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ON CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN TIME

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0. Introduction

Let H be a real Hilbert space and ψ a lower semicontinuous convex proper function from H to $(-\infty, \infty]$. Here the terminology "proper" means that $\psi \not\equiv \infty$. The subdifferential of ψ is defined as follows: For $x \in H$, the value $\partial\psi x$ is the set of all $z \in H$ such that

$$\psi(y)-\psi(x) \geqq (z, y-x) \qquad \text{for every} \quad y \in H$$

where (,) stands for the inner product of H.

H. Brezis in [1] and [2] proposed the initial value problem of the form

$$(0.1) \qquad \begin{cases} \dfrac{d^2}{dt^2}u+\partial\psi u \ni f \\ u(0)=a\,, \quad \dfrac{d}{dt}u(0)=b\,. \end{cases}$$

In [1] he stated that in the particuler case where $\psi = I_K$ is the indicator function of a closed convex set K, the solution u represents, roughly speaking, the trajectory of an optical ray caught in K and reflecting at the boundary of K. Then $-\partial\psi u = -\partial I_K u$ may be regarded as the repulsive power at the boundary of K. In case H is finite dimensional, M. Schatzman made a deep investigation on this problem in [3] and [5] and established a general existence theorem as well as various results on the uniqueness and non-uniqueness of solutions. By a simple example in which ψ is the indicator function a closed convex set K she showed that the uniqueness of the solution does not hold in general and the solution which reflects optically on the boundary of K is an energy conserving solution. Moreover she obtained that even the energy conserving solutoin is not necessarily unique.

In case H is infinite dimensional, to the author's best knowledge, it seems to be extremely difficult to solve this porblem in a general situation. Hence as the first step of the study of this problem we are concerned with the case where the subdifferential operator $\partial\psi$ is expressed as

$$(0.2) \qquad \partial\psi = A + \partial I_K ,$$

where A is a positive definite self-adjoint operator and I_K is the indicator function of a closed convex set K with non empty interior. M. Schatzman showed in [2] the existence of local solutions of (0.1) in the case of (0.2) and for some specific initial data.

Clearly if f is continuous, the solution u of (0.1) in the csae (0.2) is twice continuously differentiable so long as $u(t)$ lies in the interior of K since $\partial I_K u(t)=0$ then. However, for some reason as was illustrated in M. Schatzman [5] in a finiet dimensinal case a reflection occurs if $u(t)$ reaches the boundary of K, and this causes discontinuity of $\frac{du}{dt}$. Thus we cannot expect the existence of a twice continuously differnetial solution. Hence, following M. Schatzman [5] we seek a function satisfying the equation with $\frac{d^2u}{dt^2}$ and $\partial I_K u$ considered as measures with values in H.

In Theorem 1 we will show the existence of the solutions of (0.1) in a slightly more general case than (0.2), namaly, the case of $\partial\psi=\partial\phi+\partial I_K$. Here ϕ is a lower semicontinuous proper convex function and coercive in a dense subspace V such that $V\subset H\subset V^*$, and K is a closed convex subset which is contained in a closed subspace L of finite codimension and has interior points in the relative topology of L. Assuming that the imbedding $V\rightarrow H$ is compact and $\partial\phi$ is single valued, continuous in some weak sense, we will show the existence of global solutions of the above problem satisfying the prescribed initial conditions. The solution is obtained as a limit of a subsequence of the solutions of the above problems with the Yosida approximations $\partial\phi_\lambda$, $\partial I_{K,\lambda}$ in place of $\partial\phi$, ∂I_K.

In the subsequent part of the paper it will be always assumed that K has interior points and the boundary of K is so smooth that there exists the outward unit normal vector satisfying a uniform Lipschitz condition in each bounded subset.

In Theorem 2 we will show the existence of an energy conserving solution of (0.1) in the case of (0.2). To prove this theorem we consodier the following sequence of functions

$$\rho_\lambda(t) = \int_0^t ||\partial I_{K,\lambda} u_\lambda(s)|| ds$$

where u_λ are the solutions of the above problems with ∂I_K replaced by its Yosida approximation $\partial I_{K,\lambda}$ and apply Helly's choice theorem to the above sequence of functions. This enables us to extract a subsequence $\{u_{\lambda_j}\}$ so that $u_{\lambda_j}\rightarrow u(t)$, $A^{1/2}u_{\lambda_j}\rightarrow A^{1/2}u(t)$, $\frac{d}{dt}u_{\lambda_j}(t)\rightarrow\frac{d}{dt}u(t)$ from which it readily follows that u satisfies the energy equality since as is easily seen u_λ are the energy conserving solutions of the approximate equations with $\partial I_{K,\lambda}$ in place of ∂I_K.

Since the energy conserving solution is not necessarily unique (see [5]), to obtain the uniqueness theorem, we are required to consider some specific class of energy conserving solutions. Hence we introduce a class of energy conserving solutions called herein "$\{t_i\}$-energy conserving solutions". Let $\{t_i\}$ be a dense and countable sequence in the interval $[0, T]$. Roughly speaking a $\{t_i\}$-energy conserving solution is an energy conserving solution such that the integral of the size of the replusive power from 0 to t_i is minimal for each i in the energy conserving solutions. It should be admitted that this class of solutions depends also on the order of the elements of the sequence $\{t_i\}$.

In Theorem 3 we will study a linear functional assoicated with the solution which plays an important role in the definition of $\{t_i\}$-energy conserving solution and establish a fairly concrete integral expression of the linear functional playing the part of the measure $\partial I_K u$.

In Theorem 4 we will show that the existence and uniqueness theorems of $\{t_i\}$-energy conserving solutions are established.

The outline of the present paper is as follows. In section 1 we list notations and properties of some operators. In section 2 we list definitions and state the assumptions and our main theorems. In section 3, 4, 5 and 6 we prove Theorem 1, 2, 3 and 4 respectively. Finally, section 7 contains some examples.

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1. Preliminaries

We first list some notations and known results which will be used throughout this paper. Let H be a real Hilbert space with inner product $(\ ,\ )$ and norm $\|\cdot\|$, and V a real reflexive Banach space such that V is a dense subspace of H and the inclusion mapping $V\rightarrow H$ is continuous. Identifying H with its dual space we may consider $V\subset H\subset V^*$. The paring between V and V^* is also denoted by $(\cdot, \cdot)$. The norms of V and V^* are denoted by $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$ respectively.

For a normed space X, $C([0, T]; X)$ (resp. $WC([0, T]; X)$) denotes the space of all strongly continuous (resp. weakly continuous) functions from $[0, T]$ to X. $C^j([0, T]; X)$ is the space of all functions from $[0, T]$ to X whose all derivatives up to order j all belong to $C([0, T]; X)$.

$L_q(0, T; X)$, $1\leqq q<\infty$, is the space of all measurable functions from $[0, T]$ to X such that $(\|u\|_X)^q$ is integrable on $[0, T]$, where $\|\cdot\|_X$ is the norm of X, and $L_\infty(0, T; X)$ is the space of all essentially bounded, measurable functions in $[0, T]$ with values in X. Similarly we denote by $W_q^m(0, T; X)$ the totality of mesurable functions from $[0, T]$ to X such that all derivatives in the sense of distributions up to order m belong to $L_q(0, T; X)$.

By dist(x, S) we denote the distance between a point x of H and a subset S of H. Let K be a closed convex subset of H. Then for any $x\in H$ there exists a unique point $P_K x$ of K satisfying $\|x-P_K x\|=\text{dist}(x, K)$. P_K is called the projection operator on K. If K is contained in a closed subspace L of H, then

$$P_L P_K = P_K P_L = P_K .$$

By $\mathring{K}$ and $bdy(K)$ we denote the interior and the boundary of K in H respectively. $\mathring{K}_L$ and $bdy_L(K)$ are the interior and the boundary of K in the relative topology of L respectively if $K\subset L$.

Let A be a positive definite self adjoint linear operator in H and $A^{1/2}$ the 1/2-fractional power of A. We here employ the complexification $\bar{H}$ of H such that

1) each $z\in\bar{H}$ is represented as
$z=\alpha+\sqrt{-1}\beta$ for some $\alpha, \beta\in H$, and

2) the inner product $((\ ,\ ))$ is defined by

$$((\alpha+\sqrt{-1}\beta, \theta+\sqrt{-1}\gamma)) = (\alpha, \theta)+(\beta, \gamma) + \sqrt{-1}\{(\beta, \theta)-(\alpha, \gamma)\} .$$

We then extend the operator A to an operator $\bar{A}$ in $\bar{H}$ by

$$\text{Domain } (\bar{A}) = \{\alpha+\sqrt{-1}\beta ; \alpha, \beta\in D(A)\} ,$$
$$\bar{A}(\alpha+\sqrt{-1}\beta) = A\alpha+\sqrt{-1}A\beta .$$

It is easy to see that the operator $\bar{A}$ is positive self adjoint in $\bar{H}$.

Let $\{U(t)\}$ be the (C_0)-group $\bar{H}$ on generated by $\sqrt{-1}(\bar{A})^{1/2}$. In the following we write

$$D=\sqrt{-1}(\bar{A})^{1/2}, \quad S(t)=2^{-1}\{U(t)-U(-t)\}D^{-1} ,$$
$$C(t)=2^{-1}\{U(t)+U(-t)\},$$

for simplicit simplicity in notations. In view of the first property 1) of $\bar{H}$, $C(t)x$ is represented as

$$C(t)x=\alpha(t)+\sqrt{-1}\beta(t) \qquad \alpha(t)\in H,\ \beta(t)\in H$$

for each $x\in H$, and it is easily seen from the definition of $\bar{A}$ that the function β is a solution of the initial-value problem

$$\begin{cases} \dfrac{d^2}{dt^2}\beta(t)+A\beta(t)=0 , \\ \beta(0)=0 , \quad \dfrac{d}{dt}\beta(0)=0 . \end{cases}$$

This implies $\beta(t)\equiv 0$ because of the uniqueness of the solution of the above problem, and hence $C(t)x\in H$. Similarly, $S(t)x\in H$ for any $x\in H$. We denote the norm of $\bar{H}$ by $|||\cdot|||$.

Let $\phi(\cdot)$ be a proper, convex and lower semicontinuous function from V to $(-\infty, \infty)$ and let $\partial\phi$ be its subdifferential operator defined by

$$\partial\phi x = \{f\in V^*;\ \phi(y)-\phi(x)\geqq(f, y-x) \qquad \text{for any} \quad y\in V\}.$$

Let $I_K(\cdot)$ be the indicator function of K defined by

$$I_K(x) = \begin{cases} 0 & \text{if} \quad x\in K \\ \infty & \text{if} \quad x\notin K . \end{cases}$$

The subdifferential operator ∂I_K of $I_K(\cdot)$ is defined by

$$D(\partial I_K) = \{x\in H;\ \text{there exists } z\in H \text{ such that } (y-x, z)\leqq I_K(y)-I_K(x) \qquad \text{for any} \quad y\in K\},$$

$$\partial I_K x = \{z\in H;\ (y-x, z)\leqq 0 \qquad \text{for any} \quad y\in K\}.$$

We put

$$I_{K,\lambda}(x) = (2\lambda)^{-1}||x-P_K x||^2 \qquad \text{forfor any} \quad x\in H$$

where λ is a positive number. We see that $I_{K,\lambda}(\cdot)$ is a convex, Frechet differentiable function on H and has a single valued subdifferential operator $\partial I_{K,\lambda}$ which is represented as

$$\partial I_{K,\lambda}x = \lambda^{-1}(x-P_K x) .$$

For $x\in bdy(K)$ the set $\partial I_K(x)$ is equal to the union of the set of all exterior normal vectors at the boundary point x and a 0-vector's set. In particular, if the boundary of K holds some smoothness, we know that there exists only one unit normal vector $n(x)$ at the boundary point x such that

$$\partial I_K x = \{\lambda n(x);\ \lambda\in[0, \infty)\}.$$

Let $\phi(\cdot)$ satisfy the coerciveness condition in V. Then if ϕ is the convex from H to $(-\infty, \infty]$ defined by

$$\Phi(x) = \begin{cases} \phi(x) & \text{if} \quad x\in V \\ \infty & \text{if} \quad x\in H-V=\{g\in H;\ g\notin V\} , \end{cases}$$

it follows that Φ is lower semicontinuous on H by the coerciveness condition and its subdifferential operator $\partial\Phi$ is defined with domain $D(\partial\Phi)=\{x\in V\colon \partial\phi x\in H\}$. Moreover $\partial\Phi(x)=\partial\phi(x)$ for any $x\in D(\partial\Phi)$. For every $\lambda>0$ a convex Frechet differentiabel function Φ_λ is defined by

$$\Phi_\lambda(x) = (2\lambda)^{-1}\|x-J_\lambda x\|^2+\Phi_\lambda(J_\lambda x) \qquad \text{for any} \quad x\in H$$

where $J_\lambda=(I+\lambda\partial\Phi)^{-1}$ and I is the identity operator on H. Let $\partial\Phi_\lambda$ be the Yosida approximation of $\partial\Phi$, namely,

$$\partial\Phi_\lambda(x) = \lambda^{-1}(x-J_\lambda x) \qquad \text{for any} \quad x\in H\,.$$

Then it is known that $\partial\Phi_\lambda$ is the subdifferential operator of Φ_λ and

$$\partial\Phi_\lambda(x) = \partial\Phi(J_\lambda x)\,.$$

2. Assumptions and main results

In this section we list definitions and state assumptions and theorems.

