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ON 3-FOLD IRREGULAR BRANCHED COVERING SPACES OF PRETZEL KNOTS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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It is well-known that any orientable closed 3-manifold is a 3-fold irregular branched covering space of a 3-sphere branched along a knot. It is an interesting problem to know which 3-manifold can be a 3-fold irregular branched covering space of a given knot. In this paper we consider those of pretzel knots.

For the permutation group S_3 on $\{0, 1, 2\}$, let $a=(01)$, $b=(02)$, $c=(12)$, $x=(012)$, $y=(021)$. Then there are relations $a^2=b^2=c^2=1$, $ab=bc=ca=x$, $ba=ac=cb=y$. Especially, we remark the following relations:

$$\begin{array}{llll} aba^{-1}=c\,, & aca^{-1}=b\,, & axa^{-1}=y\,, & aya^{-1}=x\,, \\ bab^{-1}=c\,, & bcb^{-1}=a\,, & bxb^{-1}=y\,, & byb^{-1}=x\,, \\ cac^{-1}=b\,, & cbc^{-1}=a\,, & cxc^{-1}=y\,, & cyc^{-1}=x\,, \\ xax^{-1}=b\,, & xbx^{-1}=c\,, & xcx^{-1}=a\,, & xyx^{-1}=y\,, \\ yay^{-1}=c\,, & yby^{-1}=a\,, & ycy^{-1}=b\,, & yxy^{-1}=x\,. \end{array}$$

A knot group G has a Wirtinger presentation:

$$(x_1, x_2, \cdots, x_n; r_1, r_2, \cdots, r_{n-1}) \qquad (1)$$

where each relator r_i indicates the relation form $r_i=x^{\varepsilon}_{j(i)}x_ix^{-\varepsilon}_{j(i)}x^{-1}_{i+1}$ $(\varepsilon=\pm 1)$ at a crossing as in Fig. 1.

Then a homomorphism from a knot group G to S_3 satisfies a condition as follows.

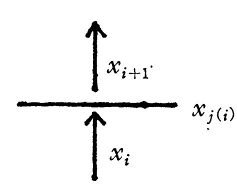


Fig. 1

Proposition 1. *Let the above* (1) *be a Wirtinger presentation of a knot group G. Then a homomorphism h from G to S_3 satisfies one of the followings.*

(i) $h(x_i)=a$ *(or b, c)* $\quad (i=1, 2, \cdots, n)$,

(ii) $h(x_i)=x$ *(or y)* $\quad (i=1, 2, \cdots, n)$.

Proposition 2. *Let* $(x_{11}, \cdots, x_{1n_1}, \cdots, x_{m1}, \cdots, x_{mn_m}; r_1, \cdots, r_k)$ *be a Wirtinger* *presentation of a link group of an m-component link, where* $x_{i1}, x_{i2}, \cdots, x_{in_i}$ *represent meridians of the i-th component* ($i=1, 2, \cdots, m$). *Then a homomorphism h from this link group to* S_3 *satisfies one of the followings.*

(i) $h(x_{ij})=a$ (*or* b, c) $\quad (j=1, 2, \cdots, n_i)$,

(ii) $h(x_{ij})=x$ (*or* y) $\quad (j=1, 2, \cdots, n_i)$.

If all generators are mapped to a (or b, c) by h, then the branched covering space corresponding to h is the disjoint union of a 3-sphere and the 2-fold regular branched covering space of the knot. And if all generators are mapped to x (or y) by h, then that is the 3-fold regular branched covering space of the knot. So, the branched covering space corresponding to h is 3-fold irregular, iff h satisfies the condition (i) of Proposition 1 and there exist generators x_i and x_j with $h(x_i)\neq h(x_j)$.

First, we consider the image of meridians by h at twists, especially for typical cases as in Fig. 2.

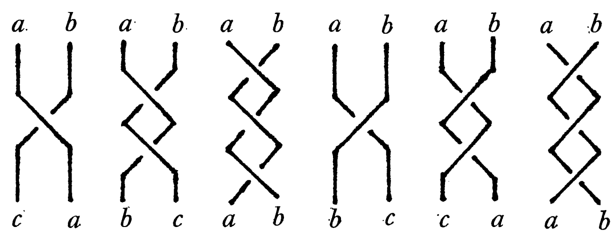


Fig. 2

Since $a^2=b^2=c^2=1$, we can ignore the orientation of a knot or a link.

If there is a block of three half-twists at the projection of a link L, deform L by the operation cancelling three half-twists as shown in Fig. 3; we have a new link L'. Then, determining the image of meridians of L' except at the three half-twists to be the same to that by h, we have a homomorphism h' from the link group of L' to S_3. We call h' a homomorphism induced from h.

