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ON DISORDER PROBLEM WITH POINT PROCESSES

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0. Introduction

In this paper the following special optimal stopping problem called "disorder problem" is considered: on some probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ we are given an observable point process $(\xi_t)_{t\geq 0}$, and unobservable random variable Θ with values in $\boldsymbol{R}_+$. The stochastic characteristics of $(\xi_t)_{t\geq 0}$ may change at the random moment of time Θ, the probability law of Θ is known, however its value can not be observed directly. The objective is to maximize the value $E[g(\Theta, \tau)]$ by selecting a stopping time τ that is adapted to $\{\sigma(\xi_s, s\leqq t)\}_{t>0}$, for some given reward function $g(s, t)$. This kind of problems are considered in [1], [2], [3], [7], [8] and [10]

In section 2 according to the general theorem for optimal stopping problems with continuous parameter processes posed by M.E. Thompson [9], we derive the form of an optimal stopping time and the maximum expected reward function. In section 3 we restrict ourselves to the case when the expected reward process forms a monotone process and we apply the theorem of A. Irle [4] to our problem, and then we derive a form of optimal stopping time. At the end of section 3 we consider a special example and fined an optimal stopping time explicitly.

1. Statement of problem and preliminaries

Consider a measurable space $(\boldsymbol{X}, \mathcal{B})$ where $\boldsymbol{X}$ is a space of piecewise constant functions $x=(x_t)$, $t\geqq 0$, such that $x_0=0$ and $x_t=x_{t-}+(0 \text{ or } 1)$, $\mathcal{B}$ is a σ-algebra $\sigma\{x_s;\, s\geqq 0\}$. On $(\boldsymbol{X}, \mathcal{B})$ we are given complete probability measures μ^1 and μ^2, which satisfy Assumption I given bellow, and they are absolutely continuous with respect to each other.

Let $(\mathcal{B}_t)$, $t\geqq 0$, be an increasing family of right continuous sub σ-algebra of $\mathcal{B}$ such that

$$\mathcal{B}_t = \bigcap_{\varepsilon>0} \sigma\{x_s;\, s\leqq t+\varepsilon\} \vee \boldsymbol{Q}\,,$$

where

$$\boldsymbol{Q} = \{A \,|\, \mu^1(A) = 0 \quad \text{or} \quad \mu^1(A) = 1,\ A\in\mathcal{B}\}\,.$$

We define the family of probability measure $\{\mu_s, s\geqq 0\}$ as follows:

$$\mu_s(A)=\int \mu^2(A\,|\,\mathcal{B}_s)d\mu^1\,, \qquad A\in\mathcal{B}\,.$$

In the sequel we suppose that the following holds: for each $A\in\mathcal{B}$, $\mu_s(A)$, $\mu.(A)$: $\boldsymbol{R}_+\to[0, 1]$, is $\mathcal{B}(\boldsymbol{R}_+)$ measurable, where $\mathcal{B}(\boldsymbol{R}_+)$ is the Borel field of $\boldsymbol{R}_+$.

According to 18.3 in [5], for each $s\geqq 0$ the process $X^s=(x_t, \mathcal{B}_t, \mu_s)$, $t\geqq 0$, on the probability space $(\boldsymbol{X}, \mathcal{B}, \mu_s)$ is a point process. Let $\tau_i(x)=\inf\{s\geqq 0: x_s=i\}$, setting $\tau_i(x)=\infty$ if $\lim_{t\to\infty} x_t<i$, and let $\tau_\infty(x)=\lim_{i\to\infty}\tau_i(x)$.

