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CONDITIONS AGAINST RAPID DECREASE OF OSCILLATORY INTEGRALS AND THEIR APPLICATIONS TO INVERSE SCATTERING PROBLEMS

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Introduction

Analysing singularities of distributions, we often examine the following integral with a parameter $\sigma>0$:

$$I(\sigma)=\int_{\boldsymbol{R}^n} e^{-i\sigma\varphi(x)}\rho(x;\sigma)dx \quad (\text{or } \int e^{i\sigma\varphi(x)}\rho(x;\sigma)dx)\,,$$

where $\varphi(x)$ is a real-valued C^∞ function and $\rho(x;\sigma)$ is a C^∞ function with an asymptotic expansion

$$\rho(x;\sigma)\sim\rho_0(x)+\rho_1(x)\,(i\sigma)^{-1}+\rho_2(x)\,(i\sigma)^{-2}+\cdots \quad (\text{as } \sigma\to\infty)\,.$$

In this paper we study conditions for the integral $I(\sigma)$ not to decrease rapidly as $\sigma\to\infty$, and solve some inverse scattering problems.

As is well known, if stationary points of $\varphi(x)$ are non-degenerate (i.e. det $(\partial_x^2\varphi(x))\neq 0$ when $\partial_x\varphi(x)=0$), $I(\sigma)$ is expanded asymptotically as $\sigma\to\infty$, and we can know whether $I(\sigma)$ decreases rapidly as $\sigma\to\infty$. Also when the stationary points are degenerate, the asymptotic expansion of $I(\sigma)$ is obtained if $\varphi(x)$ is analytic (cf. Varchenko [16], Duistermaat [1], etc.), and then we can know it through the expansion. But it seems difficult to do so when all derivatives of $\varphi(x)$ vanish at some points, whose case we take into consideration. In our methods we do not employ the asymptotic expansion of $I(\sigma)$. In the previous paper [13], the author examined the case that $n=2$ and $\rho_1(x)=0$ $(j\geqq 1)$: If $\rho_0(x)\geqq 0$ on $\boldsymbol{R}^2$ and $\rho_0(x_0)>0$ for a degenerate stationary point x_0 of $\varphi(x)$, then $(1+|\sigma|)^m I(\sigma)\notin L^2(\boldsymbol{R}^1)$ for some $m<2^{-1}$ (cf. Theorem 1 of [13]). Improving the methods in [13], whose idea is due to [8], we shall obtain similar results also in the case of $n\geqq 3$.

Let supp$[\rho(\cdot\,;\sigma)]$ and supp$[\rho_j]$ $(j\geqq 0)$ be contained in a compact set D in $\boldsymbol{R}^n$. We set

$$E(s)=\{x\colon \varphi(x)\leqq s\} \qquad (s\in\boldsymbol{R})\,,$$

$$(0.1) \qquad g_j(s) = \int_{E(s)} \rho_j(x)dx \qquad (j = 0, 1, \cdots).$$

One of our main results is the following

Theorem 1. *Let all ρ_j $(j \geqq 0)$ be real-valued. Then, for every $m \in \boldsymbol{R}$ we have*

$$\sigma^m I(\sigma) \in L^2(1, \infty)$$

if and only if for every integer $N(\geqq 1)$

$$\tilde{g}_N(s) \equiv g_0(s) + \sum_{j=1}^{N} \int_{s_0}^{s} \frac{(s-t)^{j-1}}{(f-1)!} g_j(t)dt \in C^N(\boldsymbol{R}^1).$$

The following theorem, derived from the above theorem, seems useful to estimate singular points of distributions.

Theorem 2. *Let all $\rho_j (j \geqq 0)$ be real-valued, and let $\rho_0(x) \geqq 0$ on $\boldsymbol{R}^n$. If ρ_0 satisfies*

$$\rho_0(x) > 0 \quad on \ E(\min_{x \in D} \varphi(x)),$$

then for some $m(\in \boldsymbol{R})$ depending only on the dimension n we have

$$\sigma^m I(\sigma) \notin L^2(1, \infty).$$

Theorem 1 implies that decreasingness of $I(\sigma)$ is connected with smoothness of the measure $|E(s)|$. This is seen also from the discussions in Vasil'ev [17] or Kaneko [3] (cf. §2 in Chapter I of [3]). Our methods in the proof of Theorem 2 (and in the author [13]) are based on analysis of $|E(s)|$.

In the latter of the present paper we shall consider some inverse scattering problems, and solve them by means of the above results. In §2 we deal with the scattering by a bounded obstacle $\mathcal{O}$ $(\subset \boldsymbol{R}^n, n \geqq 2)$ with a C^∞ boundary $\partial\mathcal{O}$. Assume that the domain $\Omega = \boldsymbol{R}^n - \mathcal{O}$ is connected, and consider the initial-boundary value problem

$$\begin{cases} \Box u(t, x) = 0 & \text{in } \boldsymbol{R}^1 \times \Omega \quad (\Box = \partial_t^2 - \Delta), \\ u(t, x') = 0 & \text{on } \boldsymbol{R}^1 \times \partial\Omega \quad (\partial\Omega = \partial\mathcal{O}), \\ u(0, x) = f_1(x) & \text{on } \Omega, \\ \partial_t u(0, x) = f_2(x) & \text{on } \Omega. \end{cases}$$

We denote by $k_-(s, \omega)$ $(k_+(s, \omega)) \in L^2(\boldsymbol{R}^1 \times S^{n-1})$ the incoming (outgoing) translation representation of the data (f_1, f_2) (cf. Lax and Phillips [6], [7]). The operator $S: k_- \to k_+$ is called the scattering operator and represented by a distribution kernel $S(s, \theta, \omega)$ called the scattering kernel:

$$(Sk_-)(s, \theta) = \iint S(s-t, \theta, \omega) k_-(t, \omega) dt d\omega$$

(cf. Majda [8] or §1 of the author [14]).

Majda [8] showed in the case of $\mathcal{O} \subset \boldsymbol{R}^3$ (i.e. $n=3$) that for any fixed $\omega \in S^2$

$$(0.2) \qquad \begin{aligned} &\text{(i)} \quad \operatorname{supp} S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)], \\ &\text{(ii)} \quad S(s, -\omega, \omega) \textit{ is singular (not } C^\infty) \textit{ at } s = -2r(\omega), \end{aligned}$$

where $r(\omega) = \min_{x \in \mathcal{O}} x \cdot \omega$. He reduced proof of the above (ii) to verifying that the integral of the form

$$\int_{\boldsymbol{R}^2} e^{-i\sigma\varphi(x)} \rho(x; \sigma) dx$$

does not decrease rapidly as $\sigma \to \infty$ (cf. §2 of Majda [8] or §4 of the author [14]). His methods are not applicable to the case of $n>3$, one of whose reasons is that the stationary points of the phase function $\varphi(x)$ are not necessarily non-degenerate.

Using Theorem 2, we can prove that (0.2) is valid also when $n>3$:

Theorem 3. *For any fixed ω and $\theta \in S^{n-1}$ with $\omega \neq \theta$, we have*
(i) $\operatorname{supp} S(\cdot, \theta, \omega) \subset (-\infty, -r(\omega-\theta)]$,
(ii) *$S(s, \theta, \omega)$ is singular at $s = -r(\omega-\theta)$.*

In §3 we consider the scattering by inhomogeneity of media expressed by the equation

$$(0.3) \qquad \begin{cases} \partial_t^2 u(t, x) - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x) \partial_{x_j} u(t, x)) = 0 & \text{in } \boldsymbol{R}^1 \times \boldsymbol{R}^n, \\ u(0, x) = f_1(x) & \text{on } \boldsymbol{R}^n, \\ \partial_t u(0, x) = f_2(x) & \text{on } \boldsymbol{R}^n, \end{cases}$$

where $a_{ij}(x)$ are real-valued C^∞ functions satisfying

$$\begin{aligned} &a_{ij}(x) = a_{ji}(x), \quad x \in \boldsymbol{R}^n, \\ &a_{ij}(x) = 0 \ (i \neq j), \quad a_{ii}(x) = 1 \qquad \text{when } |x| \geqq r_0, \\ &\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geqq \delta |\xi|^2, \quad x \in \boldsymbol{R}^n, \quad \xi \in \boldsymbol{R}^n. \end{aligned}$$

We can apply the scattering theory of Lax and Phillips [6], [7] to the equation (0.3). For this scattering the author in [15] has obtained the results corresponding to (0.2), but they are not satisfactory in the case of $n \geqq 3$. By means of Theorem 2 we get rid of the restriction to the dimension n.