Let H, V, ϕ be the ones stated in the previous section. We assume that $\partial\phi$ is a single valued, everywhere defined and bounded operator from V to V^*, and that $\phi(\cdot)$ satisfies the following coerciveness condition

$$\lim_{\|x\|_V\to\infty} \phi(x)/\|x\|_V = \infty\,. \tag{2.1}$$

Next we suppose that $f(t, x)$ is a continuous function from $[0, T]\times H$ to H to H satisfying

$$\begin{cases} \|f(t, x)-f(t, y)\|\leqq h(t)\|x-y\| \\ \qquad\qquad\qquad \text{for any } x, y\in H \text{ and any } t\in[0, T] \\ \|\frac{d}{dt}f(t, x)\|\leqq h(t)(1+\|x\|) \qquad \text{for any} \quad x\in H \end{cases} \tag{2.2}$$

where $h(\cdot)$ is a positive integrable function of $t\in[0, T]$.

In this paper we consider the following type of equation

$$\begin{cases} \frac{d^2}{dt^2}u(t)+\partial\phi u(t)+\partial I_K u(t)\ni f(t, u(t))\,, \\ u(0) = a\,, \quad \frac{d}{dt}u(0) = b\,. \end{cases} \tag{2.3}$$

With regard to this type of problem we employ the notion of solution on $[0, T]$ defined as follows

Definition 2.1. We say that a function $u\in C([0, T]; H)$ is a solution of the problem (2.3) if the following conditions are satisfied:

1) $u\in W^1_\infty(0, T; H)\cap WC([0, T]; V)$.
2) For any $t\in[0, T]$, $u(t)$ belongs to $V\cap K$.
3) The right derivative $\frac{d^+}{dt}u(t)$ and the left derivative $\frac{d^-}{dt}u(t)$ exist on $[0, T]$

both in the weak topology of H and in the strong topology of V^* (with necessary modifications at 0 and T).

4) We have

$$2^{-1}\|\frac{d}{dt}u(t)\|^2+\phi(u(t))\leqq 2^{-1}\|b\|^2+\phi(a)+\int_0^t\left(\frac{d}{dt}u(s),\ f(s,\ u(s))\right)ds$$

for any $t\in[0,\ T]$) with necessary modifications at 0 and T).

5) There exists a linear continuous functional F on $C([0,\ T];\ H)$ such that

$$F(v-u)\leqq 0 \qquad \text{for any} \quad v\in C([0,\ T];\ K)$$

and for any $v\in W_1^1([0,\ T];\ H)\cap C([0,\ T];\ V)$

$$F(v)=\int_0^T\left(\frac{d}{ds}u(ds),\ \frac{d}{ds}v(s)\right)ds-$$
$$\int_0^T(\partial\phi u(s)-f(s,\ u(s)),\ v(s))ds+(b,\ v(0))-\left(\frac{d^-}{dt}u(T),\ v(T)\right).$$

6) The initial condition is satisfied in the following sense:

$$u(0)=a \quad \text{and} \quad b-\frac{d^+}{dt}u(0)\in\partial I_K a\,.$$

REMARK. Vaguely speaking, the functional F is a element of the set $\partial I_K u$ in the dual space of $C([0,\ T];\ H)$.

We state the assumption and the existence theorem for the solutions of (2.3) as mentioned above.

ASSUMPTION A–1.

1) There exists a closed linear subspace L of H such that the closed convex set K is contained in L and has interior points in L.

2) The orthogonal complement $L^\perp$ of L is of finite dimension and is contained in V.

3) For any sequence of functions $\{u_n\}$ in $W_\infty^1(0,\ T;\ H)\cap L_\infty(0,\ T;\ V)$ such that $\{u_n\}$ is bounded in $L_\infty(0,\ T;\ V)$ and converges to some u in the strong topology of $L_2(0,\ T;\ H)$ as $n\to\infty$, a subsequence $\{u_{n_j}\}$ can be extracted so that $\partial\phi_{n_j}\to\partial\phi u$ in the weak star topology of $L_\infty(0,\ T;\ V^*)$. In particular, the sequence $\{\partial\phi u_n(\cdot)\}$ is bounded in $L_\infty(0,\ T;\ V^*)$.

4) For any $\alpha\in L$ and any $u,\ v\in V$ such that $\|u\|_V\leqq R$ and $\|v\|_V\leqq R$, the following inequality holds:

$$|(\partial\phi u-\partial\phi v,\ \alpha)|\leqq C_1\|u-v\|$$

where C_1 is a constant depending only on α and R.

Theorem 1. *Assume that H is separable, and that the injection mapping of V into H is compact. Let the initial values a and b be given in $V\cap K$ and H, respectively. Then under the assumption A-1 there exists at least one solution of* (2.3) *on* $[0, T]$.

In what follows we consider the case in which $\partial\phi=A$ is a positive definite self adjoint linear operator in H. In this case $\phi(u)=2^{-1}||A^{1/2}u||^2$, and $V=D(A^{1/2})$ endowed with the graph norm of $A^{1/2}$. Then the problem (2.3) is written as

$$(2.4)\qquad \begin{cases} \dfrac{d^2}{dt^2}u(t)+Au(t)+\partial I_K u(t)\ni f(t, u(t)) \\ u(0)=a\,, \quad \dfrac{d}{dt}u(0)=b\,. \end{cases}$$

REMARK. **Theorem 1′.** *Replacing in Theorem* 1 *the assumption A–1 by conditions listed below and assuming $a\in D(A)$, we have the same conclusion as in Theorem* 1 *for the problem* (2.4):

1) *For the subspace L condition* 1) *of A–1 is satisfied.*

2) *The orthogonal complement $L^\perp$ is spaned by a infinite set $\{p_j\}_{j=0}^\infty$ of orthonormal eigenvectors of A.*

3) *The function h stated in* (2.2) *belongs to $L_\infty(0, T)$.*

We employ the following notion of the energy conserving solution of (2.4) (c.f Schatzman [5]).

DEFINITION 3.2. We say that u is an energy conserving solution of (2.4) if satisfies the following requirements:

1) u is a solution of (2.4) in the sense of Definition 1.1.

2) u belongs to $C([0, T]; V)$.

3) $\dfrac{d^+}{dt}u(t)$ and $\dfrac{d^-}{dt}u(t)$ are respectively right an left-continuous on $[0, T]$ in the strong topology of H (with necessary modifications at 0 and T).

4) We have

$$2^{-1}\,||\frac{d^\pm}{dt}u(t)||^2+2^{-1}(Au(t), u(t))=$$
$$2^{-1}||b||^2+2^{-1}(Aa, a)+\int_0^t\left(\frac{d}{dt}u(s), f(s, u(s))\right)ds$$

for any $t\in[0, T]$ (with necessary modifications at 0 and T).

We then state the assumption and the existence theorem for energy conserving solutions of (2.4).

ASSUMPTION A–2.

1) The closed convex set K has interior points.
2) For any $x \in bdy(K)$, $\partial I_K x$ forms a closed convex set

$$\{\lambda n(x);\ \lambda \geqq 0,\ n(x) \in \partial I_K x \quad \text{and} \quad ||n(x)|| = 1\}.$$

3) For any $x, y \in bdy(K)$ such that $||x|| \leqq R$ and $||y|| \leqq R$

$$||n(x) - n(y)|| \leqq N ||x - y||$$

where R is any positive number and N is a constant depending only on R.

Theorem 2. *Let $a \in V \cap K$ and $b \in H$. Under the assumption A–2 the problem* (2.4) *admits at least one energy conserving solution.*

We here give a representation theorem for the linear functionals F introduced in (5) of Definition 2.1.

Theorem 3. *Suppose that assumption A–2 holds. Let u be a solution of* (2.3) *and F be the associated linear functional as in* 5) *of Definition* 2.1. *Then functional F is represented as*

$$F(v) = \int_0^T (\bar{n}(s), v(s)) d\rho_u(s)$$

for $v \in C([0, T]; H)$, where

$$\bar{n}(u(t)) = \begin{cases} n(u(t)) & \text{if} \quad u(t) \in bdy(K), \\ 0 & \text{if} \quad u(t) \notin bdy(K), \end{cases}$$

and ρ_u is a left continuous increasing function on $[0, T]$ *such that* $\rho_u(0) = 0$ *and* $0 \leqq \rho_u(t) \leqq ||F||$ *for each* $t \in [0, T]$. *If $u(t)$ belongs to the interior $\mathring{K}$ of K, $d\rho_u = 0$ in some neighborhood at t.*
Moreover the function ρ_u is uniquely determined by u.

REMARK. Vaguely speaking, if $u(t) \in bdy(K)$, $-\bar{n}(u(t))$ is the direction of the repulsive power at the boundary point $u(t)$ and $\rho_u(t+0) - \rho_u(t)$ is its size.

In order to study the uniqueness of the energy conserving solution, we shall introduce a restricted class of solutions of (2.4) by using the increasing function ρ_u as mentioned in Theorem 3.

Let $\{t_i\}_{i=1}^{\infty}$ be a dense subsetof $[0, T]$, and define

$$M_0 = \{v \in C([0, T]; H);\ v \text{ is the energy conserving solution of (2.4) on } [0, T]\},$$

$$M_1 = \{v \in M_0;\ \min_{w \in M_0} \rho_w(t_1) = \rho_v(t_1)\},$$

$$
\begin{aligned}
M_2 &= \{v \in M_1;\ \operatorname*{Min}_{w \in M_1} \rho_w(t_2) = \rho_v(t_2)\}\ , \\
&\vdots \\
M_i &= \{v \in M_{i-1};\ \operatorname*{Min}_{w \in M_{i-1}} \rho_w(t_i) = \rho_v(t_i)\}\ , \\
&\vdots
\end{aligned}
$$

inductively. If M_j is empty for some j, we regard M_k as empty sets for all $k \geqq j$.

Definition 2.3. We call an element of $\bigcap_{i=0}^{\infty} M_i$ a $\{t_i\}$-energy conserving solution of (2.4).

Theorem 4. *Under assumption A–2 there exists a unique $\{t_i\}$-energy conserving solution of* (2.4) *for each pair of initial values $a \in V \cap K$ and $b \in H$.*

3. Existence of the solution

In this section we discuss the existence of the solutions of (2.3) and give the proof of Theorem 1.

Throughout this section we assume that all of the conditions listed in the assumption A–1 hold.

We begin by introducing for each $\lambda > 0$ the following equation:

$$
(3.1) \qquad \begin{cases} \dfrac{d^2}{dt^2} u_\lambda(t) + \partial\Phi_\lambda u_\lambda(t) + \partial I_{K,\lambda} u_\lambda(t) = f(t, u_\lambda(t)) \\ u_\lambda(0) = a \in V \cap K \quad \text{and} \quad \dfrac{d}{dt} u_\lambda(0) = b \in H. \end{cases}
$$

Lemma 3.1. *The equation* (3.1) *has a unique solution $u_\lambda \in C^2([0, T]; H)$.*

Proof. Since the operators $\partial\Phi_\lambda$, $\partial I_{K,\lambda}$ and $f(t, \cdot)$ are all Lipschitz continuous in H, this lemma is easily shown.

Lemma 3.2. *For any $t \in [0, T]$, the following inequality holds:*

$$
\begin{aligned}
&\|u_\lambda(t)\|^2 + \|\frac{d}{dt} u_\lambda(t)\|^2 + I_{K,\lambda}(u_\lambda(t)) + \Phi_\lambda(u_\lambda(t)) \\
&\leqq C(1 + \|a\|^2 + \|b\|^2 + \Phi_\lambda(a))\ ,
\end{aligned}
$$

where C is a constant depending only on h and T.