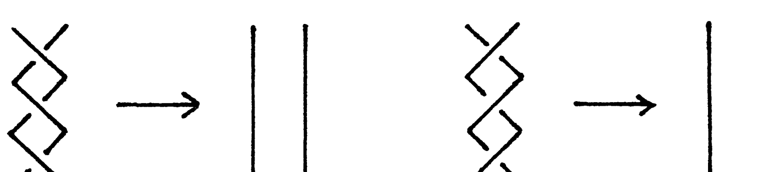


Fig. 3

Since it is easily seen that the inverse of the above operation is also an operation cancelling three half-twists after a slight deformation of the projection of L' as Fig. 4, we have

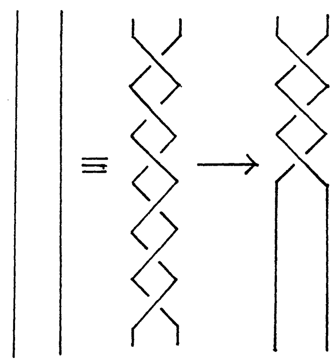


Fig. 4

Proposition 3. *Let L' be a link obtained from a link L by operation cancelling three half-twists, and G and G' be the link group of L and L'. Then the followings are equivalent.*

(a) *There exists a homomorphism h from G to S_3 satisfying the condition* (i) *of Proposition* 2.

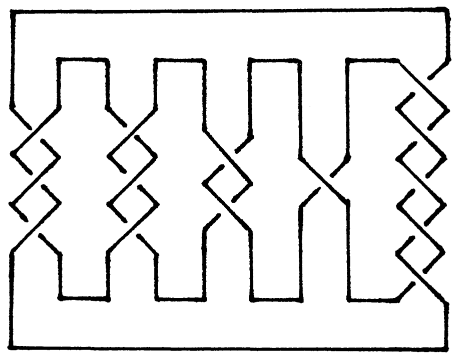
(b) *There exists a homomorphism h' from G' to S_3 satisfying the condition* (i) *of Proposition* 2.

We regard a part of the three half-twists as a *trivial tangle* i.e. a pair of a 3-ball and two proper arcs which are trivial and separated in the 3-ball. Since the irregular 3-fold branched covering space of a trivial tangle is a 3-ball (Burde [2]), we have

Proposition 4 (Montesinos [5]). *Let L' be a link obtained from L by operation cancelling three half-twists, and G and G' be the link group of L and L'. Suppose that a homomorphism h from G to S_3 exists and h' is a homomorphism induced by h. Furthermore, at the three half-twists deformed by operation, we suppose that the images of medidians of the two arcs by h are distinct and transpositions. Then the* 3*-fold irregular branched covering space of a* 3*-sphere branched along L corresponding to h is homeomorphic to the irregular* 3*-fold branched covering space of a* 3*-sphere branched along L' corresponding to h'.*

From Proposition 4, we can deside *all* 3-fold irregular branched covering spaces of a 3-sphere branched along a pretzel knot. *A pretzel knot* is a knot consisting of a row of 2-strand braids of $q_1, q_2, \cdots, q_m$ half-twists, which we denote by $k(q_1, q_2, \cdots, q_m)$. We assume $q_i \neq 0$ for $i=1, 2, \cdots, m$. Fig. 5 shows $k(3, 3, -2, -1, -5)$.

Theorem. *Each* 3*-fold irregular branched covering space of a* 3*-sphere branched along a pretzel knot, if it exists, is isomorphic to a* 3*-sphere, a lens space of type* $(p, 1)$ *for some non-negative integer p, or a connected sum of those spaces.*



$k(3, 3, -2, -1, -5)$

Fig. 5

Proof. We note that a pretzel knot does not always have an irregular 3-fold cover.

First, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by h are distinct. In this case, we see each $q_i = \pm 1$ (*mod* 3) from Fig. 2. So, by operations cancelling three half-twists, this pretzel knot can be deformed to $k(1, 1, \cdots, 1)$ or $k(-1, -1, \cdots, -1)$. Moreover, on each operation cancelling three half-twists, the condition of Proposition 4 is satisfied. Here the number of "1"'s or "-1"'s in the above $k(1, 1, \cdots, 1)$ or $k(-1, -1, \cdots, -1)$ is a multiple of three, and we can obtain a trivial link of 2-components from this link by operations cancelling three half-twists such that the image of meridians of components by h are distinct. Since the 3-fold irregular branched covering space of a 3-sphere branched along a trivial link