Let us review the results of [15]. We set

$$\lambda_0^-(x, \xi) = -\{\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j\}^{1/2}.$$

Denote by $(q^-(t; s, x, \xi), p^-(t; s, x, \xi))$ the solution of the equation

$$\begin{cases} \dfrac{dq^-}{dt} = -\partial_\xi \lambda_0^-(q^-, p^-)\,, & \dfrac{dp^-}{dt} = \partial_x \lambda_0^-(q^-, p^-)\,, \\ q^-|_{t=s} = x\,, & p^-|_{t=s} = \xi\,, \end{cases}$$

and for ω, $\theta \in S^{n-1}$ set

$$M_\omega(\theta) = \{y : y\cdot\omega = -r_0,\ \lim_{t\to\infty} p^-(t; -r_0, y, \omega) = \theta\}\ ,$$

$$s_\omega(\theta) = \sup_{y\in M_\omega(\theta)} \{\lim_{t\to\infty}(q^-(t; -r_0, y, \omega)\cdot\theta - t)\}\ ,$$

$$\tilde{M}_\omega(\theta) = \{y \in M_\omega(\theta) : s_\omega(\theta) = \lim_{t\to\infty}(q^-(t; -r_0, y, \omega)\cdot\theta - t)\}\ .$$

We assume that for any y $(y\cdot\omega = -r_0)$ and $\omega \in S^{n-1}$

$$\lim_{t\to\infty} |q^-(t; -r_0, y, \omega)| = \infty\ . \tag{0.4}$$

Then singular support of the scattering kernel $S(\cdot, \theta, \omega)$ for the equation (0.3) is contained in the interval $(-\infty, s_\omega(\theta)]$ (cf. Theorem 2 in the author [15]); furthermore, when $n=2$, it is proved under some assumptions that $S(s, \theta, \omega)$ is singular at $s=s_\omega(\theta)$ (cf. Theorem 3 in [15]).

We show in §3 that this is valid also in the case of $n>2$:

Theorem 4. *Assume* (0.4) *for any* y $(y\cdot\omega = -r_0)$ *and* $\omega \in S^{n-1}$. *Fix* ω *and* $\theta \in S^{n-1}$ *with* $\omega \neq \theta$, *and let the assumption*

$$\det[\partial_x q^-(t; -r_0, y, \omega)] \neq 0 \qquad \text{for any } (t, y) \in [-r_0, \infty) \times \tilde{M}_\omega(\theta) \tag{0.5}$$

be satisfied. Then $S(s, \theta, \omega)$ *is singular at* $s = s_\omega(\theta)$.

The assumption (0.5) means that there is no caustic on $\{(t, x) : x = q^-(t; -r_0, y, \omega),\ -r_0 \leqq t < \infty,\ y \in \tilde{M}_\omega(\theta)\}$, namely, the mapping: $(t, y) \to q^-(t; -r_0, y, \omega)$ $(-r_0 \leqq t < \infty,\ y\cdot\omega = -r_0)$ is diffeomorphic on $[-r_0, \infty) \times \tilde{M}_\omega(\theta)$. In the previous paper [15] we added the assumption

$$\det[\partial_\xi p^-(t; -r_0, y, \omega)] \neq 0 \qquad \text{for any } (t, y) \in [-r_0, \infty) \times \tilde{M}_\omega(\theta)\,,$$

but this is not necessary.

1. Proofs of Theorem 1 and Theorem 2

We denote by $H^m(M)$ the Sobolev space of order m on M, and by $H^m_{loc}(M)$ the space of functions $g(x)$ satisfying $\alpha(x)g(x) \in H^m(M)$ for any $\alpha(x) \in C_0^\infty(M)$ ($C_0^\infty(M)$ is the space of C^∞ functions on M with compact support).

Lemma 1.1. *Let* $\varphi(x)$ *be a real-valued* C^∞ *function on* $\boldsymbol{R}^n$, *and let* $\rho(x)$ *be a* C^∞ *function on* $\boldsymbol{R}^n$ *with compact support. Then the function*

$$g(s) \equiv \int_{E(s)} \rho(x) dx$$

(where $E(s) = \{x: \varphi(x) \leqq s\}$*) satisfies*

(i) $g(s) = 0$ *if* $s < \min_{x \in \operatorname{supp}[\rho]} \varphi(x)$,

(ii) $g(s)$ *is constant if* $s > \max_{x \in \operatorname{supp}[\rho]} \varphi(x)$,

(iii) $g(s) \in H^m_{\mathrm{loc}}(\boldsymbol{R}^1)$ *for any* $m < 2^{-1}$.

Proof. Set

$$H(s) = \begin{cases} 1 & \text{for } s \geqq 0, \\ 0 & \text{for } s < 0. \end{cases}$$

Then it follows that $H(s) \in H^m_{\mathrm{loc}}(\boldsymbol{R}^1)$ for any $m < 2^{-1}$, and so $H(s - \varphi(x))$ becomes a $H^m_{\mathrm{loc}}(\boldsymbol{R}^1)$-valued continuous function on $\boldsymbol{R}^n_x$. Therefore, noting that $g(s) = \int_{\boldsymbol{R}^n} \rho(x) H(s - \varphi(x)) dx$, we obtain (iii). If $s < \min_{x \in \operatorname{supp}[\rho]} \varphi(x)$ we have $E(s) \cap \operatorname{supp}[\rho] = \phi$, which proves (i). If $s > \max_{x \in \operatorname{supp}[\rho]} \varphi(x)$, $E(s)$ contains $\operatorname{supp}[\rho]$, which yields (ii). The proof is complete.

Proof of Theorem 1. It follows from (iii) of Lemma 1.1 that the function $g_j(s)$ defined in (0.1) belongs to $L^2_{\mathrm{loc}}(\boldsymbol{R}^1)$. Therefore we have

$$\int_{s_0}^{s} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t) dt \in H^j_{\mathrm{loc}}(\boldsymbol{R}^1_s) \qquad (j \geqq 1),$$

$$\partial_s^j \int_{s_0}^{s} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t) dt = g_j(s).$$

Hence the function $\tilde{g}_N(s)$ $\left(= g_0(s) + \sum_{j=1}^{N} \int_{s_0}^{s} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t) dt\right)$ satisfies

$$\partial_s^N \tilde{g}_N(s) = \sum_{j=0}^{N} \partial_s^{N-j} g_j(s). \tag{1.1}$$

We define $\tilde{I}(\sigma)$ by

$$\tilde{I}(\sigma) = \begin{cases} I(\sigma) & \text{for } \sigma > 0, \\ \overline{I(-\sigma)} & \text{for } \sigma < 0. \end{cases}$$

Then $\sigma^m I(\sigma) \in L^2(1, \infty)$ if and only if $(1 + |\sigma|)^m \tilde{I}(\sigma) \in L^2(\boldsymbol{R}^1)$. Furthermore, since $\rho_j(x)$ are assumed real-valued, it follows that for any integer $N (\geqq 0)$

$$\tilde{I}(\sigma) = \sum_{j=0}^{N} \int_{\boldsymbol{R}^n} e^{-i\sigma\varphi(x)} \rho_j(x) dx (i\sigma)^{-j} + 0(|\sigma|^{-N-1}). \tag{1.2}$$

Here $0(|\sigma|^k)$ means that $|0(|\sigma|^k)| \leqq C|\sigma|^k$ $(|\sigma| \geqq 1)$ for some constant C independent of σ.