The compensator $A^s=(A^s_t(x), \mathcal{B}_t)$ of this process is to be defined as follows:

$$A^s_t(x)=\sum_{i\geq 1} A^s_t(x, i)\,, \qquad x\in\boldsymbol{X}\,,$$

where

$$A^s_t(x, i)=\int_0^{t\wedge\tau_i(x)}[1-F^s(u-;\,x, i)]^{-1}dF^s(u;\,x, i)\,, \quad \text{and}$$

$$F^s(t;\,x, 1)=\mu_s\{y\colon \tau_1(y)\leqq t,\, y\in\boldsymbol{X}\}\,,$$

$$F^s(t;\,x, i)=\mu_s\{y\colon \tau_i(y)\leqq t,\, y\in\boldsymbol{X}\,|\,\tau_{i-1}(x), \cdots, \tau_1(x)\}\,, \qquad i\geqq 2\,.$$

Denote $A^\infty_0(x)=A_t(x)$ and $A^0_t(x)=\tilde{A}_t(x)$, the compensators of the point processes $(x_t, \mathcal{B}_t, \mu^1)$ and $(x_t, \mathcal{B}_t, \mu^2)$ respectively. We suppose that the following assumption is satisfied:

ASSUMPTION I. *For any $x\in\boldsymbol{X}$, $A_t(x)$ and $\tilde{A}_t(x)$ are continuous with respect to $t\in[0, \infty)$, and there exist some non-negative predictable processes $(\lambda_t(x), \mathcal{B}_t)$ and $(\tilde{\lambda}_t(x), \mathcal{B}_t)$ such that*

$$\tilde{A}_t(x)=\int_0^t \lambda_u(x)dA_t(x)\,, \quad A_t(x)=\int_0^t \tilde{\lambda}_u(x)d\tilde{A}_u(x)\,, \qquad t<\tau_\infty\,,$$

$$\int_0^{\tau_\infty}[1-\sqrt{\lambda_u(x)}]^2 dA_u(x)<\infty, \qquad a.e.\ (\mu^1 \text{ and } \mu^2)\,.$$

For each $s\geqq 0$, according to the definition of $A^s_t(x)$, it holds that

$$A^s_t(x)=\int_0^t \lambda^s_u(x)dA_u(x)\,, \quad A_t(x)=\int_0^t \tilde{\lambda}^s_u(x)dA^s_u(x)\,, \quad \text{and}$$

$$\int_0^{\tau_\infty}[1-\sqrt{\lambda^s_u(x)}]^2 dA_u(x)<\infty, \quad \int_0^{\tau_\infty}[1-\sqrt{\tilde{\lambda}^s_u(x)}]^2 dA^s_u(x)<\infty\,,$$

and $A^s_t(x)$ is continuous, where

$$\lambda^s_u(x)=\begin{cases}1 & u\leqq s\\ \lambda_u(x) & u>s\end{cases} \quad \text{and} \quad \tilde{\lambda}^s_u(x)=\begin{cases}1 & u\leqq s\\ \tilde{\lambda}_u(x) & u>s\,.\end{cases}$$

In Theorem 19.7 of [5] it, is shown that under Assumption I the measure μ^1, μ^2 and μ_s are absolutely continuous with respect to each other and

$$\frac{d\mu_s}{d\mu^1}(t, x) = \left(\frac{d\mu^1}{d\mu_s}(t, x)\right)^{-1} = \exp\left\{\int_0^t \log \frac{dA_u^s(x)}{dA_u(x)} dx_u - [A_t^s(x) - A_t(x)]\right\}.$$

Let $\Lambda(x; s, t)$ denote $\frac{d\mu_s}{d\mu^1}(t, x)$, which is adapted to $\mathcal{B}_t$. In addition, let $F(t)$, $t \geqq 0$, be a given distribution function and ν be the Lebesgue-Stieltjes measure induced by $F(t)$ on the measurable space $(\boldsymbol{R}_+, \mathcal{B}(\boldsymbol{R}_+))$.