Proof. Taking the inner products of both sides of (3.1) with $\frac{d}{dt} u_\lambda(t)$ and integrating the resultant equality over $[0, t]$, we have

$$
\begin{aligned}
(3.2) \qquad & 2^{-1}\|\frac{d}{dt} u_\lambda(t)\|^2 + I_{K,\lambda} u_\lambda(t) + \Phi_\lambda(u_\lambda(t)) \\
&= 2^{-1}\|b\|^2 + \Phi_\lambda(a) + \int_0^t (f(s, u_\lambda(s)), \frac{d}{ds} u_\lambda(s)) ds\ .
\end{aligned}
$$

From (2.2) it follows

$$\int_0^t \left(f(s, u_\lambda(s)), \frac{d}{ds}u_\lambda(s)\right)ds$$
$$\leqq \int_0^t h(s)(1+||u_\lambda(s)||)\,||\frac{d}{ds}u_\lambda(s)||ds$$
$$\leqq \int_0^t h(s)\left(1+||u_\lambda(s)||^2+||\frac{d}{ds}u_\lambda(s)||^2\right)ds\,.$$

Since $I_{K,\lambda}u_\lambda(t)$ is nonnegative and $\Phi_\lambda(u_\lambda(t))\geqq -C_2||u_\lambda(t)||-C_3$ we have

$$||\frac{d}{dt}u_\lambda(t)||^2\leqq ||b||^2+2\Phi_\lambda(a)+||u_\lambda(t)||^2+C_2^2+C_3$$
$$+2\int_0^t h(s)\left(1+||u_\lambda(s)||^2+||\frac{d}{ds}u_\lambda(s)||^2\right)ds\,.$$

Hence noting

$$||u_\lambda(t)||^2\leqq 2\left(||a||^2+T\int_0^T ||\frac{d}{dt}u_\lambda(s)||^2 ds\right)$$

we get

$$||u_\lambda||^2+||\frac{d}{dt}u_\lambda(t)||^2\leqq \mathrm{Const}\,(||a||^2+||b||^2+\Phi_\lambda(a)+1$$
$$+\int_0^T (h(s)+1)\left(1+||u_\lambda(s)||^2+||\frac{d}{ds}u_\lambda(s)||^2\right)ds$$

for $t\in[0, T]$. Using Gronwall's lemma and the fact that h is integrable on $[0, T]$ we have

$$||u_\lambda(t)||^2+||\frac{d}{dt}u_\lambda(t)||^2\leqq \mathrm{Const}\,(||a||^2+||b||^2+\Phi_\lambda(a)+1)\,.$$

From the relation (3.2) and above estimates the assertion of the lemma is obtained.

Lemma 3.3. *Let x_0 belong to K_L and R be any positive number. Then for any $x\in B(x_0, R)$ we have*

$$(\partial I_{K,\lambda}x,\ x-x_0)\geqq \mathrm{Const}\,||P_L\partial I_{K,\lambda}x||\ ||x-x_0||\,,$$

where Const stands for a positive constant independent of x and λ, and $B(x_0, R)$ is the ball of radius R centered at x_0.

Proof. Put $\partial I_{k,\lambda}x=z$ and $P_L z=z_1$. If $z_1=0$, the conclusion is clear. Hence assume $z_1\neq 0$. Set

$$(3.3) \qquad z_0 = z_1 - (z_1, x-x_0)\|x-x_0\|^{-2}(x-x_0) .$$

Since $P_L P_K = P_K P_L = P_K$, P_L is a self adjoint operator and $(P_L x - P_K x, x_0 - P_K P_L x) \leqq 0$ it follows that

$$\begin{aligned}(z_1, x-x_0) &= \lambda^{-1}(P_L(x-P_K x), x-x_0) \\ &= \lambda^{-1}(P_L x - P_K x, P_L x - x_0) \\ &= \lambda^{-1}\{(P_L x - P_K x, P_L x - P_L P_K x) - (P_L x - P_K x, x_0 - P_K P_L x)\} \\ &\geqq \lambda^{-1}\|P_L x - P_L P_K x\|^2 \geqq 0 .\end{aligned}$$

Since

$$\begin{aligned}(P_{L^\perp} z, x-x_0) &= \lambda^{-1}(P_{L^\perp}(x-P_K x), x-x_0) \\ &= \lambda^{-1}(P_{L^\perp}(x-x_0), x-x_0) \geqq 0 .\end{aligned}$$

and $z = P_L z + P_{L^\perp} z$, we have

$$0 \leqq (z_1, x-x_0) \leqq (z, x-x_0) .$$

On the other hand, from (3.3) it follows that

$$\|z_0 - z_1\| = (z_1, x-x_0)\|x-x_0\|^{-1} .$$

We now assume the following relation and derive a contradiction:

$$(3.4) \qquad (z_1, x-x_0) \leqq \mathrm{dist}(x_0, bdy_L(K))(4R)^{-1}\|x-x_0\| \cdot \|z_1\| .$$

From the estimates mentioned above we have

$$\|z_0 - z_1\| < \mathrm{dist}(x_0, bdy_L(K))(4R)^{-1}\|z_1\| .$$

If $\mathrm{dist}(x_0, bdy_L(K)) > R$, $P_L x$ would belong to $\mathring{K}$, and so we would have $\partial I_{K,\lambda} x = z \in L^\perp$. This contradicts $z_1 = P_L z \neq 0$. Hence $\mathrm{dist}(x_0, bdy_L(K)) \leqq R$. From this we have

$$(3.5) \qquad \|z_0\| \geqq (1 - \mathrm{dist}(x_0, bdy_L(K))(4R)^{-1})\|z_1\| > 0 .$$

Put

$$w = x_0 + \mathrm{dist}(x_0, bdy_L(K)) z_0 \|z_0\|^{-1} .$$

From $P_L w \in K$ it follows $(P_L x - P_K x, P_L w - P_K P_L x) \leqq 0$. Then

$$\begin{aligned}(z_1, x-w) &= \lambda^{-1}(P_L x - P_L P_K x, P_L x - P_L w) \\ &\geqq \lambda^{-1}\{(P_L x - P_L P_K x, P_L x - P_L w) \\ &\quad + (P_L x - P_K P_L x, P_L w - P_K P_L x)\} \\ &= \lambda^{-1}\|P_L x - P_L P_K x\|^2 \geqq 0 .\end{aligned}$$

Hence, noting that $(z_1, z_0)||z_0||^{-1}=||z^0||$, we have

$$\begin{aligned}0&\leqq(z_1, x-w)\\&=(z_1, x-x_0)-\operatorname{dist}(x_0, bdy_L(K))(z_0, z_1)||z_0||^{-1}\\&=(z_1, x-x_0)-\operatorname{dist}(x_0, bdy_L(K))||z_0||\,.\end{aligned}$$

Combining (3.4), (3.5) and the above mentioned estimates yields

$$\begin{aligned}0&\leqq(z_1, x-w)\\&\leqq\operatorname{dist}(x_0, bdy_L(K))\{(4R)^{-1}||x-x_0||\cdot||z_1||-||z_0||\}\\&\leqq\operatorname{dist}(x_0, bdy_L(K))\{(4R)^{-1}||x-x_0||^{-1}-1\\&\quad+(4R)^{-1}\operatorname{dist}(x_0, bdy_L(K))\}||z_1||=I\,.\end{aligned}$$

Since $||x-x_0||\leqq R$ and $\operatorname{dist}(x_0, bdy_L(K)\leqq R$, we get

$$I\leqq\operatorname{dist}(x_0, bdy_L(K))||z_1||(4^{-1}-1+4^{-1})<0\,,$$

This is impossible, and we have

$$\begin{aligned}(\partial I_{K,\lambda}x, x-x_0)&\geqq(P_L\partial I_{K,\lambda}x, x-x_0)\\&\geqq\operatorname{dist}(x_0, bdy_L(K))(4R)^{-1}||x-x_0||\,||P_L\partial I_{K,\lambda}x||\,.\end{aligned}$$

Lemma 3.4. *If the initial value b belongs to L we have*

$$\overline{\lim_{\lambda\to0}}\int_0^T||\partial I_{K,\lambda}u_\lambda(s)||ds<\infty\,.$$

Proof. Let $\{p_1, p_2, \cdots, p_N\}$ be an orthonormal base of $L^\perp$. Set

$$\begin{aligned}y_\lambda^j(t)&=(u_\lambda(t)-a, p_j)=(u_\lambda(t)-P_Ku_\lambda(t), p_j)\\&=\lambda(\partial I_{K,\lambda}u_\lambda(t), p_j)\,.\end{aligned}$$

By (3.1) and the condition $b\in L$ we get

$$\begin{cases}\dfrac{d^2}{dt^2}y_\lambda^j(t)+\lambda^{-1}y_\lambda^j(t)=g_\lambda^j(t)\,,\\[2mm] y_\lambda^j(0)=0\,,\quad \dfrac{d}{dt}y_\lambda^j(0)=0\,,\end{cases}$$

where

$$g_\lambda^j(t)=(f(t, u_\lambda(t))-\partial\Phi_\lambda u_\lambda(t), p_j)\,.$$

Since $|\Phi_\lambda(u_\lambda(t))|$ is bounded on $[0, T]$ by Lemma 3.2, it follows that $|\Phi(J_\lambda u_\lambda(t))|$ is bounded on $[0, T]$. On the other hand, (2.1) implie that $||J_\lambda u_\lambda(t))||_V$ is bounded on $[0, T]$. Hence it follows from the assumption A-1, (2.2), Lemma 3.2 and the above-mentioned facts that

(3.6) $\quad |g_\lambda^j(t)-g_\lambda^j(s)| \leqq \text{Const}\,\{(1+h(s))|t-s|+\int_s^t h(\xi)d\xi\}.$

Let e belong to $D(\partial\Phi)$. Then

$$\begin{aligned}&|(\partial\Phi_\lambda a,\, p_j)|\\ &\leqq |(\partial\Phi_\lambda a-\partial\Phi_\lambda e,\, p_j)|+|(\partial\Phi_\lambda e,\, p_j)|\\ &=|(\partial\Phi J_\lambda a-\partial\Phi J_\lambda e,\, p_j)|+|(\partial\Phi_\lambda e,\, p_j)|\\ &\leqq \text{Const}(\|J_\lambda a-J_\lambda e\|+\|\partial\Phi e\|)\\ &\leqq \text{Const}(\|a-e\|+\|\partial\Phi e\|).\end{aligned}$$

Thus

$$\overline{\lim_{\lambda\to 0}}\,|g_\lambda^j(0)|<\infty.$$

Hence (3.6) and the above fact together imply that

(3.7) $\quad \overline{\lim_{\lambda\to 0}}\,|g_\lambda^j(t)|$ is uniformly bounded on $[0,\, T]$.

Since y_λ^j is explicitly represented as

$$y_\lambda^j(t)=\lambda^{-1}\int_0^t \sin(\lambda^{-1/2}(t-s))g_\lambda^j(s)ds$$

for $t>0$ and $\lambda>0$, combining (3.6) and (3.7) yields

$$|(\partial I_{K,\lambda}u_\lambda(t),\, p_j)|=\lambda^{-1}|y_\lambda^j(t)|\leqq \text{Const}$$

where the constant on the right side is independent of λ. Thus we have

(3.8) $\quad \|P_{L^\perp}\partial I_{K,\lambda}u_\lambda(t)\|\leqq \text{Const}.$

Next we see from Lemma 3.2

$$\|u_\lambda(t)-a\|\leqq T\cdot\text{Const} \qquad \text{for} \quad t\in[0,\, T].$$

Thus Lemma 3.3 implies that for $x_0\in K_L\cap V$ we have

(3.9) $\quad (\partial I_{K,\lambda}u_\lambda(t),\, u_\lambda(t)-x_0)$
$\qquad\qquad \geqq \text{Const}\,\|P_L\partial I_{K,\lambda}u_\lambda(t)\|\,\|u_\lambda(t)-x_0\|.$

Multiplying both sides of (3.1) by $u_\lambda(t)-x_0$, intergrating the resultant equality on $[0,\, T]$ and applying an intergration by parts we have

$$\begin{aligned}&\int_0^T(\partial I_{K,\lambda}u_\lambda(s),\, u_\lambda(s)-x_0)ds\\ &=(b,\, a-x_0)-\left(\frac{d}{dt}u_\lambda(T),\, u(T)-x_0\right)\\ &+\int_0^T\left\{\left\|\frac{d}{ds}u_\lambda(s)\right\|^2+(f(s,\, u_\lambda(s))-\partial\Phi_\lambda u_\lambda(s),\, u_\lambda(s)-x_0)\right\}ds.\\ &\equiv II.\end{aligned}$$

Applying the above-mentioned estimates, Lemma 3.2, (2.2) and the relation:

$$(\partial\Phi_\lambda u_\lambda,\ u_\lambda(s)-x_0)\geqq\Phi_\lambda(u_\lambda(s))-\Phi_\lambda(x_0)\,,$$

we see that II is bounded by a constant independent of λ. Therefore, using (3.9) and the fact that $||u_\lambda(t)-x_0||$ is larger than dist $(x_0,\ bdy_L(K))$ provided $P_L\partial I_{K,\lambda}u_\lambda(t)\neq 0$, we have

$$\int_0^T ||P_L\partial I_{K,\lambda}u_\lambda(s)||ds\leqq \text{Const}$$

where the constant is independent of λ. From this and (3.8) we obtain the desired assertion of the lemma.