Noting that $\delta(s-\varphi(x))$ is a $H^m(\boldsymbol{R}_s^1)$-valued continuous function of x ($m<-2^{-1}$) and equal to $\partial_s H(s-\varphi(x))$, we obtain

$$e^{-i\sigma\varphi(x)} = \int e^{-i\sigma s}\delta(s-\varphi(x))ds = F[\partial_s H(s-\varphi(x))]\,(\sigma)\,,$$

where F is the Fourier transformation in s (the above integral is in the sense of distributions). Therefore we can represent the Riemann sum $\int_{\boldsymbol{R}^n} e^{-i\sigma\varphi(x)}\rho_j(x)dx$ in the following way:

$$(1.3)\qquad \int_{\boldsymbol{R}^n} e^{-i\sigma\varphi(x)}\rho_j(x)dx = F[\partial_s\int_{\boldsymbol{R}^n} H(s-\varphi(x))\rho_j(x)dx]\,(\sigma) = F[\partial_s g_j(s)]\,(\sigma)\,.$$

(1.1), (1.2) and (1.3) yield that

$$(1.4)\qquad (i\sigma)^{N-1}\tilde{I}(\sigma) = F[\partial_s^N \tilde{g}_N(s)]\,(\sigma)+0(|\sigma|^{-2})\,.$$

Let $(1+|\sigma|)^m\tilde{I}(\sigma)\in L^2(\boldsymbol{R}^1)$ for every $m\in\boldsymbol{R}$. Then it follows from (1.4) that

$$\partial_s^N\tilde{g}_N(s)\in H^1(\boldsymbol{R}^1)\,,$$

which implies

$$\tilde{g}_N(s)\in C^N(\boldsymbol{R}^1)\,.$$

Conversely, let $\tilde{g}_N(s)\in C^N$ for every non-negative integer N. Then we have $\partial_s^{N+1}\tilde{g}_N(s)\in H_{\mathrm{loc}}^{-1}(\boldsymbol{R}^1)$, which means that $\partial_s^{N+1}\tilde{g}_N(s)\in H^{-1}(\boldsymbol{R}^1)$ since $\partial_s^{N+1}\tilde{g}_N(s)=0$ for large $|s|$ (cf. (i), (ii) of Lemma 1.1 and (1.1)). Therefore, by (1.4) we obtain $(1+|\sigma|)^{N-1}\tilde{I}(\sigma)\in L^2(\boldsymbol{R}^1)$ for every integer $N(\geqq 1)$. This shows that

$$(1+|\sigma|)^m\tilde{I}(\sigma)\in L^2(\boldsymbol{R}^1)\qquad\text{for every } m\in\boldsymbol{R}\,.$$

The proof is complete.

Proof of Theorem 2. We can assume without loss of generality that $s_0=\min_{x\in D}\varphi(x)=0$. Since $\max_{0\leqq t\leqq s}|g_j(t)|\leqq|E(s)|\max_{x\in D}|\rho_j(x)|$ ($|E(s)|=\int_{E(s)}dx$), there is a constant C independent of s such that

$$\left|\int_0^s\frac{(s-t)^{j-1}}{(j-1)!}g_j(t)dt\right|\leqq C|s|^j|E(s)|\qquad(j\geqq 1)\,.$$

Therefore we have

$$|\tilde{g}_N(s)|\geqq|g_0(s)|-\sum_{j=1}^N\left|\int_0^s\frac{(s-t)^{j-1}}{(j-1)!}g_j(t)dt\right| \geqq(\min_{x\in E(s)}\rho_0(x)-C\sum_{j=1}^N|s|^j)|E(s)|\,.$$

Since $\min_{x\in \bar{E}(0)} \rho_0(x)>0$, we obtain $\min_{x\in \bar{E}(s)} \rho_0(x)\geqq 2\delta$ for a constant $\delta>0$ independent of s if $|s|$ is small enough. Therefore, if $|s|$ is small enough, it follows that

$$|\tilde{g}_N(s)|\geqq \delta |E(s)| .$$

Take a point x_0 satisfying $\varphi(x_0)=0$ $(=\min_{x\in D} \varphi(x))$. Then there is a constant d (>0) such that

$$E(s)\supset \tilde{E}(s) = \{x: d|x-x_0|\leqq s\} ,$$

which yields $|E(s)|\geqq |\tilde{E}(s)|=\delta' s^n$ for $s\geqq 0$ (the constant δ' does not depend on s). Thus, for any sufficiently small $s\geqq 0$ we have

$$|\tilde{g}_N(s)|\geqq \delta\delta' s^n . \tag{1.5}$$

Now, assume that $\sigma^m I(\sigma)\in L^2(1, \infty)$ for every $m\in \boldsymbol{R}$. Then it follows from Theorem 1 that $\tilde{g}_N(s)\in C^N$ for any integer $N\geqq 0$. Take the N so that $N\geqq n+1$. All the derivatives $g_N(0), \partial_s g_N(0), \cdots, \partial_s^N g_N(0)$ vanish because of (i) in Lemma 1.1, and so, by the Taylor expansion, we obtain

$$|\tilde{g}_N(s)|\leqq C|s|^{n+1} .$$

This is not consistant with (1.5) as $s\to +0$. Therefore we have

$$\sigma^m I(\sigma)\notin L^2(1, \infty)$$

for some constant $m\in \boldsymbol{R}$ depending only on n.

2. Proof of Theorem 3

In this section we review some results obtained in Majda [8] and the author [14], and prove Theorem 3.

Let $v(t, x; \omega)$ be the solution of the equation

$$\begin{cases} \Box v(t, x) = 0 & \text{in } \boldsymbol{R}^1\times\Omega , \\ v(t, x') = -2^{-1}(-2\pi i)^{1-n}\delta(t-x'\cdot\omega) & \text{on } \boldsymbol{R}^1\times\partial\Omega , \\ v(t, x) = 0 & \text{for } t<r(\omega) . \end{cases} \tag{2.1}$$

Then $v(t, x; \omega)$ is a C^∞ function of x and ω with the value $\mathcal{S}'(\boldsymbol{R}_t^1)$.

Proposition 2.1. *$S(s, \theta, \omega)$ is represented of the form*

$$S(s, \theta, \omega) = \int_{\partial\Omega} \{\partial_t^{n-2}\partial_\nu v(x\cdot\theta-s, x; \omega)-\nu\cdot\theta\partial_t^{n-1}v(x\cdot\theta-s, x; \omega)\} dS_x \quad (\omega\neq\theta) ,$$

where ν is the outer unit vector normal to $\partial\Omega$ (cf. Theorem 1 in Majda [8] and Theorem 2.1 in §2 of the author [14]).

In the above proposition the integral $\int \cdot dS_x$ is in the sense of the Riemann integral with the value $\mathcal{S}'(\boldsymbol{R}^1)$. For the proof see Majda [8] and the author [14].