Let the probability space $(\Omega, \mathcal{F}, P)$ be the completion of $(\boldsymbol{X} \times \boldsymbol{R}_+, \mathcal{B} \times \mathcal{B}(\boldsymbol{R}_+), P)$, where P is defined by

$$P(A \times B) = \int_B \mu_u(A)\nu(du), \qquad A \in \mathcal{B},\ B \in \mathcal{B}(\boldsymbol{R}_+).$$

Now, we shall state our optimal stoppnig problem. We suppose that we know the whole forms of measures and distribution function μ^1, μ^2, μ_s, $s \geqq 0$, and $F(t)$. For each $\omega = (x, s) \in \boldsymbol{X} \times \boldsymbol{R}_+ = \Omega$, $x = (x_t)_{t \geq 0}$, let

$$\xi_t(\omega) = x_t \quad \text{and} \quad \Theta(\omega) = s.$$

Suppose that we observe the stochastic process $\xi = (\xi_t, \mathcal{F}_t, P)$, $t \geqq 0$, where

$$\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma\{\xi_u;\ u \leqq t + \varepsilon\} \vee \boldsymbol{Q}', \quad \boldsymbol{Q}' = \{A\colon P(A) = 0 \text{ or } P(A) = 1,\ A \in \mathcal{F}\}.$$

According to the absolute continuity of μ^1, μ^2 and μ_s with respect to each other, it follows that $\mathcal{F}_t = \mathcal{B}_t \times \{A\colon \nu(A) = 0 \text{ or } \nu(A) = 1,\ A \in \mathcal{B}(\boldsymbol{R}_+)\}$, and hence $(\mathcal{F}_t)_{t>0}$ is right continuous increasing family of σ-algebra. We assume that Θ can not be observed directly. Our objective is to find an optimal stopping time τ^* such that

$$E[g(\Theta, \tau^*)] = \sup_{\tau \in \mathfrak{M}} E[g(\Theta, \tau)],$$

for some given reward function $g(u, v)$, $g\colon \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}$, where $\mathfrak{M}$ is the set of all stopping times with respect to $(\mathcal{F}_t)_{t>0}$.

Such kind of optimal stopping problem is called disorder problems ([1], [2], [3], [7], [8], [10]), in this context the random time Θ is understood as a disorder time of some stochastic system.

From Bayes formula we have

(1) $$P(\Theta \in ds \mid \mathcal{F}_t) = \Lambda(x; s, t)\nu(ds) \Big/ \int_0^\infty \Lambda(x; s, t)\nu(ds).$$

Let $$\alpha_t(x) = \int_0^t \Lambda(x; s, t)\nu(ds), \qquad \text{then}$$

(2) $$\alpha_u(x) = \alpha_t \Lambda(x; t, u) + \int_t^u \Lambda(x; s, u)\nu(ds).$$

Lemma 1. *Let $Y_t(x)$ be any function such that $Y_t\colon \boldsymbol{X} \to \boldsymbol{R}^1$ and Y_t is $\mathcal{B}_t$ measurable, and*

$$\int_0^\infty E^{\mu_1}[|Y_u(\cdot)\Lambda(\cdot;s,u)|\,|\mathcal{B}_t]\nu(ds)<\infty \text{ a.s. } P, \qquad \text{for any} \quad u\geqq t>0\,.$$

Then

(3) $\qquad E[Y_u(\xi)|\mathcal{F}_t]=E^{\mu_1}[Y_u(\cdot)(\alpha_u(\cdot)+\nu(u,\infty))|\mathcal{B}_t]/\{\alpha_t(\xi)+\nu(t,\infty)\}\,.$

Proof.

$$\begin{aligned} E[Y_u(\xi)|\mathcal{F}_t] &= \int_0^\infty E_{\Theta=s}[Y_u(\xi)|\mathcal{F}_t]P(\Theta\in ds|\mathcal{F}_t) \\ &= \int_0^\infty E^{\mu_s}[Y_u(\cdot)|\mathcal{B}_t]P(\Theta\in ds|\mathcal{F}_t) \\ &= \int_0^\infty \frac{E^{\mu_1}[Y_u(\cdot)\Lambda(\cdot;s,u)|\mathcal{B}_t]}{E^{\mu_1}[\Lambda(\cdot;s,u)|\mathcal{B}_t]}P(\Theta\in ds|\mathcal{F}_t)\,. \end{aligned}$$

From our assumption and equation (1) and the property that $\Lambda(x;s,t)=1$ for $s>t$, it follows that the last expression is equal to the right hand side of (3). The proof is complete.