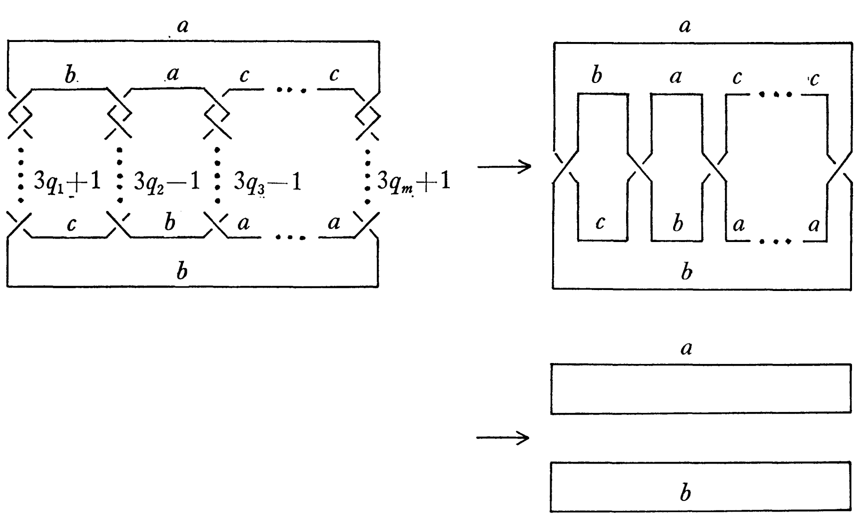


Fig. 6

of 2-components is a 3-sphere, those of the original pretzel knot is also a 3-sphere.

Secondly, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by h are same. In this case, the image of meridians of 2-strand braids by h must be (i) or (ii) shown as in Fig. 7.

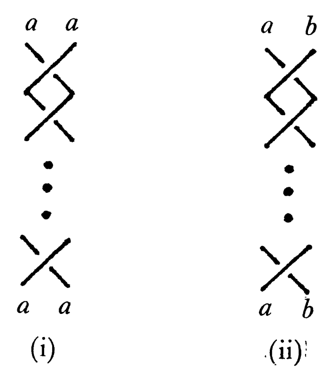


Fig. 7

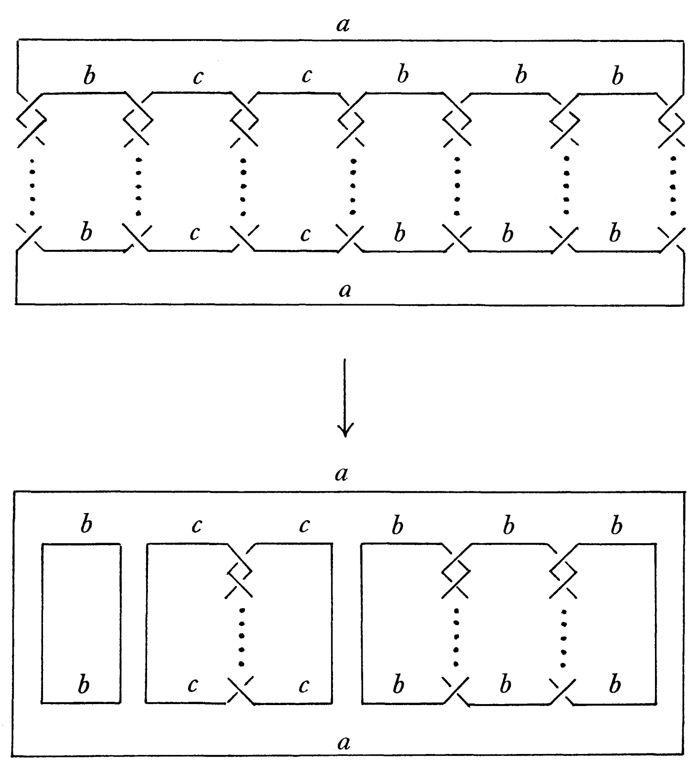


Fig. 8

Furthermore, for the case (i) the number of half-twists is arbitrary, but, for the case (ii) the number of half-twists is a multiple of three. For the case (ii), we can deform the 2-strand braid by operations cancelling three half-twists to the 2-strand braid with no half-twists and this deformation satisfies the condition of Proposition 4. But, for the case (i), we cannot do, since the condition of Proposition 4 is not satisfied. By operations cancelling three half-twists only for the case (ii), this pretzel knot can be deformed to a split link such that each component is a trivial knot, a $(p, 2)$-torus knot, or a connected sum of those knots and that every meridians of the same component are mapped by h to the same element a, b, or c in S_3.

Regarding $S^2 \times S^1$ as a lens space of type $(0, 1)$, the 2-fold branched covering space branched along a $(p, 2)$-torus knot is isomorphic to a lens space of type $(p, 1)$. Since the above link is a split sum of trivial knots, $(p, 2)$-torus knots, and their connected sum, the 3-fold irregular branched covering space of the above link corresponding to h is isomorphic to a 3-sphere, a lens space of type $(p, 1)$, or a connected sum of those spaces, and this covering space is isomorphic to the 3-fold irregular branched covering space branched along the original pretzel knot.

The proof is complete.

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