It is seen that the wave front set of $\delta(t-x\cdot\omega)|_{\boldsymbol{R}^1\times\partial\Omega}$ is non-glancing in $\{(t,x): -r(\omega-\theta)-2\eta\leqq x\cdot\theta-t\}\cap(\boldsymbol{R}^1\times\partial\Omega)$ $(\omega\neq\theta)$ if η (>0) is small enough (for description of wave front sets, see Hörmander [2], Kumano-go [5], etc.). Therefore we can construct there the solution $v(t, x; \omega)$ of (2.1) mod C^∞ by means of the Fourier integral operators (cf. §9 of Nirenberg [10]), and get information about $\partial_\nu v|_{\boldsymbol{R}^1\times\partial\Omega}$. This is indicated by Majda [8] in the case of $\theta=-\omega$ (cf. Lemma 2.1 of [8]). We have

Lemma 2.2. *There exists a first order pseudo-differential operator B on $\boldsymbol{R}^1\times\partial\Omega$ independent of t such that*

(i) *its symbol $B(\tilde{x}'; \tau, \xi')$ represented near*

$$N(\omega-\theta)=\{x: x\cdot(\theta-\omega)=r(\omega-\theta)\}\cap\partial\Omega$$

by local coordinates $(t, \tilde{x}')$, has a homogeneous asymptotic expansion $\sum_{j=0}^{\infty} B_j(\tilde{x}'; \tau, \xi')$ satisfying

(2.2) $-iB_0(\tilde{x}'; \pm 1, \mp\tilde{\theta}')>0$ *on $N(\omega-\theta)$ ($\tilde{\theta}'$ is the tangential component of θ to the plane $\{x: x\cdot(\omega-\theta)=r(\omega-\theta)\}$),*

(2.3) *$B_j(\tilde{x}'; \tau, \xi')$ are purely imaginary-valued for even j and real-valued for odd j,*

(ii) *$\partial_\nu v|_{\boldsymbol{R}^1\times\partial\Omega}$ is equal to $B(v|_{\boldsymbol{R}^1\times\partial\Omega})$ mod C^∞ in $\{(t,x): -r(\omega-\theta)-\eta\leqq x\cdot\theta-t\}\cap\boldsymbol{R}^1\times\partial\Omega$ for some small constant $\eta>0$.*

In the above lemma, "a homogeneous asymptotic expansion $\sum_{j=0}^{\infty} B_j(\tilde{x}'; \tau, \xi')$" means that $B_j(\tilde{x}'; \mu\tau, \mu\xi')=\mu^{1-j}B_j(\tilde{x}'; \tau, \xi')$ for $\mu\geqq 1$, $|\tau|+|\xi'|\geqq 1$ and that $|B(\tilde{x}'; \tau, \xi')-\sum_{j=1}^{N} B_j(\tilde{x}'; \tau, \xi')|\leqq C_N(|\tau|+|\xi'|+1)^{-N-1}$ for any non-negative integer N (for detailed description of pseudo-differential operators on manifolds, see Seeley [11], etc.); (ii) in the lemma states that $\alpha(t,x')(\partial_\nu v|_{\boldsymbol{R}^1\times\partial\Omega}-B(v|_{\boldsymbol{R}^1\times\partial\Omega}))\in C^\infty$ for any $\alpha(t, x')\in C^\infty(\boldsymbol{R}^1\times\partial\Omega)$ with $\mathrm{supp}[\alpha]\subset\{(t, x): -r(\omega-\theta)-\eta\leqq x\cdot\theta-t\}$.

Proof of Lemma 2.2. Let $\sum_{i=1}^{l}\chi_i(x)$ be a partition of unity on a neighborhood of $N(\omega-\theta)$ satisfying $\max_{1\leqq i\leqq l}|\mathrm{supp}[\chi_i]|\leqq\varepsilon_0$ (ε_0 is a sufficiently small positive constant). Then there is a constant $\varepsilon_1>0$ such that $\sum_{i=1}^{l}\chi_i(x)=1$ for any $x\in\partial\Omega$ satisfying $-r(\omega-\theta)-\varepsilon_1\leqq x\cdot\theta-x\cdot\omega$. Let $v_i(t,x)$ be the solution of the equation

$$\begin{cases} \Box v_i(t, x)=0 & \text{in } \boldsymbol{R}^1\times\Omega\,, \\ v_i(t, x')=\chi_i(x')v(t, x'; \omega) & \text{on } \boldsymbol{R}^1\times\partial\Omega\,, \\ v_i(t, x)=0 & \text{for } t<r(\omega)\,. \end{cases}$$

Since $\mathrm{supp}[v|_{\boldsymbol{R}^1\times\partial\Omega}]\subset\{(t, x'): x'\cdot\omega=t\}$, $\sum_{i=1}^{l} v_i(t, x')$ is equal to $v(t, x'; \omega)$ on $(\boldsymbol{R}^1\times\partial\Omega)\cap\{(t, x'): -r(\omega-\theta)-\varepsilon_1\leqq x'\cdot\theta-t\}$, and so, noting that the propagation speed is less than one, we have

$$v(t, x; \omega)=\sum_{i=1}^{l} v_i(t, x) \qquad \text{in } (\boldsymbol{R}^1\times\Omega)\cap\{(t, x): -r(\omega-\theta)-\varepsilon_1\leqq x\cdot\theta-t\}\ .$$

We denote by $\mathrm{WF}[f(t, x)]$ the wave front set of $f(t, x)$. It is seen that $\mathrm{WF}[v|_{\boldsymbol{R}^1\times\partial\Omega}]=\mathrm{WF}[\delta(x'\cdot\omega-t)|_{\boldsymbol{R}^1\times\partial\Omega}]=\{(t, x'; \tau, \xi'): (t, x')\in\boldsymbol{R}^1\times\partial\Omega,\ x'\cdot\omega-t=0,\ \xi'=-\tau(\omega-(\omega\cdot\nu)\nu),\ \tau\neq 0\}$ (ν is the outer unit normal to $\partial\Omega$). Hence, for any $(t, x'; \tau, \xi')\in\mathrm{WF}[v_i|_{\boldsymbol{R}^1\times\partial\Omega}]$ the equation $\tau^2-|\xi'+\lambda\nu|^2=0$ in λ has real roots, and the null-bicharacteristics associated with $\partial_t^2-\Delta$ through $\mathrm{WF}[v_i|_{\boldsymbol{R}^1\times\partial\Omega}]$ are transversal to $\boldsymbol{R}^1\times\partial\Omega$ (non-glancing). This implies that $\mathrm{sing\ supp}[\partial_\nu v_i|_{\boldsymbol{R}^1\times\partial\Omega}]\subset\mathrm{sing\ supp}[v_i|_{\boldsymbol{R}^1\times\partial\Omega}]$ (cf. Theorem 7 in §9 of (Lax and) Nirenberg [10]), and so it suffices to examine $v_i(t, x)$ only in a neighborhood $(t_i-\varepsilon_0, t_i+\varepsilon_0)\times U_i$ of (t_i, x^i) ($x^i\in\mathrm{supp}[\chi_i]\cap N(\omega-\theta)$ and $t_i=x^i\cdot\omega$).

To analyze v_i more precisely, we transform Ω in U_i into the half-space $\boldsymbol{R}_+^n=\{\tilde{x}=(\tilde{x}', \tilde{x}_0): \tilde{x}_0>0\}$. Let the derivative ∂_ν be transformed in U_i into $-\partial_{\tilde{x}_0}$. For any set M in $\boldsymbol{R}_x^n$ we denote by $\tilde{M}$ the set transformed by the coordinates $\tilde{x}$. Let $-\Delta_x$ be represented by $\tilde{x}$ of the form $\tilde{A}=\sum_{|\alpha|\leqq 2} a_\alpha(\tilde{x})\partial_{\tilde{x}}^\alpha$. Here we can assume that the coefficients $a_\alpha(\tilde{x})$ are real-valued C^∞ functions defined on $\boldsymbol{R}^n$ and constant out of $\tilde{U}_i$. Let us examine the solution $\tilde{v}(t, \tilde{x})$ of the following equation instead of $v_i(t, x)$:

$$\begin{cases} (\partial_t^2+\tilde{A})\tilde{v}(t, \tilde{x})=0 & \text{in } \boldsymbol{R}^1\times\boldsymbol{R}_+^n\ , \\ \tilde{v}(t, \tilde{x}')=g(t, \tilde{x}') & \text{on } \boldsymbol{R}^1\times\boldsymbol{R}^{n-1}\ , \\ \tilde{v}(t, \tilde{x})=0 & \text{for } t<t_i-\varepsilon_0\ , \end{cases}$$

where $g(t, \tilde{x}')=-2^{-1}(-2\pi i)^{1-n}\delta(x(\tilde{x}')\cdot\omega-t)\chi_i(x(\tilde{x}'))$. Note that $\mathrm{WF}[g(t, \tilde{x}')]$ is contained in a sufficiently small conic neighborhood of $(t_i, x^i; \pm 1, \mp\tilde{\theta}')$ ($\tilde{\theta}'$ is the component of θ (transformed by the coordinates $\tilde{x}$) tangent to the plane $\tilde{x}_0=0$), and that if $|(\tau, \tilde{\xi}')|^{-1}(\tau, \tilde{\xi}')$ is near $|(\pm 1, \mp\tilde{\theta}')|^{-1}(\pm 1, \mp\tilde{\theta}')$ the equation