Lemma 3.5. *The sequence $\{u_\lambda\}$ contains a subsequence $\{u_{\lambda_j}\}$ such that $\{u_{\lambda_j}\}$ converges uniformly to a continuous function u on $[0, T]$ with respect to the strong topology of H and the sequence $\left\{\frac{d}{dt}u_{\lambda_j}\right\}$ of the derivatives converges weakly to $\frac{d}{dt}u$ in $L_2(0, T; H)$.*

Proof. In view of the assumption of the theorem and Lemma 3.2 we see that $\{u_\lambda(t)\}$ is a precompact set in H for each t. Moreover the sequence $\{u_\lambda\}$ and $\left\{\frac{d}{dt}u_\lambda\right\}$ are uniformly bounded in H by Lemma 3.2. Hence there exists a subsequence $\{u_{\lambda_j}\}$ which converges to some element u in $C([0, T]; H)$. Since $\left\{\frac{d}{dt}u_\lambda(t)\right\}$ is uniformly bounded, it is clear that $\left\{\frac{d}{dt}u_{\lambda_j}\right\}$ converges weakly to $\frac{d}{dt}u$ in $L_2(0, T; H)$.

We denote the above-mentioend subsequence $\{u_{\lambda_i}\}$ by $\{u_\lambda\}$.

Lemma 3.6. *For each $t\in[0, T]$, $u(t)\in K\cap V$.*

Proof. By virture of Lemma 3.2 we have

$$I_{K,\lambda}(u_\lambda(t))+\Phi_\lambda(u_\lambda(t))\leqq \text{Const}\,.$$

From this and Lemma 3.5 deired assertion follows.

Lemma 3.7. *If the initial value b belongs to L, then the sequence $\{I_{K,\lambda}(u_\lambda)\}$ converges to zero in $L_1(0, T)$. Therefore there exists a subsequence $\{I_{K,\lambda_j}(u_{\lambda_j})\}$ such that*

$$\lim_{j\to\infty} I_{K,\lambda_j}(u_{\lambda_j}(t))=0 \qquad \textit{for a.e}\quad t\in[0, T]\,.$$

Proof. By the definition of $\partial I_{K,\lambda}$ we have

$$0\geqq I_{K,\lambda}(u_\lambda(t))+(\partial I_{K,\lambda}u_\lambda(t),\ u(t)-u_\lambda(t))\,.$$

On the other hand, Lemmas 3.4 and 3.5 together yield

$$\overline{\lim_{\lambda\to 0}} \int_0^T (\partial I_{K,\lambda} u_\lambda(t), u(t)-u_\lambda(t))dt = 0 .$$

Hence we have

$$\overline{\lim_{\lambda\to 0}} \int_0^T I_{K,\lambda}(u_\lambda(s))ds = 0 .$$

The remaining part of the assertion of the lemma is now obvious.

We denote the above-mentioned subsequence $\{u_{\lambda_j}\}$ by $\{u_\lambda\}$.

We next study the convergence of $\{\partial I_{K,\lambda} u_\lambda\}$.
For a while let b belong to L. We put

$$\tau_\lambda(t) = \int_0^t \partial I_{K,\lambda} u_\lambda(s)ds .$$

Lemma 3.8. *The sequence $\{\tau_\lambda(t)\}$ contains a subsequence $\{\tau_{\lambda_j}(t)\}$ which converges weakly to $\tau(t)$ for any $t\in[0, T]$, and the limit function τ is of bounded variation as a function from $[0, T]$ to H with $\tau(0)=0$.*

Proof. Let Q be a countable dense subset of H and set for each $x\in Q$ and $t\in[0, T]$,

$$\Xi_{\lambda,x}(t) = (\tau_\lambda(t), x) .$$

Then it follows from Lemma 3.4 that the total variation of $\Xi_{\lambda,x}$ on $[0, T]$ is uniformly bounded with respect to λ. Since a function of bounded variation is expressed as the difference of two nondecrasing functions, we can choose with the aid of Helly's choice theorem a subsequence $\{\Xi_{\lambda_j,x}\}$ which is convergent on $[0, T]$. Since Q is a countable set, we apply the usual diagonal procedure to extract a subsequence $\{\Xi_{\lambda_j,x}\}$ such that

$$\lim_{j\to\infty} \Xi_{\lambda_j,x}(t) = \Xi_x(t)$$

for $x\in Q$ and $t\in[0, T]$, and we see that $\Xi_x(\cdot)$ is a function of bounded variation in $[0, T]$. Moreover

$$|\Xi_x(t)-\Xi_y(t)| \leq \text{Const}\,||x-y|| \tag{3.10}$$

for $x, y\in Q$ and $t\in[0, T]$, where the constant on the right side is the constant independent of t. It then follows from (3.10) that for each $t\in[0, T]$, the mapping $x\to\Xi_x(t)$ can be extended to a continuous linear functional on H. Therefore the Riesz theorem asserts that for each $t\in[0, T]$ there exists an element $\tau(t)\in H$ such that

$$\Xi_x(t) = (\tau(t), x) \qquad \text{for} \quad x\in H .$$

Since the total variations on $[0, T]$ of $\tau_\lambda(\cdot)$ are uniformly bounded for λ, it immediately follows that $\tau(\cdot)$ is a function of bounded variation on $[0, T]$ and $\tau(0)=0$.

For simplicity in notation we denote the subsequence as mentioned above by $\{\tau_\lambda\}$.

We then put

$$F_{t,\lambda}(g) = \int_0^t (\partial I_{K,\lambda} u_\lambda(s), g(s))ds$$

for $t \in [0, T]$ and $g \in C([0, T]; H)$.

Lemma 3.9. *For any $g \in C([0, T]; H)$ the limit*

$$\lim_{\lambda\to 0} F_{t,\lambda}(g) = F_t(g)$$

exists and the limit functional F_t is a bounded linear functional on $C([0, T]; H)$.

Proof. By the relation $\partial I_{K,\lambda} u_\lambda(t) = \frac{d}{dt}\tau_\lambda(t)$ and the integration by parts we obtain

$$F_{t,\lambda}(g) = (\tau_\lambda(t), g(t)) - \int_0^t \left(\tau_\lambda(s), \frac{d}{ds} g(s)\right) ds$$

for $g \in W_1^1(0, T; H)$. Hence Lemma 3.8 implies that the limit $\lim_{\lambda\to 0} F_{t,\lambda}(g)$ exists and

$$\text{(3.11)} \qquad \begin{aligned}\lim_{\lambda\to 0} F_{t,\lambda}(g) &= (\tau(t), g(t)) - \int_0^t \left(\tau(s), \frac{d}{ds} g(s)\right) ds \\ &= F_t(g)\,.\end{aligned}$$

Since

$$|F_t(g)| = \lim_{\lambda\to 0} |F_{t,\lambda}(g)| \leqq \text{Const}\cdot \sup_{0\leqq s\leqq t} \|g(s)\|\,,$$

F_t is extended to a linear functional on $C([0, T]; H)$ and the limit $\lim_{\lambda\to 0} F_{t,\lambda}(g) = F_t(g)$ exists for any $g \in C[0, T]; H)$.

In what follows, we write $F(\cdot) = F_T(\cdot)$.

For a function $g \in C([0, T]; H)$ we introduce the scalar-valued integral

$$\text{(3.12)} \qquad \int_0^t (g(s), d\tau(s)) = \lim_{n\to\infty} \sum_{k=1}^{n-1} (g(t_k^n), \tau(t_{k+1}^n) - \tau(t_k^n))\,,$$

where $\{t_k^n\}$ is a sequence of partions of $[0, t]$ such that

$$0 = t_1^n < t_2^n < \cdots < t_n^n = t$$

and

$$\lim_{n\to\infty} \operatorname*{Max}_{k=1,2,3,\cdots,n-1} |t^n_{k+1}-t^n_k| = 0 .$$

It is easy to verify that the limit on the right side of (3.12) exists and does not depend on the choice of $\{t^n_k\}$.

Lemma 3.10. *We have*

$$F_t(g) = \int_0^t (g(s), d\tau(s))$$

for $t\in[\in 0, T]$ *and* $g\in C([0, T]; H)$.

Proof. Using Lemma 3.9, (3.12) and applying the integration by parts, we obtain the conclusion of the lemma.

Lemma 3.11. *The sequence* $\left\{\frac{d}{dt}u_\lambda(t)\right\}$ *converges weakly in* H *to* $\frac{d}{dt}u(t)$ *for a.e* $t\in[0, T]$. *Further, the sequence* $\{\partial\Phi_\lambda u\}$ *converges to* $\partial\Phi u$ *in the weak star topology of* $L_\infty(0, T; V^*)$.

Proof. In view of Lemma 3.5 we see that $\{J_\lambda u_\lambda\}$ converges pointwise to $u(t)$ with respect to the strong topology of H.
Since $\|J_\lambda u_\lambda(t)\|_V$ is uniformly bounded for λ and t by Lemma 3.2, we see with the aid of the assumption A–1 that there is a null sequence $\lambda_j\to 0$ for which

$$\partial\Phi_{\lambda_j}u_{\lambda_j} \to \partial\phi u$$

in the weak-star topology of $L_\infty(0, T; V^*)$.
Hereafter we denote this subsequence by $\{\partial\Phi_\lambda u_\lambda(t)\}$.
Multiplying both side of equation (3.1) by $\alpha\in V$, integrating the resultant relation over $[0, t]$, and using Lemma 3.5, (3.11), (2.2) and then the above-mentioned fact we infer that the limit $\lim_{\lambda\to 0}\left(\frac{d}{dt}u_\lambda(t), \alpha\right)$ exists for any $t\in(0, T)$.
On the other hand, $\left\{\frac{d}{dt}u_\lambda\right\}$ converges to $\frac{d}{dt}u$ in the weak topology of $L_2(0, T; H)$.
Thus, noting that V is dense in H, we conclude that $\left\{\frac{d}{dt}u_\lambda(t)\right\}$ converges in H to $\frac{d}{dt}u(t)$ for *a.e* $t\in[0, T]$.

We put

$$X_0 = \left\{t\in[0, T]; t \text{ is a Lebesgue point of } \frac{d}{dt}u \right.$$
$$\left. \text{and weak-}\lim_{\lambda\to 0} \frac{d}{dt}u_\lambda(t) = \frac{d}{dt}u(t)\right\} .$$

Lemma 3.12.