2. Derivation of optimal stopping rules for general disorder problems

In this section we shall derive the forms of optimal stopptig time and maximum expected reward function for the case when the expected reward process forms a general right continuous process.
For each $N=1,2,\cdots$ let $N_k=N-k2^{-N}$, $k=1,2,\cdots,N2^N$,

$$H_{N_1}(x;N)=\max\begin{cases} G_{N_1}(x)\,, \\ \int_0^{N_1}g(s,N)\Lambda(x;s,N_1)\nu(ds)+\int_{N_1}^\infty g(s,N)\nu(ds)\,, \end{cases}$$

$$H_{N_k}(x;N)=\max\begin{cases} G_{N_k}(x)\,, \\ E^{\mu_1}[H_{N_{k-1}}(\cdot;N)|\mathcal{B}_{N_k}]\,, \end{cases}$$

where $G_t(x)=\int_0^t g(s,t)\Lambda(x;s,t)\nu(ds)+\int_t^\infty g(s,t)\nu(ds)$, $x\in\boldsymbol{X}$. For each fixed dyadic rational $q\geqq 0$, $q=N-k2^{-N}$, where we select a suitable k for each N, if it exists we define

$$H_q(x)=\lim_{N\to\infty}H_{N_k}(x;N)\,.$$

For any t, that is not a dyadic rational, we select a sequence $q_n\downarrow t$, q_n are dyadic rationals, and if it exists we define

$$H_t(x)=\lim_{n\to\infty}H_{q_n}(x)\,.$$

Let
$$\hat{\sigma} = \inf \{t \geqq 0;\ H_t(\xi) \leqq G_t(\xi)\}\ ,$$
and
$$\gamma_t(x) = [\alpha_t(x) + \nu(t,\ \infty)]^{-1} H_t(x)\ .$$

The following Theorem 1 follows from Theorem 6.1, Corollary 7.1 and Theorem 7.3 of Thompson [9].

Theorem 1. *Suppose that the reward function $g(u, v)$ is bounded, and $\log \lambda_t(\xi)$ is bounded a.s.. Then $H_t(t)$, $x \in X$, exists and $(\gamma_t)_{t \geq 0}$ is the minimum dominating regular supermartingale of*

$$(E[g(\Theta,\ t)] \Big| \mathcal{F}_t]),\ t \geqq 0\ , \quad \textit{and} \quad E[\gamma_0(\xi)] = \sup_{\tau \in \mathfrak{M}} E[g(\Theta,\ \tau)]$$

In addition if $[\alpha_t(\xi) + \nu(t,\ \infty)]^{-1} G(\xi)$ is continuous to the left a.s., and $P(\hat{\sigma} < \infty) = 1$, then $\hat{\sigma}$ is an optimal stopping time:

$$E[g(\Theta,\ \hat{\sigma})] = \sup_{\tau \in \mathfrak{M}} E[g(\Theta,\ \tau)]\ .$$

Proof. From (1) and (2) it holds that if we stop the observation at time $t \geqq 0$, then we take the expected reward

$$z_t(\xi) = E[g(\Theta,\ t) \,|\, \mathcal{F}_t] = [\alpha_t(\xi) + \nu(t,\ \infty)]^{-1} G_t(\xi)\ . \tag{4}$$

The boundedness and the right continuity of z_t a.s. follow from the right continuity of ξ and the boundedness of $g(u,v)$ and $\log \lambda_t$ a.s.. Since any right continuous process is well-measurable and lower semi-continuous on the right, Theorem 6.1 of [9] is applicable to the process $(z_t(\xi),\ \mathcal{F}_t)$, $t \geqq 0$. According to Lemma 1, for $t = N - k2^{-N}$, $k = 1,\ 2,\ \cdots$, and $h = 2^{-N}$ it holds that

$$E[\gamma_{t+h}(\xi;\ N) \,|\, \mathcal{F}_t] = [\alpha_t(\xi) + \nu(t,\ \infty)]^{-1} E^{\mu 1}[H_{t+h}(\cdot\,;\ N) \,|\, \mathcal{B}_t] \,|_{x=\xi}\ ,$$

where

$$\gamma_t(x;\ N) = [\alpha_t(x) + \nu(t,\ \infty)]^{-1} H_t(x;\ N)\ .$$

Using this relation, after applying Theorem 6.1 and Corollary 7.1 of [9], the first part of this theorem follows.