$$\tau^2+\tilde{A}_0(\tilde{x}; \tilde{\xi}', \tilde{\xi}_0)=0 \tag{2.4}$$

($\tilde{A}_0(\tilde{x}, \tilde{\xi})=\sum_{|\alpha|=2} a_\alpha(\tilde{x})\tilde{\xi}^\alpha$) in $\tilde{\xi}_0$ has two real roots. Furthermore, examining the forms of these roots, by the same procedure as in Nirenberg [10] or Kumano-go [5] (see Lemma 1 in §5 of [10] or Appendix II of [5]) we can construct first order pseudo-differential operators $\xi^\pm(\tilde{x}; D_t, D_{\tilde{x}})$ on $\boldsymbol{R}_t^1\times\boldsymbol{R}_{\tilde{x}}^n$ (independent of t) with homogeneous asymptotic expansions $\sum_{j=0}^{\infty}\xi_j^\pm(\tilde{x}; \tau, \tilde{\xi}')$ such that

(i) $\xi_j^\pm(\tilde{x}; \tau, \tilde{\xi}')$ *are real-valued for even* j *and purely imaginary-valued for odd* j,

(ii) *if* $|(\tau,\xi')|^{-1}(\tau,\xi')$ *is near* $|(-1,\tilde{\theta}')|^{-1}(-1,\tilde{\theta}')$ *or* $|(1,-\tilde{\theta}')|^{-1}(1,-\tilde{\theta}')$, $\xi_0^{\pm}(\tilde{x};\tau,\xi')$ *are equal to the roots of the equation* (2.4), *and*

$$\xi_0^{\pm}(\tilde{x}^i;\pm 1,\mp\tilde{\theta}')=\mp(1-|\tilde{\theta}'|^2)^{1/2},$$

(iii) *all the null-bicharacteristic curves associated with* $D_{\tilde{x}_0}-\xi_0^{+}(\tilde{x};D_t,D_{\tilde{x}'})$ *through* $\mathrm{WF}[g(t,\tilde{x}')]$ *are transversal to the boundary* $\{\tilde{x}_0=0\}$ *and proceed in the direction* $t>0$ *as they leave the boundary,*

(iv) *if the wave front set of* $u(t,x)$ *is near the bicharacteristic curves stated in the above* (iii), *then we have*

$$(D_{\tilde{x}_0}-\xi^-(\tilde{x};D_t,D_{\tilde{x}'}))(D_{\tilde{x}_0}-\xi^+)u=\zeta(\tilde{x})(\partial_t^2+\tilde{A})u \mod C^\infty,$$

where $\zeta(\tilde{x})$ *is a* C^∞ *function on* $\boldsymbol{R}^n$ *satisfying* $\zeta(\tilde{x})<0$ *for every* $\tilde{x}$.

(iii) and (iv) imply that $\tilde{v}(t,\tilde{x}',\tilde{x}_0)$ is approximated mod C^∞ by the solution $w(\tilde{x}_0;t,\tilde{x}')$ of the equation

$$\begin{cases}(D_{\tilde{x}_0}-\xi^+(\tilde{x};D_t,D_{\tilde{x}'}))w=0\,, & \tilde{x}_0>0\,,\\ w|_{\tilde{x}_0=0}=h(t,\tilde{x}')\,.\end{cases}$$

Therefore we have

$$-\partial_{\tilde{x}_0}\tilde{v}|_{\tilde{x}_0=0}=-i\xi^+(\tilde{x}',0;D_t,D_{\tilde{x}'})(\tilde{v}|_{\tilde{x}_0=0}) \mod C^\infty.$$

Combining this with the above (i) and (ii) yields the lemma. The proof is complete.

Proof of Theorem 3. The solution $v(t,x;\omega)$ in (2.1) satisfies $\mathrm{supp}[v|_{\boldsymbol{R}^1\times\partial\Omega}]\subset\{(t,x)\colon x\cdot\omega=t\}$. Therefore, noting that the propagation speed is less than one, we see that $\mathrm{supp}[v(t,x;\omega)]\subset\{(t,x);x\cdot\omega\leqq t\}$, which yields

$$v(x\cdot\theta-s,x;\omega)=0 \qquad \text{if } s>x\cdot(\theta-\omega)\,.$$

Hence, if $s>\max_{x\in\partial\Omega}x\cdot(\theta-\omega)=-r(\omega-\theta)$ $(\omega\neq\theta)$, we obtain $S(s,\theta,\omega)=0$ from Proposition 2.1.

Next, let us prove that $S(s,\theta,\omega)$ is singular at $s=-r(\omega-\theta)$. Take $\alpha(s)\in C^\infty(\boldsymbol{R}^1)$ such that $0\leqq\alpha\leqq 1$ on $\boldsymbol{R}^1$, $\alpha(s)=1$ for $|s|\leqq 2^{-1}$ and $\alpha(s)=0$ for $|s|\geqq 1$. For any $\varepsilon>0$ set

$$\alpha_\varepsilon(s)=\alpha\left(\frac{s+r(\omega-\theta)}{2\varepsilon}\right).$$

Then we have only to prove that $\alpha_\varepsilon(s)S(s,\theta,\omega)$ is not C^∞ for any small $\varepsilon>0$. Proposition 2.1 yields

$$\alpha_\varepsilon(s)S(s,\theta,\omega)=\int_{\partial\Omega}\alpha_\varepsilon(s)(\partial_t^{n-2}\partial_\nu v)(x\cdot\theta-s,x;\omega)dS_x$$
$$-\int_{\partial\Omega}\nu\cdot\theta\alpha_\varepsilon(s)(\partial_t^{n-1}v)(x\cdot\theta-s,x;\omega)dS_x\equiv J_1(s)+J_2(s)\,.$$

Let $\bar{F}[k(s)](\sigma)=\int e^{i\sigma s}k(s)ds$. As is readily seen, it follows that

$$\bar{F}[J_2(s)](\sigma) = -2^{-1}(-2\pi i)^{1-n}\sum_{j=0}^{n-1} C_j^{n-1}(i\sigma)^{n-1-j}\int_{\partial\Omega} e^{i\sigma x\cdot(\theta-\omega)}(-\nu\cdot\theta)\cdot\alpha_\varepsilon^{(j)}(x\cdot(\theta-\omega))dS_x \tag{2.5}$$

(where $C_j^{n-1}=(n-1)!/(n-1-j)!j!$). Taking the $\varepsilon(>0)$ so that $2\varepsilon\leqq\eta$, by Lemma 2.2 we have

$$\begin{aligned}\bar{F}[J_1(s)](\sigma) &= \iint_{\boldsymbol{R}^1\times\partial\Omega} e^{i\sigma(x\cdot\theta-s)}\alpha_\varepsilon(x\cdot\theta-s)\partial_s^{n-2}[Bv\,|_{\boldsymbol{R}^1\times\partial\Omega}](s, x)dsdS_x \\ &= -2^{-1}(-2\pi i)^{1-n}\sum_{j=0}^{n-2} C_j^{n-2}\int_{\partial\Omega} {}^tB[e^{i\sigma(x\cdot\theta-s)}\alpha_\varepsilon^{(j)}(x\cdot\theta-s)]\,|_{s=x\cdot\omega}dS_x \cdot(i\sigma)^{n-2-j}\,.\end{aligned}$$