1) *The one-sided weak derivatives* $w\text{-}\frac{d^+}{dt}u(\cdot)$ *and* $w\text{-}\frac{d^-}{dt}u(\cdot)$ *exist everywhere in the intervals* $[0, T)$, $(0, T]$ *and are weakly right- and weakly left-continuous in H, respectively. Moreover* $w\text{-}\frac{d^+}{dt}u$ *and* $w\text{-}\frac{d^-}{dt}u$ *are respectively right- and left- continuous in the strong topology of* V^*. *(with necessary modifications at* 0 *and* T*).*

2) *Let* $\tau(\cdot)$ *be the weak limit of functions* $\tau_\lambda(\cdot)$ *as* $\lambda\to 0$. *Then*

$$\tau(t\pm 0)=b+w\text{-}\frac{d^\pm}{dt}u(t)-\int_0^t(\partial\phi u(s)-f(s, u(s)))\,ds$$

for any $t\in[0, T]$ *(with necessary modifications at* 0 *and* T*).*

Proof. Applying Lemmas 3.5, 3.9, and 3.11 and using the relation (3.1), we have

$$(3.13)\qquad (\tau(t), g(t))-\int_0^t\left(\tau(s), \frac{d}{ds}g(s)\right)ds$$
$$=\int_0^t(f(s, u(s)), g(s))ds+\int_0^t\left(\frac{d}{ds}u(s), \frac{d}{ds}g(s)\right)ds$$
$$-\left(\frac{d}{dt}u(t), g(t)\right)+(b, g(0))-\int_0^t(\partial\phi u(s), g(s))ds$$

for any $t\in X_0$ and any $g\in C([0, T]; V)\cap W_1^1([0, T]; H)$. Since the total variation of τ is finite, the limit

$$\lim_{s\to t, s<t}\tau(s)=\tau(t-0)$$

exists for any $t\in[0, T]$. By (2.2) and Lemma 3.2 and function $t\to\int_0^t(f(s, u(s)), \alpha)ds$ is continuous over $[0, T]$ for any element α of V. Since $\|\partial\phi u(s)\|_{V^*}$ are uniformly bounded on $[0, T]$ with respect to s, the function $t\to\int_0^t(\partial\phi u(s), \alpha)ds$ is continuous in $[0, T]$. Letting $\alpha\equiv g(t)$ in (3.13) we see that for any $t\in(0, T]$ the limite

$$\lim_{s\in X_0, s\to t, s<t}\left(\frac{d}{dt}u(s), \alpha\right)$$

exists. Therefore $w\text{-}\frac{d^-}{dt}u(t)$ exists for any $t\in(0, T]$. Noting that $\frac{d}{dt}u$ belongs to $L_\infty(0, T; H)$ and using the relation (3.12) with $g(t)\equiv\alpha$, we get

$$|(\int_0^t\partial\phi u(s)ds, \alpha)|\leqq\text{Const}\,\|\alpha\|$$

for $\alpha \in V$ and $t \in X_0$. Since V and X_0 are dense respectively in H and $(0, T]$, the integral $\int_0^t \partial\phi u(s)ds$ belongs to H for any $t \in [0, T]$. Therefore we have

$$\tau(t-0) = b - w\text{-}\frac{d^-}{dt}u(t) + \int_0^t (f(s, u(s)) - \partial\phi u(s))ds$$

for $t \in (0, T]$. Since the function $t \to (\int_0^t \partial\phi u(s)ds, \alpha)$ is continuous for $\alpha \in V$ and the function $t \to \int_0^t \partial\phi u(s)ds$ is bounded in H, $\int_0^t \partial\phi u(s)ds$ is weakly continuous in H. Since $\tau(t-0)$ is left-continuous in H, we see that $w\text{-}\frac{d^-}{dt}u$ is weakly left-continuous in H on $(0, T]$.

By the same argument as in the above, we conclude that $w\text{-}\frac{d^+}{dt}u$ is weakly right continuous in H on $[0, T]$ and the relation

$$\tau(t+0) = b - w\text{-}\frac{d^+}{dt}u(t) + \int_0^t (f(s, u(s)) - \partial\phi u(s))ds$$

holds for $t \in [0, T)$.

Moreover $\|\partial\phi u(s)\|_{V^*}$ is uniformly bounded, and so $w\text{-}\frac{d^+}{dt}u$ and $w\text{-}\frac{d^-}{dt}u$ are strongly right- and left- continuous in V^*, respectively.

Lemma 3.13. *Let F be the linear functional on $C([0, T]; H)$ stated in Definition 2.1. Then we have:*

1) $$(b, v(0)) - \left(\frac{d^-}{dt}u(T), v(T)\right) + \int_0^T \left(\frac{d}{ds}u(s), \frac{d}{ds}v(s)\right)ds - \int_0^T (\partial\phi u(s), v(s))ds + \int_0^T (f(s, u(s)), v(s))ds = F(v)$$

for any $v \in W_1^1(0, T; H) \cap C([0, T]; V)$.

2) $F(v-u) \leqq 0$ *for any* $v \in C([0, T]; K)$.

3) $$2^{-1}\left\|\frac{d^\pm}{dt}u(t)\right\|^2 + \phi(u(s)) \leqq 2^{-1}\|b\|^2 + \phi(a) + \int_0^T \left(f(s, u(s)), \frac{d}{ds}u(s)\right)ds \quad \textit{for any} \quad t \in [0, T]$$

(*with necessary modifications at* 0 *and* T).

Proof. Assertion 1) follows from (3.11), (3.13) and Lemma 3.12. Since $\{J_\lambda u_\lambda(t)\}$ converges to $u(t)$ and $\Phi(\cdot)$ is lower semicontinuous, we have

$$\varliminf_{\lambda \to 0} \Phi_\lambda(u_\lambda(u_\lambda t)) \geqq \Phi(u(t)) = \phi(u(t)).$$

Hence Assertion 3) is obtained by using (3.2), Lemma 3.11 and 1) of Lemma 3.12 and the lower semicontinuity of $\Phi(\cdot)$. Finally, Assertion 2) is obtained by applying Lemma 3.5 and 3.9 to the inequality

$$F_\lambda(v-u_\lambda)=\int_0^T(\partial I_{K,\lambda}u_\lambda(s),\ v(s)-u_\lambda(s))ds$$
$$\leqq\int_0^T I_{K,\lambda}(v(s))ds=0\,.$$

Lemma 3.14. *The function u satisfies the initial condition* 6) *stated in Definition* 2.1.

Proof. It is obvious that $u(0)=a$. Taking any $\alpha\in V$ and putting $g(\cdot)\equiv\alpha$ in (3.13), we get

$$(\tau(0+0),\ \alpha)=\left(b-w\text{-}\frac{d^+}{dt}u(0),\ \alpha\right).$$

Hence

$$\tau(0+0)=b-w\text{-}\frac{d}{dt}u(0)\,. \tag{3.14}$$

On the other hand, in virtue of Lemma 3.8, we have

$$(\tau(t),\ x-a)=\lim_{\lambda\to0}(\tau_\lambda(t),\ x-a)$$
$$=\lim_{\lambda\to0}\int_0^t(\partial I_{K,\lambda}u_\lambda(s),\ x-u_\lambda(s))ds+\lim_{\lambda\to0}\int_0^t(\partial I_{k,\lambda}u_\lambda(s),\ u_\lambda(s)-a)ds\,.$$

for $x\in K$ and $t\in[0,\ T]$. Hence, using the relation $(\partial I_{K,\lambda}u_\lambda(s),\ x-u_\lambda(s))\geqq0$ and Lemma 3.4, we get

$$(\tau(t),\ x-a)\leqq\text{Const}\sup_{0\leqq s\leqq t}\|u(s)-a\|\,.$$

From this it follows that

$$(\tau(0+0),\ x-a)\leqq0\qquad\text{for any}\quad x\in K\,.$$

Combining this with (3.14) we have

$$b-w\text{-}\frac{d^+}{dt}u(0)\in\partial I_K a\,.$$

We now give the proof of Theorem 1.

Let b be any element of H. We put $P_L b=b_0$. From Lemma 3.12, 3.13 and 3.14 we have a solution u_0 of (2.3) with the initial-value b replaced by b_0.

We denote by F_0 the linear functional associated with u_0. First we shall find a solution u of (2.3) and the associated function F.
We put $u(t)=u_0(t)$ and define $F(\cdot)$ as the linear functional $F_0(\cdot)+(b-b_0,\ \delta_0\cdot)$ where δ_0 is the Dirc measure. Then

$$\begin{aligned} F(v) &= F_0(v)+(b-b_0,\ v(0)) \\ &= (b,\ v(0))-\Big(\frac{d^-}{dt}u(T),\ v(T)\Big)+\int_0^T\Big(\frac{d}{ds}u(s),\ \frac{d}{ds}v(s)\Big)ds \\ &\quad +\int_0^T (f(s,\ u(s))-\partial\phi u(s),\ v(s))ds \end{aligned}$$

for any $v\in W_1^1(0,\ T;H)\cap C([0,\ T];\ V)$.
Since $b-b_0$ belongs to L, we have

$$F(v-u) = F_0(v-u)+(b-b_0,\ v(0)-u(0))\leqq 0$$

for any $v\in C([0,\ T];K)$. But $u=u_0$ and u_0 is the solution of (2.3); it is clear that the energy inequality of (2.3) holds for u.
Noting that $b-b_0\in\partial I_K a$, and that $\partial I_K a$ is a convex cone, we have

$$b-w\text{-}\frac{d^+}{dt}u(0) = b_0-w\text{-}\frac{d^+}{dt}u_0(0)+b_0-b\in\partial I_K a\,.$$

From the above-mentioned it is concluded that the function u is the solution of (2.3), and the proof of Theorem 1 is complete.

We next prove the Theorem 1′ stated in Remark.

Under the conditions of Remark we get

$$A_\lambda p_j = (1+\lambda\lambda_j)^{-1}\lambda_j p_j$$

where λ_i is the eigenvalue of A associated with p_j.
Let $y_\lambda^j(t)=(u_\lambda(t)-a,\ p_j)$ be the function as defined in the proof of Lemma 3.4. Then, by the method employed the proof of Lemma 3.4 and by the equation (3.4), we have

$$\begin{cases} \dfrac{d^2}{dt^2}y_\lambda^j+\{(\lambda^{-1}+\lambda_j(1+\lambda\lambda_j)^{-1}\}\,y_\lambda^j = (f(\cdot,\ u_\lambda(\cdot))-A_\lambda a,\ p_j) \\ y_\lambda^j(0) = 0\,, \quad \dfrac{d}{dt}y_\lambda^j(0) = 0\,. \end{cases}$$

Using a method similar to the proof of Lemma 3.4 we get

$$\begin{aligned} &|(1+2\lambda\lambda_j)\{\lambda(1+\lambda\lambda_j)\}^{-1}y_\lambda^j(t)| \\ &\quad\leqq |(f(0,\ a),\ p_j)|+|(f(t,\ u_\lambda(t)),\ p_j)|+|(A_\lambda a,\ p_j)| \\ &\quad +\int_0^t \Big|\Big(\frac{d}{ds}f(s,\ u_\lambda(s)),\ p_j\Big)\Big|ds\,. \end{aligned}$$

From this together with Bessel's inequality we obtain

$$4^{-1}\sum_{j=0}^{\infty}|\lambda^{-1}y_\lambda^j(t)|^2 \leqq \|f(0, a)\|^2 + \sup_{0\leqq t\leqq T}\|f(t, u_\lambda(t))\|^2$$
$$+\|A_\lambda a\|^2 + \int_0^t \|\frac{d}{ds}f(s, u_\lambda(s))\|^2 ds .$$

Thus condition (2.2) and Lemma 3.2 together imply

$$\|P_{L^\perp}\partial I_{K,\lambda}u_\lambda(t)\|^2 = \sum_{j=0}^{\infty}|\lambda^{-1}y_\lambda^j(t)|^2 \leqq \text{Const} .$$

Condition A–1, 3) is clearly satisfied in the present case and conditions A–1 2), 4) are needed in the proof of Lemma 3.4. Consequently, we can obtain the desired conclusion of Remark by following each step of the proof of Theorem 1.

4. Energy conserving solutions

In this section we discuss the existence of energy conserving solutions which belong to $W_\infty^1(0, T; H) \cap C([0, T]; V)$. Throughout this section we assume that all of the conditions listed in the assumption A–2 are satisfied. We begin by preparing some lemmas concerning the closed set $bdy(K)$.

Lemma 4.1. *Let R be any positive number. For any $x, y \in bdy(K) \cap B(0, R)$ there exists a positive constant N_R, depending only on R, such that*

$$0 \leqq (n(x), x-y) \leqq N_R\|x-y\|^2$$

and

$$|(n(x)+n(y), x-y)| \leqq N_R\|x-y\|^2 .$$

Proof. From the assumption A–2 it follows that the function $n(x)$ from $bdy(K) \cap B(0, R)$ to H is Lipschitz continuous. We denote the Lipschitz constant by N_R. From the convexity of K we see that for $x, y \in bdy(K) \cap B(0, R)$

$$(n(y), x-y) \leqq 0 \leqq (n(x), x-y) . \tag{4.1}$$

Thus

$$\begin{cases} (n(x), x-y) \leqq (n(x)-n(y), x-y) , \\ (n(y), x-y) \geqq (n(x)-n(y), y-x) . \end{cases} \tag{4.2}$$

The first part of the lemma is then proved by combining (4.1), (4.2) and the Lipschitz continuity of $n(x)$. Next, (4.1) yields

$$(n(y), x-y) \leqq (n(x)+n(y), x-y) \leqq (n(x), x-y) . \tag{4.3}$$

Thus the remaining part of the lemma is easily proved by the first part and (4.3).