In order to see that the second part of this theorem holds, it is enough to note that $(z_t)_{t \geq 0}$ satisfies the conditions of Theorem 7.3 of [9].
The proof is complete.

The following Corollary deals with the special problem $D_{a,b}$ (see [2], [3], [10]).

PROBLEM $D_{a,b}$. *For some fixed $a,\ b \geqq 0$, find τ^* such that*

$$P(\Theta - a \leqq \tau \leqq \Theta + b) = \sup_{\tau \in \mathfrak{M}} P(\Theta - a \leqq \tau \leqq \Theta + b)\ .$$

Corollary. *Suppose that* $\Lambda(x; s, t)$, $\Lambda(\cdot; s, t)$: $\boldsymbol{X}\rightarrow\boldsymbol{R}_+$, *is* $\mathcal{B}_{s,t}=\sigma\{x_u; s\leqq u\leqq t, x\in\boldsymbol{X}\}\vee\boldsymbol{Q}$*-measurable, and* $\log\lambda_t$ *is bounded a.s.. If we set* $g(u, v)=I_{([u-a,u+b])}(v)$, *then* $H_t(x)$, $x\in\boldsymbol{X}$, *exists and is* $\mathcal{B}_{t-b,t}$*-measurable, and*

$$E[\gamma_0(\xi)]=\sup_{\tau\in\mathfrak{M}}P(\Theta-a\leqq\tau\leqq\Theta+b)\,.$$

In addition if $[\alpha_t(\xi)+\nu(t, \infty)]^{-1}G_t(\xi)$ *is continuous to the left a.s., and* $P(\tilde{\sigma}<\infty)=1$, *then* $\tilde{\sigma}$ *is an optimal stopping time for the problem* $D_{a,b}$.
In this case $\{\omega: \tilde{\sigma}(\omega)<t\}\in\mathcal{F}_{t-b,t}=\sigma\{\xi_u; t-b\leqq u\leqq t\}\vee\boldsymbol{Q}'$, $t\geqq 0$.

3. Derivation of an optimal stopping rule for the monotone reward process

In this section we shall derive the following Theorem 2, which is derived by applying the general theorem for optimal stopping problems with continuous parameter monotone processes by A. Irle [4]. At the end of this section we shall derive an explicit form of an optimal stopping time for a special example.

As in the proof of Theorem 1, applying Lemma 1 to the expected reward process $(z_t)=(E[g(\Theta, t)\,|\,\mathcal{F}_t])$, $t\geqq 0$, defined by (4), we have

$$E[z_{t+\Delta}(\xi)\,|\,\mathcal{F}_t]=[\alpha_t(\xi)+\nu(t, \infty)]^{-1}V_t(\xi; \Delta)\,,\qquad \Delta\geqq 0\,, \tag{5}$$

where

$$V_t(x; h)=\int_0^t g(s, t+h)\Lambda(x; s, t)\nu(ds)+\int_t^\infty g(s, t+h)\nu(ds)\,.$$

Let $(G_t(\xi))$, $t\geqq 0$, be the process defined in section 2.

Condition I. (i) *There exists a* $\tilde{V}_t(\xi)=\sup_{h>0}V_t(\xi; h)$ *a.s., and a version* (V_t), $t\geqq 0$, *of* $(\tilde{V}_t)$, $t\geqq 0$, *such that the process* (I_{C_t}), $t\geqq 0$, *with* $C_t=\{\omega: G_t\geqq V_t\}$ *is right continuous.*
(ii) $C_t\subset C_{t+\Delta}$ *holds for any* t, $\Delta\geqq 0$ *and* $\bigcup_{t\geqq 0}C_t=\Omega$.