Here ${}^tB$ denotes the transposed operator of B (i.e. $\langle {}^tBf, g\rangle=\langle f, Bg\rangle$ for any f and $g\in C_0^\infty(\boldsymbol{R}^1\times\partial\Omega)$). Let us note that the symbol of ${}^tB$ expressed near supp $[\alpha_\varepsilon(x\cdot\theta-t)]\cap(\boldsymbol{R}^1\times\partial\Omega)$ by the local coordinates $(t, \tilde{x}')$, has a homogeneous asymptotic expansion $\sum_{j=0}^{\infty}{}^tB_j(\tilde{x}'; \tau, \xi')$ such that ${}^tB_j(\tilde{x}'; \tau, \xi')$ are real-valued for odd j and purely imaginary valued for even j and that $-i\,{}^tB_0(\tilde{x}'; \pm1, \mp\tilde{\theta}')= -iB_0(\tilde{x}'; \mp1, \pm\tilde{\theta}')\leqq 0$ for $\tilde{x}'\in\tilde{N}(\omega-\theta)$, which follows from Lemma 2.2. By the methods of stationary phases (cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we can expand ${}^tB[e^{i\sigma(x\cdot\theta-s)}\alpha_\varepsilon^{(j)}(x\cdot\theta-s)]$ asymptotically (as $\sigma\to\infty$) in the same way as in Proposition 4.1 of the author [12]. Therefore we obtain the asymptotic expansion

$$\bar{F}[J_1](\sigma)\sim -2^{-1}(-2\pi i)^{1-n}\sum_{j=0}^{\infty}(i\sigma)^{n-1-j}\int_{\partial\Omega} e^{i\sigma x\cdot(\theta-\omega)}\beta_j(x)dS_x \quad (\text{as } \sigma\to\infty)\,, \tag{2.6}$$

where $\beta_j(x)$ are real-valued C^∞ functions on $\partial\Omega$ with $\mathrm{supp}[\beta_j]\subset\mathrm{supp}[\alpha_\varepsilon(x\cdot(\theta-\omega))]\cap\partial\Omega$, and $\beta_0(x)$ is non-negative valued and satisfies

$$\beta_0(x) = -i\,{}^tB_0(\tilde{x}'(x); -1, \tilde{\theta}')\alpha_\varepsilon(x\cdot(\theta-\omega))>0 \qquad \text{for } x\in N(\omega-\theta)\,.$$

Combining (2.5) and (2.6) yields that for any integer $N(>0)$

$$\bar{F}[\alpha_\varepsilon(s)S(s, \theta, \omega)](\sigma) = -2^{-1}(-2\pi i)^{1-n}(i\sigma)^{n-1}\int_{\boldsymbol{R}^{n-1}} e^{-i\sigma x(\tilde{x}')\cdot(\omega-\theta)}\cdot\{\sum_{j=0}^{N-1}\rho_j(\tilde{x}')(i\sigma)^{-j}\}d\tilde{x}'+0(\sigma^{-N})\,.$$

Here $\tilde{x}'$ is the local coordinates on $\partial\Omega$ near $N(\omega-\theta)$ and

$$\rho_j(\tilde{x}') = \beta_j(x(\tilde{x}'))+(-\nu\cdot\theta)\alpha_\varepsilon^{(j)}(x(\tilde{x})\cdot(\theta-\omega)) \quad (\alpha_\varepsilon^{(j)}=0, j\geqq n)\,.$$

Noting that $\rho_0(\tilde{x}')>0$ when the phase function $x(\tilde{x}')\cdot(\omega-\theta)$ is minimum, and applying Theorem 2, we obtain for some constant $m\in\boldsymbol{R}$

$$\sigma^m\bar{F}[\alpha_\varepsilon(s)S(s, \theta, \omega)](\sigma)\notin L^2(1, \infty)\,,$$

which shows that $\alpha_\varepsilon(s)S(s, \theta, \omega)$ is not C^∞. The proof is complete.

3. Proof of Theorem 4

We use the same notations as for the scattering by obstacles in §2. The scattering operator S for the equation (0.3) is represented as follows (see Theorem 1 and (3.1) of the author [15]):

Proposition 3.1. *Set*

$$S_0(s, \theta, \omega) = \int_{\boldsymbol{R}^n} (\partial_t^{n-2}\Box w)(x\cdot\theta - s, x)dx\,,$$
$$Kk = F^{-1}[(\operatorname{sgn}\sigma)^{n-1}(Fk)(\sigma)]\,,$$

where $w(t, x)$ is the solution of the equation

$$\begin{cases} (\partial_t^2 - A)w(t, x) = 0 \quad \left(Aw = \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j}w)\right) & \text{in } \boldsymbol{R}^1\times\boldsymbol{R}^n\,, \\ w(-r_0, x) = -2^{-1}(-2\pi i)^{1-n}\delta(-r_0 - x\cdot\omega) & \text{on } \boldsymbol{R}^n\,, \\ \partial_t w(-r_0, x) = -2^{-1}(-2\pi i)^{1-n}\delta'(-r_0 - x\cdot\omega) & \text{on } \boldsymbol{R}^n\,. \end{cases}$$

Then we have

$$(Sk)(s, \theta) = \iint S_0(s-t, \theta, \omega)k(t, \omega)dtd\omega + (Kk)(s, \theta)\,.$$

Note that $S_0(s, \theta, \omega) = S(s, \theta, \omega)$ if $\omega \neq \theta$.

To prove Theorem 4, we have only to show that for any small $\varepsilon(>0)$ there exist a real number m and a function $\rho(s)\in C_0^\infty(s_\omega(\theta)-2\varepsilon, s_\omega(\theta)+2\varepsilon)$ such that

$$(1+|\sigma|)^m \bar{F}[\rho(s)S(s, \theta, \omega)](\sigma) \notin L^2(\boldsymbol{R}^1)\,.$$

Let $\gamma(x)\in C_0^\infty(\boldsymbol{R}^n)$ with $\gamma(x)=1$ in a neighborhood of $\tilde{M}_\omega(\theta)$, and denote by $\tilde{w}(t, x)$ the solution of the equation

$$\begin{cases} (\partial_t^2 - A)\tilde{w}(t, x) = 0 & \text{in } \boldsymbol{R}^1\times\boldsymbol{R}^n\,, \\ \tilde{w}(-r_0, x) = \gamma(x)w(-r_0, x) & \text{on } \boldsymbol{R}^n\,, \\ \partial_t\tilde{w}(-r_0, x) = \gamma(x)\partial_t w(-r_0, x) & \text{on } \boldsymbol{R}^n\,. \end{cases}$$

The author [15] showed that if $\tilde{t}$ is large enough we have for any integer $N(>0)$

$$\bar{F}[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1}e^{-i\sigma\tilde{t}}\sum_{j=0}^{N-1}(i\sigma)^{n-1-j}\mathcal{F}'[\beta_j(x)\{\tilde{w}(\tilde{t}, x) + (i\sigma)^{-1}\partial_t\tilde{w}(\tilde{t}, x)\}](-\sigma\theta) + 0(\sigma^{-N+N_0})$$

as $\sigma\to\infty$ (N_0 is an integer independent of N) (cf. (4.5) in [15]). Here, $\mathcal{F}'$ denotes the Fourier transformation in x, and the functions $\beta_j(x)\in C_0^\infty(\boldsymbol{R}^n)$ are all real-valued.