In what follows we assume $N_R \geqq 1$ and set

$$K_0^R = \{x \in K \cap B(0, R);\ \mathrm{dist}(x, bdy(K)) < N_{R+1}^{-1}\}.$$

Lemma 4.2. *Let z be any point of K_0^R. Then there is one and only one point x belonging to $bdy(K)$ such that*

$$\mathrm{dist}(z, bdy(K)) = \|x-z\| \qquad \textit{and} \quad x-z \in \partial I_K x .$$

Proof. Pui $\alpha = \mathrm{dist}(z, bdy(K))$ and $x+n(x)=c(x)$ for $x_1 \in bdy(K)$. By the definition of K_0^R there exists an element $x \in bdy(K)$ such that $\|x_1-z\| < N_{R+1}^{-1}$. Let x_2 be the point of intersection of $bdy(K)$ and the segment connecting the point $c(x_1)$ and z. Inductively, we denote a sequence $\{x_n\}_{n=1}^{\infty}$ in such a way that x_{n+1} is the point of intersection of $bdy(K)$ and the segment connecting the point $c(x_n)$ and z for each n. Then we know the following inequalitis

$$\begin{aligned}&\|x_n-z\|+\|n(x_n)\| \geqq \|c(x_n)-z\| \\ &\quad = \|x_{n+1}-z\|+\|c(x_n)-x_{n+1}\| \geqq \|x_{n+1}-z\|+\|n(x_n)\| .\end{aligned}$$

Thus $\|x_n-z\| \geqq \|x_{n+1}-z\| \geqq \alpha$, and we have

$$\lim_{n\to\infty} \|x_n-z\| = \beta \geqq \alpha , \quad \lim_{n\to\infty} \|c(x_n)-z\| = \beta+1 .$$

Put $x_n - z = \beta n(x_n) + \varepsilon_n$. Then

$$(\beta+1)n(x_n)+\varepsilon_n = c(x_n)-z .$$

Since $\|\beta n(x_n)+\varepsilon_n\|$ and $\|(\beta+1)n(x_n)+\varepsilon_n\|$ tend respectively to β and $(\beta+1)$ as $n \to \infty$, we get

$$\lim_{n\to\infty} (n(x_n), \varepsilon_n) = 0 .$$

Thus

$$\lim_{n\to\infty} \varepsilon_n = 0 . \tag{4.4}$$

On the other hand

$$\|x_n-x_n\|^2 = \beta(n(x_n)-n(x_m), x_n-x_m)+(\varepsilon_n-\varepsilon_m, x_n-x_m)$$

for m, n sufficiently large. Since

$$\beta(n(x_n)-n(x_m), x_n-x_m) \leqq \beta N_{R+1}\|x_n-x_m\|^2 \quad \text{and} \quad \beta N_{R+1} < 1$$

we have

$$\|x_n-x_m\|^2 \leqq (1-\beta N_{R+1})^{-1}(\|\varepsilon_n\|+\|\varepsilon_m\|)\|x_n-x_m\| .$$

Hence we see (4.4) that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. We put

$$\lim_{n\to\infty} x_n = x_\infty .$$

Then $x_\infty - z = \beta n(x_\infty)$. We now show that $\beta = \alpha$. Assume to the contrary that $\alpha = \beta$. Then we can choose $\bar{x}_1$ such that

$$\bar{x}_1 \in bdy(K) \quad \text{and} \quad \|\bar{x}_1 - z\| < \beta .$$

Using the same method as in the above argument we find a boundary point $\bar{x}_\infty$ of K such that

$$\bar{x}_\infty - z = \bar{\beta} n(x_\infty) \quad \text{and} \quad \alpha \leqq \bar{\beta} < \beta < N_{R+1}^{-1} .$$

On the other hand, Since $(n(\bar{x}_\infty),\ x_\infty - \bar{x}_\infty) \leqq 0$, we have

$$\begin{aligned} (4.5) \qquad \|x_\infty - \bar{x}_\infty\|^2 &= \beta(n(x_\infty) - n(\bar{x}_\infty).\ x - \bar{x}_\infty) \\ &\quad + (\beta - \bar{\beta})(n(\bar{x}_\infty),\ x_\infty - \bar{x}_\infty) < \beta N_{R+1} \|x_\infty - \bar{x}_\infty\|^2 . \end{aligned}$$

But $\beta N_{R+1} < 1$, and so $x_\infty = \bar{x}_\infty$. Thus we must have $\beta = \bar{\beta}$, which is a contradiction. Thus $\beta_\infty = \text{dist}(x,\ bdy(K))$. Finally, we can prove the uniqueness of the point x_∞ by using the same method as in the derivation (4.5).

For any $0 < \delta < 1$ we define

$$K_\delta^R \equiv \{x \in B(0,\ R);\ \text{dist}(x,\ bdy(K)) < \delta N_{R+1}^{-1}\} .$$

Let z belong to K_δ^R and define

$$r(z) = \begin{cases} \text{the point } x \text{ as in Lemma 4.2} & \text{if } z \in K \\ P_K z & \text{if } z \notin K . \end{cases}$$

Lemma 4.3. *For $z_1,\ z_2 \in K_\delta^R$ we have*

$$\|r(z_1) - r(z_2)\| \leqq 2(1-\delta)^{-1} \|z_1 - z_2\| .$$

Proof. Let $z_1,\ z_2 \in K$.
Then $r(z_i) - z_i = \text{dist}(z_i,\ bdy(K)) n(r(z_i))$ for $i = 1,\ 2$. On the other hand

$$|\text{dist}(z_1,\ bdy(K) - \text{dist}(z_2,\ bdy(K))| \leqq \|z_1 - z_2\| ,$$

and so we have

$$\|r(z_1) - r(z_2)\| \leqq 2\|z_1 - z_2\| + \delta N_{R+1}^{-1} N_{R+1} \|r(z_1) - r(z_2)\| .$$

Hence

$$\|r(z_1) - r(z_2)\| \leqq 2(1-\delta)^{-1} \|z_1 - z_2\| .$$

Next let $z_1 \in K$ and $z_2 \in H - K$.
Then $r(z_2) - z_2 = -\text{dist}(z_2,\ bdy(K)) n(r(z_2))$ and $\text{dist}(z,\ bdy(K)) + \text{dist}(z,\ bdy(K))$

$\leqq \|z_1 - z_2\|$. Hence the application of the same method as above implies the desired estimate.

Finally, the assertion of the lemma is clear for the case in which both z_1 and z_2 belong to $H-K$.

We now cosider the following equation:

$$(4.6)\qquad \begin{cases} \dfrac{d^2}{dt^2}u_\lambda + Au_\lambda + \partial I_{K,\lambda}u_\lambda = f(\cdot, u_\lambda) \\ u_\lambda(0) = a \in V \cap K, \quad \dfrac{d}{dt}u_\lambda(0) = b \in H. \end{cases}$$

Lemma 4.4. *For any* $\lambda > 0$ *the initial value problem* (4.6) *has a unique solution* u_λ *such that*

$$u_\lambda \in C([0, T]; V) \cap C^1(0, T]; H) \cap C^2([0, T]; V^*).$$

Moreover we have for any $t \in [0, T]$,

$$(4.7)\qquad \|u_\lambda(t)\|^2 + \|\frac{d}{dt}u_\lambda(t)\|^2 + I_{K,\lambda}(u_\lambda(t)) + (Au_\lambda(t), u_\lambda(t)) \leqq \text{Const}\ \{1 + \|a\|^2 + \|b\|^2 + (Aa, a)\}\ ;$$

$$(4.8)\qquad \int_0^T \|\partial I_{K,\lambda}\, u_\lambda(s)\|\, ds \leqq \text{Const};$$

and

$$(4.9)\qquad \|\frac{d}{dt}u_\lambda(t)\|^2 + (Au_\lambda(t), u_\lambda(t)) + 2I_{K,\lambda}(u_\lambda(t)) = \|b\|^2 + (Aa, a) + 2\int_0^t (f(s, u_\lambda(s)), \frac{d}{ds}u_\lambda(s))\, ds.$$

Proof. Since $D(A)$ is dence in $D(A^{1/2})$ and since $D(A^{1/2})$ is dense in H, there exist sequences $\{a_j\}_{j=1}^\infty$ in $D(A)$ and $\{b_j\}_{j=1}^\infty$ in $D(A^{1/2})$ such that

$$\|A^{1/2}(a_j - a)\| + \|b_j - b\| \leqq j^{-2},$$
$$\|a_j - a\| \leqq j^{-2}.$$

Let u_0 be a solution of the initial value problem

$$\begin{cases} \dfrac{d^2}{dt^2}u + Au = 0 \\ u(0) = a_1, \quad \dfrac{d}{dt}u(0) = b_1. \end{cases}$$

We then define a sequence $\{u_j\}_{j=0}^{\infty}$ of "approximate" solutions in an inductive manner by

$$\text{(4.10)}\qquad \begin{cases} \dfrac{d^2}{dt^2}u_j+Au_j = E(\cdot, u_{j-1})\,, \\ u_j(0) = a_j\,,\ \ \dfrac{d}{dt}u_j(0) = b_j\,,\ \ j = 1, 2, 3, \cdots, \end{cases}$$

where $E(t, x)=f(t, x)-\partial I_{K,\lambda}\, x$.
By (2.2) and the Lipschitz continuity of $\partial I_{K,\lambda}$ we have

$$\text{(4.11)}\qquad \begin{cases} \|E(t, x)-E(t, y)\|\leqq h_\lambda(t)\|x-y\|\,, \\ \|\dfrac{d}{dt}E(t, x)\|\leqq h_\lambda(t)\,(\|x\|+1) \end{cases}$$

where $h_\lambda(t)=h(t)+\lambda^{-1}$. Using the well-known result for the linear hyperbolic equation repeatedly, we get solutions of (4.10) in such a way that

$$u_j\in W_\infty^1(0, T; V)\cap W_\infty^2(0, T; H)$$

for all nonnegative integers j. Now (4.10) implies the relations

$$\frac{d^2}{dt^2}(u_j-u_{j-1})+A(u_j-u_{j-1}) = E(\cdot, u_{j-1})-E(\cdot, u_{j-2})$$

for $j=1, 2, 3, \cdots$. Taking the inner product of $\dfrac{d}{dt}(u_j-u_{j-1})$ and both sides of the above equality and then intergating the resulting equation with respect to t, we have

$$\text{(4.12)}\qquad 2^{-1}\|\frac{d}{dt}(u_j(t)-u_{j-1}(t))\|^2+2^{-1}(A(u_j(t)-u_{j-1}(t)),\ u_j(t)-u_{j-1}(t))$$
$$= 2^{-1}\varepsilon_j'+\int_0^t(E(s, u_{j-1}(s))-E(s, u_{j-2}(s)),\ \frac{d}{ds}(u_j(s)-u_{j-1}(s)))\,ds$$

where $\quad \varepsilon_j' = \|b_j-b_{j-1}\|^2+\|A^{1/2}(a_j-a_{j-1})\|^2\,.$

From (4.11), (4.12) and the positivity of A it follows that

$$\|\frac{d}{dt}(u_j(t)-u_{j-1}(t))\|^2$$
$$\leqq\varepsilon_j'+2\int_0^t h_\lambda(s)\|u_{j-1}(s)-u_{j-2}(s)\|\ \|\frac{d}{ds}(u_j(s)-u_{j-1}(s))\|ds\,.$$

Hence Gronwall's inequality ([2; p. 157]) yields

$$||\frac{d}{dt}(u_j(t)-u_{j-1}(t))||\leqq(\varepsilon_j')^{1/2}+\int_0^t h_\lambda(s)||u_{j-1}(s)-u_{j-2}(s)||\,ds\,.$$

Combining this with the estimate

$$||u_{j-1}(s)-u_{j-2}(s)||\leqq||a_{j-1}-a_{j-2}||+\int_0^s||\frac{d}{d\xi}(u_{j-1}(\xi)-u_{j-2}(\xi))||d\xi\,,$$

we have

$$||\frac{d}{dt}(u_j(t)-u_{j-1}(t))||\leqq\varepsilon_j+C_\lambda'\int_0^t||\frac{d}{ds}(u_{j-1}(s)-u_{j-2}(s))||ds\,,$$

where $C_\lambda'=\int_0^T h_\lambda(s)\,ds$ and $\varepsilon_j=\varepsilon_j'^{1/2}+C_\lambda'||a_{j-1}-a_{j-2}||$.