Let $\sigma=\inf\{t: G_t\geqq V_t\}=\inf\{t: \omega\in C_t\}$.

Condition II. (i) $P(\sigma<\infty)=1$.

(ii) *For any increasing sequence* (τ_n), $n=1, 2, \cdots$, *of* $\mathcal{F}_t$*-stopping times* $\tau_n\leqq\sigma$ *it holds that*

$$E[z_{\sup\tau_n}]\geqq\overline{\lim_{n\to\infty}}\,E[z_{\tau_n}]\,. \tag{6}$$

Theorem 2. *Suppose that the conditions I, II and* $E[z_\tau]>-\infty$ *hold for any bounded* $\mathcal{F}_t$*-stopping time* τ. *Then there exists the minimum dominating supermartingale* $(\gamma)_{t\geqq 0}$ *for* $(z_t)=(E[g(\Theta, t)\,|\,\mathcal{F}_t])$, $t\geqq 0$, *and*

(i) $E[g(\Theta, \sigma)] \geqq \sup \{E[g(\Theta, \tau)] : \tau\ \mathcal{F}_t\text{-stopping time}, \ \underline{\lim} \int_{(\tau > t)} \gamma_t^- \, dP = 0\}$
$\geqq \sup \{E[g(\Theta, \tau)] : \tau\ \mathcal{F}_t\text{-stopping time}\}$.

(ii) *If* $(\gamma_t)_{t\geq 0}$ *is regular then*

$$E[g(\Theta, \sigma)] = \sup_{\tau \in \mathfrak{M}} E[g(\Theta, \tau)] .$$

Proof. Since the expected reward process $(z_t)_{\geq 0}$ is right continuous, the existence of the minimum dominating supermartigale follows from Mertens [6]. From (4) and (5) the assertion of (i) and (ii) follow by making use of Corollary of § 2 and Theorem of § 3 in Irle [4]. The proof is complete.

EXAMPLE. Suppose that the reward function $g(\theta, t)$ is given by

$$g(\theta, t) = \begin{cases} -\exp[c(t-\theta)] & 0 \leqq \theta < t, \\ -K & \theta \geqq t, \end{cases} \qquad c, K > 0,$$

and the compensators $(A_t(x))_{t\geq 0}$ and $(\tilde{A}_t(x))_{t\geq 0}$ are given by

$$A_t(x) = \lambda_1 \int_0^t f(u)\,du , \quad \tilde{A}_t(x) = \lambda_2 \int_0^t f(u)\,du .$$

Suppose that $\lambda_2 > \lambda_1 > 0$ and $f(u)$ is a non-negative bounded function such that

$$c\Delta/(\lambda_2 - \lambda_1) \geqq \int_t^{t+\Delta} f(u)\,du , \qquad \text{for any} \quad t, \Delta \geqq 0 . \tag{7}$$

In this case the point processes $(x_t, \mathcal{B}_t, \mu^i)$, $i=1, 2$, are Poisson processes with intensity $\lambda_i f(u)$, $i=1, 2$, respectively. According to section 1 we have

$$\Lambda(x; s, t) = (\lambda_2/\lambda_1)^{x_t - x_s} \exp\left[(\lambda_1 - \lambda_2) \int_s^t f(u)\,du\right] .$$

Suppose that $F(t) = 1 - e^{-\alpha t}$, $\alpha > 0$. In this case from (4) and (5) we have

$$\begin{aligned}
&z_t(x) = [\alpha_t + \nu(t, \infty)]^{-1} G_t(x) , \\
&E[z_{t+h}(\xi) \mid \mathcal{F}_t] = [\alpha_t(\xi) + \nu(t, \infty)]^{-1} V_t(\xi; h), \quad \text{with} \\
&G_t(x) = -\int_0^t e^{c(t-s)} \beta^{x_t - x_s} e^{\gamma(t,s)} \alpha e^{-\alpha s} ds - K e^{-\alpha t} , \\
&V_t(x; h) = -e^{ch} \int_0^t e^{c(t-s)} \beta^{x_t - x_s} e^{\gamma(t,s)} \alpha e^{-\alpha s} ds \\
&\qquad + \frac{\alpha}{c+\alpha} [e^{-\alpha(t+h)} - e^{ch - \alpha t}] - K e^{-\alpha(t+h)} , \qquad \text{for} \quad h \geqq 0 ,
\end{aligned}$$