We take $\tilde{t}$ so large as to have (i) and (ii) stated in the following

Lemma 3.2. *Let r_1 be an arbitrary constant ($\geqq r_0$), and set*

$$\psi(x; t) = q^-(t; -r_0, x, \omega)\cdot\theta .$$

Then, for any $\varepsilon(>0)$ there is a constant $\tilde{t}_0$ such that for any fixed $\tilde{t}\geqq\tilde{t}_0$

(i) $\max_{\substack{|x|\leqq r_1 \\ x\cdot\omega=-r_0}} \psi(x; \tilde{t})\leqq s_\omega(\theta)+\tilde{t}+\varepsilon$,

(ii) *all points at which $\psi(x;\tilde{t})$ is maximum ($x\cdot\omega=-r_0$, $|x|\leqq r_1$), are contained in ε-neighborhood $(\tilde{M}_\omega(\theta))_\varepsilon$ of $\tilde{M}_\omega(\theta)$ ($(\tilde{M})_\varepsilon=\{x\colon \mathrm{dis}(x, \tilde{M})<\varepsilon\}$).*

This lemma will be proved later. Choose the $\rho(s)$ so that $\rho(s)\geqq 0$ on $\boldsymbol{R}^1$ and $\rho(s)>0$ on $[s_\omega(\theta)-\varepsilon, s_\omega(\theta)+\varepsilon]$. Then it is seen from the form of $\beta_0(x)$ (cf. (4.4) and (4.6) in [15]) and the above lemma that

$$\text{(3.1)} \qquad \beta_0(x)\geqq 0 \quad \text{on } \boldsymbol{R}^n \quad \text{and} \quad \beta_0(q^-(\tilde{t}; -r_0, y, \omega))>0$$
$$\text{for any} \quad y\in(\tilde{M}_\omega(\theta))_\varepsilon \quad (y\cdot\omega = -r_0) .$$

We take the $\gamma(x)$ so that $\gamma(x)\geqq 0$ on $\boldsymbol{R}^n$, $\gamma(x)>0$ on $(\tilde{M}_\omega(\theta))_\varepsilon$ and $\mathrm{supp}[\gamma]\subset(\tilde{M}_\varepsilon(\theta))_{2\varepsilon}$.

By the same procedure as in Nirenberg [10], Kumano-go [5] (cf. §5 of [10] or Appendix II of [5]), we can construct a symbol $\lambda(x, \xi)$ with a homogeneous asymptotic expansion $\sum_{j=0}^{\infty}\lambda_j(x, \xi)$ such that

$$\lambda_0(x, \xi) = \{\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\}^{1/2} ,$$
$$-\partial_t^2+A = (D_t+\lambda(x, D_x))\,(D_t-\lambda(x, D_x)) \quad \text{modulo a smoothing operator}$$

(cf. Corollary 2.5 in the author [15] also). Furthermore we see that $\lambda_j(x, \xi)$ *are real-valued for even j and purely imaginary valued for odd j* since the coefficients $a_{ij}(x)$ are all real-valued (recall the construction of $\xi^{\pm}(\tilde{x}'; \tau, \tilde{\xi}')$ in §2). Consider the Cauchy problem

$$\begin{cases} (D_t-\lambda(x, D_x))u(t, x) = 0 & \text{in } \boldsymbol{R}^1\times\boldsymbol{R}^n , \\ u|_{t=0} = u_0(x) & \text{on } \boldsymbol{R}^n , \end{cases}$$

and denote by $E(t)$ the operator: $u_0\to u(t, \cdot)$. Then $\tilde{w}(\tilde{t}, x)$ and $\partial_t\tilde{w}(\tilde{t}, x)$ are represented as follows:

$$\begin{aligned} \tilde{w}(\tilde{t}, x) &= 2^{-1}E(\tilde{t}+r_0)\,(\tilde{w}(-r_0, \cdot)-i\tilde{\mu}\partial_t\tilde{w}(-r_0, \cdot))\,(x) \\ &\quad +2^{-1}E(-\tilde{t}-r_0)\,(\tilde{w}(-r_0, \cdot)+i\tilde{\mu}\partial_t\tilde{w}(-r_0, \cdot))\,(x) , \\ \partial_t\tilde{w}(\tilde{t}, x) &= 2^{-1}E(\tilde{t}+r_0)i\lambda(\tilde{w}(-r_0, \cdot)-i\tilde{\mu}\partial_t\tilde{w}(-r_0, \cdot))\,(x) \\ &\quad +2^{-1}E(-\tilde{t}-r_0)i\tilde{\lambda}(\tilde{w}(-r_0, \cdot)+i\tilde{\mu}\partial_t\tilde{w}(-r_0, \cdot))\,(x) , \end{aligned}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are pseudo-differential operators whose symbols coincide with

$\lambda(x,\xi)$ and $\mu(x,\xi)$ ($\mu(x,D_x)$ is the parametrix of $\lambda(x,D_x)$) respectively in a neighborhood of supp$[\gamma(x)]$ and vanish for large $|x|$. Therefore, noting that

$$\mathcal{F}'[\beta_j E(-\tilde{t}-r_0)(\tilde{w}(-r_0,\cdot)+i\tilde{\mu}\partial_t\tilde{w}(-r_0,\cdot))](-\sigma\theta)=0(\sigma^{-\infty}),$$
$$\mathcal{F}'[\beta_j E(-\tilde{t}-r_0)\tilde{\lambda}(\tilde{w}(-r_0,\cdot)+i\tilde{\mu}\partial_t\tilde{w}(-r_0,\cdot))](-\sigma\theta)=0(\sigma^{-\infty})$$

as $\sigma\to\infty$ (cf. §4 of the author [15]), we have

$$\tilde{F}[\rho(s)S(s,\theta,\omega)](\sigma)=2^{-1}e^{-i\sigma\tilde{t}}\sum_{j=0}^{N-1}(i\sigma)^{n-1-j}\mathcal{F}'[2^{-1}\beta_j E(\tilde{t}+r_0)(1+\sigma^{-1}\tilde{\lambda}) \cdot(\tilde{w}(-r_0,\cdot)-i\tilde{\mu}\partial_t\tilde{w}(-r_0,\cdot))](-\sigma\theta)+0(\sigma^{-N+N_0}).$$

The assumption (0.5) implies that if WF$[u_0]$ is contained in a conic neighborhood of $\tilde{M}_\omega(\theta)\times\{-\omega\}$ (WF$[\tilde{w}(-r_0,\cdot)-i\tilde{\mu}\partial_t\tilde{w}(-r_0,\cdot)]$ is contained there) $E(\tilde{t}+r_0)u_0$ is represented by the Fourier integral operator:

$$E(\tilde{t}+r_0)u_0(x)=(2\pi)^{-n}\int e^{i\phi(\tilde{t}+r_0,x,\xi)}a(\tilde{t}+r_0,x,\xi)\hat{u}_0(\xi)d\xi \quad \mathrm{mod}\ C^\infty$$

(cf. the proof of Theorem 2.6 in the author [15]). Moreover note that $\mathcal{F}'[\delta^{(k)}(-r_0-x\cdot\omega)](B\eta)=(-i\eta_1)^k e^{ir_0\eta_1}\delta(\eta')$ ($\eta=(\eta_1,\eta')$), where $B=(b_1,\cdots,b_n)$ is an orthogonal matrix with $b_1=\omega$. Then, introducing change of the variables $x=q^-(\tilde{t};-r_0,y,\omega)$ $(=q^-(y))$ near $x=q^-(\tilde{t};-r_0,\tilde{M}_\omega(\theta),\omega)$ ($y=(y_0,y')$ is orthogonal coordinates with $y_0=x\cdot\omega$), we obtain

$$\begin{aligned}&\mathcal{F}'[2^{-1}\beta_j E(\tilde{t}+r_0)(1+\sigma^{-1}\tilde{\lambda})(\tilde{w}(-r_0,\cdot)-i\tilde{\mu}\partial_t\tilde{w}(-r_0,\cdot))](-\sigma\theta)\\&=\int e^{i\sigma x\cdot\theta}\tilde{\gamma}(x)\beta_j(x)\int_0^{\tilde{\tau}_\sigma}e^{i\phi(\tilde{t}+r_0,x,-\tau\omega)}a(\tilde{t}+r_0,x,-\tau\omega)e^{-i\tau r_0}d\tau dx+0(\sigma^{-\infty})\\&=\int_{\boldsymbol{R}^{n-1}}dy'\int_{-\infty}^{\infty}dy_0\int_0^{\tilde{\tau}}\sigma d\tau e^{i\sigma\{q^-(y)\cdot\theta-\tau(y_0+r_0)\}}\beta_j(q^-(y))\gamma(y)\\&\qquad\cdot a(\tilde{t}+r_0,q^-(y),-\sigma\tau\omega)\left|\det\frac{\partial q^-}{\partial y}\right|+0(\sigma^{-\infty})\qquad(\text{as }\sigma\to\infty)\end{aligned}$$