Therefore we obtain

$$||\frac{d}{dt}(u_j(t)-u_{j-1}(t))||\leqq\sum_{i=0}^{j-3}\varepsilon_{j-i}(C_\lambda't)^i(i!)^{-1}+C_\lambda'M(C_\lambda't)^{j-3}((j-3)!)^{-1}\,,$$

where $M=\operatorname*{Max}_{0\leqq t\leqq T}||u_2(t)-u_1(t)||$.

Since $\varepsilon_{j-1}\leqq\text{Const}(j-i-2)^{-2}$ and $C_\lambda'\leqq\text{Const}$ independing of λ we see

$$\sum_{j=4}^{\infty}\sum_{i=0}^{j-3}\varepsilon_{j-1}(C_\lambda'\,t)^i\,(i!)^{-1}<\infty\,.$$

Then we obtain

$$\sum_{j=4}^{\infty}||\frac{d}{dt}(u_j(t)-u_{j-1}(t))||\leqq\text{Const}\,.$$

Thus we conclude that $\left\{\frac{d}{dt}u_j(t)\right\}$ is uniformly convergent on $[0, T]$. Moreover from the above result and $\lim_{j\to\infty}a_j=a$ it follows that $\{u_j(t)\}$ converges to some function $u_\lambda(t)$ on $[0, T]$ and the convergence is uniform for $t\in[0, T]$. It thus follows from (4.12) and the above result that

$$\lim_{j\to\infty}A^{1/2}u_j(t)=A^{1/2}u_\lambda(t)\qquad\text{uniformly on}\quad[0, T]\,.$$

Since $A^{1/2}u_\lambda$ is continuous and $V=D(A^{1/2})$, we infer that $Au_\lambda\in C([0, T]; V^*)$. Further, $f(\cdot, u_\lambda(\cdot))\in C([0, T]; H)$, and so $u_\lambda(\cdot)\in C^2([0, T]; V^*)$. Therefore u_λ is the solution of (4.6). Multiplying both sides of (4.10) by $\frac{d}{dt}u_j$ and integrating the resultant equality over $[0, t]$ we have

$$2^{-1}\|\frac{d}{dt}u_j(t)\|^2+2^{-1}(Au_j(t),\,u_j(t))+I_{K,\lambda}(u_j(t))$$
$$+\int_0^t(\partial I_{K,\lambda}\,u_{j-1}(s)-\partial I_{K,\lambda}\,u_j(s),\,\frac{d}{ds}u_j(s))ds=2^{-1}\|b_j\|^2$$
$$+2^{-1}(Aa_j,\,a_j)+I_{K,\lambda}(a_j)+\int_0^t(f(s,\,u_{j-1}(s)),\,\frac{d}{ds}u_j(s))\,ds\,.$$

Combining the above results, we obtain (4.9) and (4.7). Assertion (4.8) is verified in the same way as in Lemma 3.4.

We here employ the complexification $\bar{H}$ of H and the extension $\bar{A}$ in $\bar{H}$ of A as mentioned in Section 1. Let $\{C(t)\}$ and $\{S(t)\}$ be the cosine function generated by $D(=\sqrt{-1}\,A^{1/2})$ and the associated sine function, respectively. Recall that $C(t)x$ as wall as $S(t)x$ belong to H for $t\geqq 0$ and $x\in H$. Moreover, we have

$$\text{(4.13)}\qquad \begin{cases} |||2^{-1}\{U(t)\pm U(-t)\}|||\leqq\|x\|\,, \\ |||2^{-1}\{U(t)\pm U(-t)\}D^{-1}|||\leqq\|A^{-1/2}\|\,\|x\| \qquad \text{for} \quad x\in H\,. \end{cases}$$

Now let R' be the square root of the right side of (4.7) and put $R=R'T+\|a\|$. Then the solution $u_\lambda(t)$ of (4.6) takes its values in $B(0,R)$ for $\lambda>0$ and $0\leqq t\leqq T$.

Suppose for the moment that the initial value a belong to $bdy(K)$. For $\delta\in(0,\,1)$ and $\lambda>0$, set

$$T_{1,\lambda}=\operatorname*{Sup}_t\,\{t\leqq T\colon u_\lambda(s)\in K_\delta^R \text{ for any } 0\leqq s\leqq t\}\,.$$

Then the energy estimate (4.7) ensures that there is a positive number T_1 such that

$$T_{1,\lambda}\geqq T_1\geqq\delta N_{R+1}^{-1}\,R'^{-1} \qquad \text{for any} \quad \lambda>0\,.$$

We then consider the equation (4.7) on the interval $[0,\,T_1]$. First we recall that $\partial I_{K,\lambda}(u_\lambda(t))$ is represented as

$$\text{(4.14)}\qquad \partial I_{K,\lambda}\,u_\lambda(t)=l_\lambda(t)\,n(r_\lambda(t))\,,$$

where $r_\lambda(t)=r(u_\lambda(t))$ and $l_\lambda(t)=\|\partial I_{K,\lambda}\,u_\lambda(t)\|$.
Further, the solutions u_j of (4.10), which belong to $W_\infty^1(0,\,T;\,V)\cap W_\infty^2(0,\,T;\,H)$, are expressed as

$$u_j=C(t)a_j+S(t)b_j-\int_0^t S(t-s)\{\partial I_{K,\lambda}\,u_{j-i}(s)-f(s,\,u_{j-1}(s))\}\,ds\,.$$

Noting that

$$\partial I_{K,\lambda}\, u_{j-1}(s) = \lambda^{-1}\|u_{j-1}(s) - P_K\, u_{j-1}\| n(r(u_{j-1}(s)))$$

converges to $l_\lambda(s)\, n(r_\lambda(s))$ as $j\to\infty$, we have

$$u_\lambda(t) = a(t) + W(t, u_\lambda) - \int_0^t S(t-s)\, l_\lambda(s)\, n(r_\lambda(s))\, ds\,, \tag{4.15}$$

where $a(t) = C(t)a + S(t)b$ and

$$W(t, u_\lambda) = \int_0^t S(t-s) f(s, u_\lambda(s))\, ds\,.$$

Similarly, we obtain

$$\frac{d}{dt}\, u_\lambda(t) = \frac{d}{dt}\, a(t) + \frac{d}{dt}\, W(t, u_\lambda) - \int_0^t C(t-s)\, l_\lambda(s)\, n(r_\lambda(s))\, ds \tag{4.16}$$

by computing the derivatives of u_j and taking the limit as $j\to\infty$.

Lemma 4.5. *The sequence $\{u_\lambda\}_{\lambda>0}$ contains a subsequence $\{u_{\lambda_j}\}$ convergent uniformly in the strong topology of H to a continuous function $u(t)$ on* $[0, T_1]$.

Proof. Put

$$\rho_\lambda(t) = \int_0^t l_\lambda(s)\, ds \qquad \text{for} \quad t \geqq 0 \text{ and } \lambda > 0\,.$$

Then $\rho_\lambda(\cdot)$, $\lambda>0$, are uniformly bounded functions on $[0, T_1]$ by (4.8). Applying Helly's theorem, we find the subsequence $\{\rho_{\lambda_j}(\cdot)\}$ such that

$$\lim_{\lambda_j\to 0} \rho_{\lambda_j}(t) = \rho(t) \qquad \text{for} \quad t\in[0, T_1] - Q_0 \tag{4.17}$$

where $\rho(t)$ is a left-continuous, increasing function on $[0, T_1]$ and Q_0 is some countable set in $[0, T_1]$. Now in view of (4.15) we get

$$\begin{aligned} u_{\lambda_j}(t) - u_{\lambda_k}(t) &= \{W(t, u_{\lambda_j}) - W(t, u_{\lambda_k})\} \\ &\quad - \int_0^t S(t-s)\, \{n(r_{\lambda_j}(s)) - n(r_{\lambda_k}(s))\}\, l_{\lambda_j}(s)\, ds \\ &\quad - \int_0^t S(t-s)\, n(r_{\lambda_k}(s))\, \frac{d}{ds}(\rho_{\lambda_j}(s) - \rho_{\lambda_k}(s))\, ds \\ &= I_1 - I_2 - I_3\,. \end{aligned} \tag{4.18}$$

By 3) of the assumption A–2 and Lemma 4.3 we have

$$\|I_2\| \leqq \text{Const} \int_0^t l_{\lambda_j}(s) \|u_{\lambda_j}(s) - u_{\lambda_k}(s)\|\, ds\,. \tag{4.19}$$

Also we infer from (2.2) that

$$(4.20)\qquad \|I_1\| \leqq \text{Const} \int_0^t h(s)\|u_{\lambda_j}(s)-u_{\lambda_k}(s)\|\, ds\,.$$

Using $\rho_{\lambda_j}(0)=\rho_{\lambda_k}(0)=0$ and applying an integration by parts in I_3 we have

$$\begin{aligned} I_3 &= \int_0^t C(t-s)\, n(r_{\lambda_k}(s))\, (\rho_{\lambda_j}(s)-\rho_{\lambda_k}(s))\, ds \\ &\quad -\int_0^t S(t-s)\, \frac{d}{ds}\, n(r_{\lambda_k}(s))\, (\rho_{\lambda_j}(s)-\rho_{\lambda_k}(s))\, ds \\ &\equiv I_4+I_5\,. \end{aligned}$$

The first term I_4 is estimated as

$$\|I_4\| \leqq \int_0^t |\rho_{\lambda_j}(s)-\rho_{\lambda_k}(s)|\, ds\,,$$

and 3) of the assumption A–2, Lemma 4.3 and (4.7) together imply

$$\left\|\frac{d}{ds}\, n(r_{\lambda_k}(s))\right\| \leqq \text{Const}\,.$$

Therefore the norm of I_3 is bounded by

$$\omega_{j,k}(t) \equiv \text{Const} \int_0^t |\rho_{\lambda_j}(s)-\rho_{\lambda_k}(s)|\, ds\,.$$

Combining (4.18), (4.19), (4.20) and the above estimate gives

$$\|u_{\lambda_j}(t)-u_{\lambda_k}(t)\| \leqq \omega_{j,k}(t)+\text{Const} \int_0^t \{h(s)+l_{\lambda_j}(s)\}\, \|u_{\lambda_j}(s)-u_{\lambda_k}(s)\| ds\,.$$

Hence Grownall's lemma yields

$$\begin{aligned} (4.21)\qquad \|u_{\lambda_j}(t)-u_{\lambda_k}(t)\| &\leqq \omega_{j,k}(t) \\ &+\int_0^t \omega_{j,k}(s)\, \{h(s)+l_{\lambda_j}(s)\} \exp \int_s^t \{h(\xi)+l_{\lambda_j}(\xi)\}\, d\xi\, ds\,. \end{aligned}$$

We now show by use of (4.21) that $\{u_{\lambda_j}\}$ converges. First (4.17) impies that

$$(4.22)\qquad \lim_{j,k\to\infty} \omega_{j,k}(t) = 0 \qquad \text{uniformly on} \quad [0, T_1]\,.$$

Since $\int_0^{T_1} \{h(s)+l_{\lambda_j}(s)\}\, ds \leqq \text{Const}$, it follows from (4.21) and (4.22) that

$$\lim_{j\to\infty} u_{\lambda_j}(t) = u(t) \qquad \text{uniformly on} \quad [0, T_1]\,.$$

In what follows we write u_λ and λ for u_{λ_j} and λ_j, respectively.

Lemma 4.6. *We have*

$$u(t)\in K\cap V \qquad \textit{for} \quad t\in[0, T_1]$$

and

$$\lim_{\lambda\to 0} n(r_\lambda(t)) = n(r(t))$$

where $r(t)=r(u(t))$ *and the convergence is uniform on* $[0, T_1]$ *with respect to* t.

Proof. The assertion of the lemma follows immediately from (4.7), 3) of the assumption A–2, Lemmas 4.3 and 4.7.

Lemma 4.7. $\left\{\frac{d}{dt} u_\lambda(t)\right\}$ *converges strongly in* H *to* $\frac{d}{dt}u(t)$ *for a.e* $t\in[0, T_1]$.