where

$$\beta = \lambda_2/\lambda_1 \quad \text{and} \quad \gamma(t, s) = (\lambda_1 - \lambda_2) \int_s^t f(u)\,du , \qquad 0 \leqq s \leqq t .$$

For $t \geq 0$ and $x \in \boldsymbol{X}$ let

$$h_t(x) = \begin{cases} 0 & \text{if } \alpha/(c+\alpha) \geqq K\,, \\ \max\{0,\ \tilde{h}_t(x)\} & \text{if } \alpha/(c+\alpha) < K\,, \end{cases}$$

where

$$(8)\quad \tilde{h}_t(x) = \left[-\alpha t + \log \frac{(c+\alpha)K-\alpha}{c\{(c+\alpha)\int_0^t e^{c(t-s)}\beta^{x_t-x_s}e^{\gamma(t,s)}e^{-\alpha s}ds + e^{-\alpha t}\}}\right](c+\alpha)^{-1}.$$

Then it holds that

$$(9)\qquad V_t(x) = V_t(x;\ h_t(x)) = \sup_{h\geq 0} V_t(x;\ h)\,.$$

From the right continuity of $(h_t(\xi))$, $t \geqq 0$, it follows that the set

$$(10)\qquad C_t = \{G_t \geqq V_t\} = \{\omega\colon (\int_0^t e^{c(t-s)}\beta^{\xi_t-\xi_s}e^{\gamma(t,s)}\alpha e^{-\alpha s}ds)(e^{ch_t}-1) + Ke^{-\alpha t}(e^{-\alpha h_t}-1) + \frac{\alpha e^{-\alpha t}}{c+\alpha}(e^{ch_t}-e^{-\alpha h_t}) \geqq 0\}\,,\qquad t \geqq 0\,,$$

is also right continuous. Suppose that $\alpha/(c+\alpha) < K$. Take any $\omega \in C_t$, let $\xi = \xi(\omega)$, since $G_t(\xi) - V_t(\xi,\ h_t(\xi)) \geqq 0$ and (9) holds, it follows that for any $\Delta > 0$

$$\begin{aligned} &[G_{t+\Delta}(\xi) - V_{t+\Delta}(\xi;\ h_{t+\Delta}(\xi))] - [G_t(\xi) - V_t(\xi;\ h_t(\xi))] \\ &\geqq [G_{t+\Delta}(\xi) - V_{t+\Delta}(\xi;\ h_{t+\Delta}(\xi))] - [G_t(\xi) - V_t(\xi;\ h_{t+\Delta}(\xi))] \\ &\geqq (e^{ch_{t+\Delta}}-1)[\int_t^{t+\Delta} e^{c(t+\Delta-s)}\beta^{\xi_{t+\Delta}-\xi_s}e^{\gamma(t+\Delta,s)}\alpha e^{-\alpha s}ds \\ &+ (\beta^{\xi_{t+\Delta}-\xi_t}-1)\int_0^t e^{c(t-s)}\beta^{\xi_t-\xi_s}e^{\gamma(t,s)}\alpha e^{-\alpha s}ds] \\ &+ Ke^{-\alpha t}(1-e^{-\alpha\Delta})(1-e^{-\alpha h_{t+\Delta}}) - \frac{\alpha e^{-\alpha t}}{\alpha+c}(1-e^{-\alpha\Delta})(e^{ch_{t+\Delta}}-e^{-\alpha h_{t+\Delta}}) \geqq 0 \end{aligned}$$

Hence it follows that $\omega \in C_{t+\Delta}$, in other words $C_t \subset C_{t-\Delta}$, $\Delta \geqq 0$. Where we make use of the fact that $\beta > 1$ and

$$(11)\qquad \exp[c\Delta + \gamma(t+\Delta,\ t)] \geqq 1\,,\qquad t,\ \Delta \geqq 0\,,$$

which follows from (7). In case $\alpha/(c+\alpha) \geqq K$, we have $C_t = C_{t+\Delta}$, $\Delta \geqq 0$.