($\tilde{\gamma}(x)\in C_0^\infty(\boldsymbol{R}^n)$, $\tilde{\gamma}(x)=1$ on a neighborhood of q^- (supp$[\gamma]$), and $\tilde{\tau}$ is a positive constant independent of σ). The function $\Phi(y_0,\tau)=q^-(y_0,y')\cdot\theta-\tau(y_0+r_0)$ has the stationary point $(y_0,\tau)=(-r_0,p^-(-r_0,y')\cdot\theta)$, at which its Hesse matrix equals $\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$. Expanding $\int_{-\infty}^{\infty}\int_0^{\tilde{\tau}}e^{i\sigma\Phi(y_0,\tau)}\beta_j\gamma\cdots dy_0 d\tau$ (as $\sigma\to\infty$) by the methods of stationary phases (e.g., cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we have the asymptotic expansion

$$\tilde{F}[\rho(s)S(s,\theta,\omega)](\sigma)=e^{-i\sigma\tilde{t}}(i\sigma)^{n-1}\int_{x\cdot\omega=-\gamma_0}e^{i\sigma q^-(\tilde{t};-r_0,x,\omega)\cdot\theta}\cdot\{\sum_{j=0}^{N-1}\rho_j(x)(i\sigma)^{-j}\}dx+0(\sigma^{-N+N_0}) \tag{3.2}$$

(N_0 is an integer independent of $N=1,2,\cdots$). Here ρ_j are C^∞ functions with

$\mathrm{supp}[\rho_j]\subset\mathrm{supp}[\gamma]$ and all real-valued, which follows from the fact that the symbol $a(\tilde{t}, x, \xi)$ has a homogeneous asymptotic expansion $\sum_{k=0}^{\infty} a_k(\tilde{t}, x, \xi)$ such that $a_k(\tilde{t}, x, \xi)$ *are real-valued for even* k *and purely imaginary valued for odd* k; furthermore ρ_0 is of the form

$$\rho_0(y) = \gamma(y)\beta_0(q^-(\tilde{t}; -r_0, y, \omega))a_0(\tilde{t}+r_0, q^-(\tilde{t}; -r_0, y, \omega), -\omega)\,|\det\frac{\partial q^-}{\partial y}|\,.$$

Combining this with (3.1) and (ii) of Lemma 3.2, we see that $\rho_0(x)\geqq 0$ on $\boldsymbol{R}^n$ and $\rho_0(x)>0$ for any x at which the function

$$\varphi(x) = -q^-(\tilde{t}; -r_0, x, \omega)\cdot\theta \quad (x\cdot\omega = -r_0)$$

is minimum. Thus, applying Theorem 2 to (3.2), we obtain

$$\sigma^m \bar{F}[\rho S](\sigma)\notin L^2(1, \infty)$$

for some constant $m\in\boldsymbol{R}$, which proves Theorem 4.

Proof of Lemma 3.2. We denote by y the variables on $\boldsymbol{R}^{n-1}=\{x: x\cdot\omega=-r_0\}$. It follows from (0.4) that for a large constant t_0 independent of t, y and ω

$$q^-(t; -r_0, y, \omega) = q^-(t_0; -r_0, y, \omega)+(t-t_0)p^-(t_0; -r_0, y, \omega),\quad t\geqq t_0, y\in\boldsymbol{R}^{n-1}\,.$$

Fix $\tilde{y}\in M_\omega(\theta)$ arbitrarily and take a neighborhood $U(\tilde{y})$ of $\tilde{y}$ such that

$$|q^-(t_0; -r_0, y, \omega)-q^-(t_0; -r_0, \tilde{y}, \omega)|\leqq\varepsilon/2 \qquad \text{for any } y\in U(\tilde{y})\,,$$
$$|t_0\{p^-(t_0; -r_0, y, \omega)-p^-(t_0; -r_0, \tilde{y}, \omega)\}|\leqq\varepsilon/2 \qquad \text{for any } y\in U(\tilde{y})\,.$$

Then, in view of the definitions of $M_\omega(\theta)$ and $s_\omega(\theta)$ we have for any $y\in U(\tilde{y})$ and $\tilde{t}\geqq t_0$

$$\begin{aligned}\psi(y; \tilde{t})&\leqq q^-(t_0; -r_0, \tilde{y}, \omega)\cdot\theta - t_0 p^-(t_0; -r_0, \tilde{y}, \omega)\cdot\theta+\tilde{t}p^-(t_0; -r_0, y, \omega)\cdot\theta+\varepsilon\\ &\leqq s_\omega(\tilde{t})+\varepsilon+\tilde{t}\,.\end{aligned}$$

On the other hand, for any neighborhood U of $M_\omega(\varepsilon)$ it follows that $\delta=\inf_{\substack{y\notin U\\ |y|\leqq r_1}}\{1-p^-(t_0; -r_0, y, \omega)\cdot\theta\}>0$, which yields that $\psi(y; t)\leqq(C-\delta t)+t$ for any $y\notin U$ ($|y|\leqq r_1$) and $t\geqq t_0$ (C is a constant independent of y and t). This means that

$$\psi(y; \tilde{t})\leqq s_\omega(\theta)-1+\tilde{t}$$

if $y\notin U$, $|y|\leqq r_1$ and $\tilde{t}$ is large enough. Therefore we obtain the lemma.

References

[1] J.J. Duistermaat: *Oscillatory integrals, Lagrange immersions and unfolding of singularities,* Comm. Pure Appl. Math. **27** (1974), 204–281.

[2] L. Hörmander: *Fourier integral operators* I, Acta Math. **127** (1971), 79–183.
[3] A. Kaneko: *Newton's diagram, singularities and oscillatory integrals,* Lecture note of Sophia Univ. No 11 (1981) (in Japanese).
[4] H. Kumano-go: *A calculus of Fourier integral operators on $\boldsymbol{R}^n$ and the fundamental solution for an operator of hyperbolic type*, Comm. Partial Differential Equations **1** (1976), 1–44.
[5] H. Kumano-go: Pseudo-differential operators, M.I.T. Press (1982).
[6] P.D. Lax and R.S. Phillips: Scattering theory, Academic Press (1967).
[7] P.D. Lax and R.S. Phillips: *Scattering theory for the acoustic equation in an even number of space dimensions*, Indiana Univ. Math. J. **22** (1972), 101–134.
[8] A. Majda: *A representation formula for the scattering operator and the inverse problem for arbitrary bodies*, Comm. Pure Appl. Math. **30** (1977), 165–194.
[9] M. Matsumura: *Asymptotic behavior at infinity for Green's functions of first order systems with characteristics of nonuniform multiplicity,* Publ. Res. Inst. Math. Sci. Kyoto Univ. **12** (1976), 317–377.
[10] L. Nirenberg: Lectures on linear partial differential equations, Regional Conference Series in Math. No 17, A.M.S., Providence, R.I., 1973.
[11] R.T. Seeley: *Topics in pseudo-differential operators*, Pseudo-Diff. Operators (C.I.M.E., Stresa 1968), Edizioni Cremonese, Rome (1969), 169–305.
[12] H. Soga: *Mixed problems for the wave equation with a singular oblique derivative,* Osaka J. Math. **17** (1980), 199–232.
[13] H. Soga: *Oscillatory integrals with degenerate stationary points and their application to the scattering theory,* Comm. Partial Differential Equations **6** (1981), 273–287.
[14] H. Soga: *Singularities of the scattering kernel for convex obstacles,* J. Math. Kyoto Univ. **22** (1983), 729–765.
[15] H. Soga: *Singular support of the scattering kernel for the acaustic equation in inhomogeneous media,* Comm. Partial Differential Equations **9** (1984), 467–502.
[16] A.N. Varchenko: *Newton polyhedra and estimation of oscillating integrals,* Funkc. anal. i prilož. **10**-3 (1976), 13–38=Functional Anal. Appl. **10** (1976), 175–196.
[17] V.A. Vasil'ev: *Asymptotic exponential integrals, Newton's diagram and the classification of minimal points,* Funkc. anal. i prilož. **13**-4 (1979), 1–12=Functional Anal. Appl **13** (1979), 239–247.

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