorphism for all endomorphisms s. As $\mathrm{End}(A_i)$ is local, $1_A - pstv$ is an isomorphism. By Beck [1, Lemma 1.1], $1_M - ste_i$ is also an isomorphism and so te_i is an element of the radical. The other case works similarly.

Corollaries.

(a) *For all* $t \in J' \cap End(M)$, $1-t$ *is a monomorphism.*

(b) $J' \cap End(M)$ *does not contain nonzero idempotents.*

(c) *Lemma* 1 *is also true for arbitrary local idempotents and for finite sums of orthogonal local idempotents.*

(d) *Suppose* J *is a finite subset of* I, *take* $x \in \bigoplus_J A_j$ *and* $d := \sum_J e_j$. *Then* $x = (1 - dt)(1 - dtd)^{-1}(x)$. (*Condition* (§)).

Proofs. The definition of J' does not depend on a particular decomposition of M and this implies the first statement of (c). The second statement of (c) and (d) are obtained by a straightforward calculation. For (a), take $0 \neq x \in M$. There exists a finite subset J of I with $x \in \bigoplus_J A_j$. By (c), td is in the radical and $1 - td$ is an isomorphism, where $d = \sum_J e_j$. Thus $(1-t)(x) = (1-td)(x) \neq 0$. (b) follows from that, as $1-e$ is not monic for each nonzero idempotent e.

Having (a) in mind, in order to complete the proof of "(4)⇒(5)" it is enough to show $1-t$ is an epimorphism for all $t \in J' \cap \mathrm{End}(M)$. This is where (4) turns up. The idea is to apply the König-Graph-Lemma somehow.

Lemma 2. *Let* $(A_i \colon I)$ *be a locally-semi-T-nilpotent family and take* $t \in J' \cap End(M)$. *Then* $1-t$ *is an epimorphism.*

Proof. For an arbitrary $j \in I$ and $x \in e_j M$ there is constructed a $f_x \in \mathrm{End}(M)$ with $(1-t)f_x(x) = x$. Then $e_i M \subset (1-t)M$ for all $i \in I$ and $1-t$ is onto. Let $j \in I$ and $x \in e_j M$ be as above. Sequences $(f_n \colon \boldsymbol{N})$, $(g_n \colon \boldsymbol{N})$, $(h_n \colon \boldsymbol{N})$, $(d_n \colon \boldsymbol{N})$ with elements in $\mathrm{End}(M)$ and $(K_n \colon \boldsymbol{N})$, $(I_n \colon \boldsymbol{N})$ with subsets of I are constructed by induction, having the following properties:

(A) d_n is an idempotent

(B) $K_n \cap I_{n-1} = \emptyset$ and $\{j\} \cup K_1 \cup \cdots \cup K_n = I_n$

(C) $1 - g_n = (1-t) f_n$

(D) $g_n(x) = \prod_{1 \leq i \leq n} \sum_{k_i \in K_i} e_{k_i}(1-d_i) t h_i(x)$

$n=1$ Define $d_1 := e_j$, $I_0 := \{j\}$, $h_1 := (1 - d_1 t d_1)^{-1}$, $f_1 := h_1$, $g_1 := 1 - (1-t) f_1$. (A) and (C) are valid per def. Now, $(1-g_1)(x) = (1-t)h_1(x) = (1-d_1 t)h_1(x) - (1-d_1)th_1(x)$. As by Condition (§), $(1-d_1 t)h_1(x) = x$, follows $g_1(x) =$

$(1-d_1)th_1(x)$ and so $j \notin \mathrm{supp}(g_1(x))=:K_1$ (for $y \in M$, $\mathrm{supp}(y)$ is the finite set of $i \in I$ with $e_i(y) \neq 0$). $g_1(x) = \sum_{k_1 \in K_1} e_{k_1}(1-d_1)th_1(x)$. Take $I_1:=K_1 \cup I_0$ and get (D) and (B).

$n \rightsquigarrow n+1$ Define $d_{n+1}:=\sum_{i \in I_n} e_i$, $h_{n+1}:=(1-d_{n+1}td_{n+1})^{-1}$, $f_{n+1}:=f_n+h_{n+1}g_n$, $g_{n+1}:=1-(1-t)f_{n+1}$, $K_{n+1}:=\mathrm{supp}(g_{n+1}(x))$, $I_{n+1}:=K_{n+1} \cup I_n$. Again, (A) and (C) are valid per def. For the rest:

$$\begin{aligned}(1-g_{n+1})(x) &= (1-t)(f_n+h_{n+1}g_n)(x) \\ &= \underbrace{x-g_n(x)+\underbrace{(1-d_{n+1}t)h_{n+1}g_n(x)}_{=\,g_n(x) \text{ by Condition } (\S)}}_{=\,0}-(1-d_{n+1})th_{n+1}g_n(x)\end{aligned}$$

From $g_{n+1}(x)=(1-d_{n+1})th_{n+1}g_n$ one gets (D) by insertion. It is easy to see that $K_{n+1} \cap I_n = \emptyset$, which gives (B).

The construction is now complete. All summands in (D) are nonisomorphisms (as compositions with t) between certain A_i, and none of these A_i occur twice. Now by locally-semi-T-nilpotency and the König-Graph-Lemma there exists a natural number m with $g_m(x)=0$. (C) implies $x=(1-t)f_m(x)$.

References

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