Now we shall show that $\sigma = \inf\{t\colon \omega \in C_t\}$ is bounded. Since $\beta > 1$ and (11) holds, from (8) it follows that

$$\begin{aligned} \tilde{h}_t(x) &\leqq \left[-\alpha t + \log\{\alpha(K(c+\alpha)-\alpha)\} - \log \alpha c - \log\left\{\frac{c}{\alpha}(1-e^{-\alpha t})+1\right\}\right] \Big/ (c+\alpha) \\ &= \chi(t) \qquad \text{for any} \quad x \in \boldsymbol{X}\,. \end{aligned}$$

Let t^* be the unique root of $\chi(t) = 0$. Since the number t^* is bounded, and since $\tilde{h}_t(x) \leqq \chi(t) \leqq 0$ for any $t \geqq t^*$ and $x \in \boldsymbol{X}$, it holds that $h_t(x)=0$ for any $t \geqq t^*$ and $x \in \boldsymbol{X}$. Hence we have

(12) $\quad \sigma(\omega) \leqq t^* \vee 0 \quad$ and so σ is a bounded random variable.

We shall show that Condition II-(ii) holds. In this case the reward process $(z_t, \mathcal{F}_t)_{t \geq 0}$ is a continuous process, and σ is a bounded stopping time, it holds that $\lim_{n\to\infty} z_{\tau_n} = z_{\lim_{n\to\infty} \tau_n} = z_{\sup \tau_n}$, a.s. for any increasing sequence (τ_n), $n=1, 2, \cdots$, such that $\tau_n \leqq \sigma$ a.s.. On the other hand from (12) and the form of the reward function $g(\theta, t)$ it holds that $0 \geqq z_\tau \geqq -K-e^{ct^*}$ a.s., for any $\tau \leqq \sigma$ a.s.. Thus, from Lebesgue's bounded convergence theorem (6) holds with equality.

Consequently, we see that all of the conditions in Theorem 2 are satisfied for this example. Thus we can conclude that the stopping time $\sigma = \inf\{t: \omega \in C_t\}$, C_t is defined by (10), is optimal in the sense of the assertion (i) of Theorem 2. The example is complete.

References

[1] R.F. Anderson & A. Friedman: *A quality control problem and quasivariational inequalities*, Arch. Rational Mech. Anal. **63** (1977), 205–252.

[2] T. Bojdecki: *Probability maximizing approach to optimal stopping and its application to a disorder problem*, Stochastics **3** (1979), 61–71.

[3] T. Bojdecki & J. Hosza: *On generalized disorder problem*, Stochastic Process. Appl. **18** (1984), 349–359.

[4] A. Irle: *Monotone stopping problems and continuous time processes*, Z. Wahrsch. Verw. Gebiete **48** (1979), 49–56.

[5] R.S. Liptser & A.N. Shiryaev: Statistics of random process II, Springer-Verlag, New York-Heiderberg-Berlin, 1977.

[6] J.F. Mertens: *Théorie des processus stochastiques généraux applications aux surmartingales*, Z. Wahrsch. Verw. Gebiete **22** (1972), 45–69.

[7] A.N. Shiryaev: Statistical sequential analysis, Translations of Mathematical Monographs 38, Amer. Math. Soc., 1973.

[8] A.N. Shiryaev: *On optimal methods in quickest detection problems*, Theory Probab. Appl. **8** (1963), 22–46.

[9] M.E. Thompson: *Continuous parameter optimal stopping problems*, Z. Wahrsch. Verw. Gebiete **19** (1971), 302–318.

[10] M. Yoshida: *Probability maximizing approach to a detection problem with continuous Markov processes*, Stochastics **11** (1984), 173–188.

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