Proof. In view of (4.16) we write

$$\begin{aligned}(4.23)\qquad \frac{d}{dt} u_\lambda(t) &= \frac{d}{dt} a(t)+\frac{d}{dt} W(t, u_\lambda)\\ &\quad -\int_0^t C(t-s)\, \{n(r_\lambda(s))-n(r(s))\}\, l_\lambda(s)\, ds-\int_0^t C(t-s)\, n(r(s))\, l_\lambda(s)\, ds\\ &\equiv \frac{d}{dt} a(t)+\frac{d}{dt} W(t, u_\lambda)+I_1+I_2\,.\end{aligned}$$

Then Lemma 4.6 yields

$$(4.24)\qquad \lim_{\lambda\to 0} ||I_1|| = 0$$

and Lemma 4.5 ensures that

$$(4.25)\qquad \lim_{\lambda\to 0} \frac{d}{dt} W(t, u_\lambda) = \int_0^t C(t-s) f(s, u(s))ds\,.$$

On the other hand, $V\equiv D(A^{1/2})$ is dense in H, and so there exist exists a sequence of functions $\{g_j\}$ in $C^1(([0, T_1]; H)\cap C([0, T_1]; V)$ such that

$$(4.26)\qquad \operatorname*{Sup}_{0\leqq t\leqq T_1} ||g_j(t)-n(r(t))||\leqq j^{-1}\,.$$

In order to estimate I_2 we write

$$\begin{aligned}I_2 &= -\int_0^t C(t-s)\, \{n(r(s))-g_j(s)\}\, l_\lambda(s)\, ds\\ &\quad -\int_0^t C(t-s)\, g_j(s) \frac{d}{ds} \rho_\lambda(s)\, sd = I_3+I_4\,.\end{aligned}$$

Then the first term I_3 is estimated as

$$(4.27)\qquad ||I_3||\leqq \text{Const}/j \qquad \text{for} \quad j = 1, 2, 3, \cdots\,.$$

The second term I_4 is transformed to the following from by integration by parts and $\rho_\lambda(0)=0$;

$$I_4 = -g_j(t)\,\rho_\lambda(t)+\int_0^t C(t-s)\,\frac{d}{ds}\,g_j(s)\,\rho_\lambda(s)\,ds$$
$$-\int_0^t S(t-s)\,D^2 g_j(s)\,\rho_\lambda(s)\,ds\,.$$

The application of (4.17) then implies

$$\lim_{\lambda\to 0} I_4 = -g_j(t)\,\rho(t)+\int_0^t C(t-s)\,\frac{d}{ds}\,g_j(s)\,\rho(s)\,ds$$
$$-\int_0^t S(t-s)\,D^2 g_j(s)\,\rho(s)\,ds$$
$$=-\int_0^t C(t-s)\,g_j(s)\,d\rho(s)\,.$$

Hence we infer from (4.26) that

$$(4.28)\qquad \|\int_0^t C(t-s)\,n(r(s))\,d\rho(s)+\lim_{\lambda\to 0} I_4\|\leqq \rho(t)/j$$

for any $j\geqq 1$.
Using (4.23), (4.24), (4.25), (4.27) and (4.28) and letting $j\to\infty$ we see that the $\lim_{\lambda\to\infty}\frac{d}{dt}\,u_\lambda(t)$ exists for any $t\in(0,\,T_1]-Q_0$ and the assertion of the lemma is now obtained by combining Lemma 4.5 and the above-mentioned estimates.

Lemma 4.8. *We have*

$$u(t) = a(t)+W(t,\,u)-\int_0^t S(t-s)\,n(r(s))\,d\rho(s)$$

for $t\in[0,\,T_1]$ *and*

$$\frac{d}{dt}\,u(t) = \frac{d}{dt}\,a(t)+\frac{d}{dt}\,W(t,\,u)-\int_0^t C(t-s)\,n(r(s))\,d\rho(s)$$

for a.e $t\in[0,\,T_1]$

Proof. The assertion of the lemma is readily shown by (4.15), (4.16), Lemma 4.5, Lemma 4.7 and together with the argument employed in the proof.

Lemma 4.9. *We have*

$$\lim_{\lambda\to 0} A^{1/2}\,u_\lambda(t) = A^{1/2}\,u(t) \qquad in \quad C([0,\,T_1];\,H)\,.$$

In particular, $A^{1/2}\,u$ *belongs to* $C([0,\,T_1];\,H)$.

Proof. By virtue of (4.15) we have

$$(4.29)\quad \begin{aligned} &A^{1/2} u_{\lambda_p}(t) - A^{1/2} u_{\lambda_q}(t) \\ &= \int_0^t S'(t-s)\, \{n(r_{\lambda_p}(s)) - n(r_{\lambda_q}(s))\}\, l_{\lambda_q}(s)\, ds \\ &+ \int_0^t S'(t-s)\, n(r_{\lambda_q}(s)) \frac{d}{ds} (\rho_{\gamma_p}(s) - \rho_{\gamma_q}(s))\, ds \\ &- \int_0^t S'(t-s)\, \{f(s, u_{\lambda_p}(s)) - f(s, u_{\lambda_q}(s))\}\, ds \\ &\equiv I_1 + I_2 + I_3 , \end{aligned}$$

where $S'(t) = 2^{-1} \sqrt{-1}\, \{U(t) - U(-t)\}$.

Using the same method as in the derivation of (4.19) and (4.20), we obtain

$$||I_1|| + ||I_3|| \leqq \text{Const} \int_0^t \{h(s) + l_{\lambda_p}(s)\}\, ||u_{\lambda_p}(s) - u_{\lambda_q}(s)|| ds .$$

Hence we infer from Lemma 4.5 that

$$(4.30)\quad \lim_{\lambda_p, \lambda_q \to 0} ||I_1|| + ||I_3|| = 0 ,$$

uniformly on $[0, T_1]$.

Next, we write

$$\begin{aligned} I_2 &= \int_0^t S'(t-s)\, \{n(r_{\lambda_q}(s)) - n(r(s))\}\, \{l_{\lambda_p}(s) - l_{\lambda_q}(s)\}\, ds \\ &+ \int_0^t S'(t-s)\, n(r(s)) \frac{d}{ds} \{\rho_{\lambda_p}(s) - \rho_{\lambda_q}(s)\}\, ds \\ &= I_4 + I_5 . \end{aligned}$$

As to the first term I_4 we see from Lemmas 3.4 and 4.6 that

$$(4.31)\quad \lim_{\lambda_p, \lambda_q \to 0} ||I_4|| = 0 \quad \text{uniformly in } [0, T_1] .$$

The second term I_5 is written as

$$\begin{aligned} I_5 &= \int_0^t S'(t-s)\, \{n(r(s)) - g_j(s)\} \frac{d}{ds} \{\rho_{\lambda_p}(s) - \rho_{\lambda_q}(s)\}\, ds \\ &+ \int_0^t S'(t-s)\, g_j(s) \frac{d}{ds} \{\rho_{\lambda_p}(s) - \rho_{\lambda_q}(s)\}\, ds = I_6 + I_7 \end{aligned}$$

where g_j is the function in $C^1([0, T_1]; H) \cap C([0, T_1]; V)$ satisfying (4.26).

For the term I_6 we have

$$(4.32)\quad \operatorname*{Sup}_{0 \leqq 0 \leqq T_1} ||I_6|| \leqq \text{Const}/j .$$

By integration by parts and (4.17) we have

$$\lim_{\lambda_p, \lambda_q \to 0} \|I_7\| = 0 \quad \text{uniformly on } [0, T_1] .$$

Combining this with (4.29), (4.30), (4.31) and (4.32) and letting $j \to \infty$, we obtain the assertion of the lemma.

Lemma 4.10. *The function* $t \to \int_0^t U(t-s)\, n(r(s))\, d\rho(s)$ *has both of the left and right limits on* $(0, T_1]$ *and* $[0, T_1)$. *Moreover this function is left continuous on* $(0, T_1]$.

Proof. We put

$$\begin{aligned} &\int_0^t U(t-s)\, n(r(s))\, d\rho(s) \\ &= \int_0^t U(t-s)\{n(r(s))-g_j(s)\}\, d\rho(s) + \int_0^t U(t-s)\, g_j(s)\, d\rho(s) \\ &= I_1 + I_2 . \end{aligned}$$

where g_j is a function satisfying (4.26). Since each is a contraction mappn mapping on $\bar{H}$, (4.26) yields

$$|||I_1||| \leqq \rho(t)/j .$$

By integration by parts we have

$$I_2 = g_j(t)\, \rho(t) + \int_0^t U(t-s)\, Dg_j(s)\, \rho(s)\, ds - \int_0^t U(t-s) \frac{d}{ds} g_j(s)\, \rho(s)\, ds .$$

Noting that ρ has both the left and right limits we infer that I_2 has both the left and right limits as well. Thus the function stated in the lemma possesses the left and right limits. Further, since ρ is left-continuous, we see that the function is left-continuous on $(0, T_1]$.

Lemma 4.11. *The one-sided derivatives* $\frac{d^-}{dt} u$ *and* $\frac{d^+}{dt} u$ *are left and right continuous on* $(0, T_1]$ *and* $[0, T_1)$, *respectively.*

Proof. The derivatives $\frac{d}{dt} a(\cdot)$ and $\frac{d}{dt} W(\cdot, u)$ are continuous, and so the assertion follows from Lemma 4.8 and 4.10.

Lemma 4.12. *The function u satisfies all conditions stated in Definition* 2.2 on $[0, T_1]$.

Proof. The proof is obtained by applying Lemma 3.7, 4.5, 4.6, 4.7, 4.9, 4.11 and (4.9).

In what follows we simply write $d\mathring{\rho}(t_0)=0$ when $d\rho(\cdot)=0$ in some neighborhood of t_0.

Lemma 4.13. *If $u(t_0)$ belongs to $\mathring{K}$, then $d\mathring{\rho}(t_0)=0$.*

Proof. Lemma 4.5 implies that there exists a positive constant δ such that

$$\inf_{t\in[t_0-\delta,t_0+\delta]} dist(u(t), bdy(K)) \geqq 2^{-1}\, dist(u(t_0), bdy(K)),$$

and that if δ is sufficiently small, then $u_\lambda(t)$ belong to $\mathring{K}$ for all $t\in[t_0-\delta, t_0+\delta]$. From the definitions of l_λ and ρ_λ in (4.14) and the proof of Lemma 4.5 we have $l_\lambda(t)=0$ and $\rho_\lambda(t)=\rho_\lambda(t_0)$ respectively for any $t\in[t_0-\delta, t_0+\delta]$. Letting λ tend to 0 implies that

$$\rho(t)=\rho(t_0) \qquad \text{for any} \quad t\in[t_0-\delta, t_0+\delta],$$

which means that $d\mathring{\rho}(t_0)=0$.

We here recall the definitions of the mapping $\bar{n}$ and numbers R' and R;

$$\bar{n}(u(t)) = \begin{cases} n(u(t)), & \text{if } u(t)\in bdy(K), \\ 0 & \text{if } u(t)\notin bdy(K), \end{cases} \tag{4.33}$$

$$R' = \{\text{the right side of } (4.7)\}^{1/2} \quad \text{and} \quad R = R'T+\|a\|.$$

Lemma 4.14. *We have the relations*

$$u(t) = a(t)+W(t, u)-\int_0^t S(t-s)\,\bar{n}(u(s))\,d\rho(s),$$

$$\frac{d^-}{dt}u(t) = \frac{d}{dt}a(t)+\frac{d}{dt}W(t,u)-\int_0^t C(t-s)\,\bar{n}(u(s))\,d\rho(s)$$

for any $t\in(0, T_1]$ where $T_1\leqq T$ and $T_1\geqq \mathrm{Min}\{\delta N_{R+1}^{-1} R'^{-1}, T\}$. Moreover we have the energy estimates

$$\|u(t)\|^2+\|\frac{d^\pm}{dt}u(t)\|^2+(Au(t), u(t))\leqq \mathrm{Const}\{1+\|a\|^2+\|b\|^2+(Aa, a)\}, \tag{4.34}$$

$$\|\frac{d^\pm}{dt}u(t)\|^2+(Au(t), u(t)) = \|b\|^2+(Aa, a)+2\int_0^t (f(s,u(s)), \frac{d}{ds}u(s))\,ds \tag{4.35}$$

for $t\in[0, T_1]$ (with necessary modifications at 0 and T_1).

Proof. The integral repesentations of $u(t)$ and $\frac{d}{dt}u(t)$ are readily obtained from Lemma 4.8, 4.13 and (4.33). The energy estimates (3.43) and (4.35) follows from (4.7), (4.9), Lemmas 4.5, 4.7, 4.9 and 4.11.

Lemma 4.15. *Let the initial values a and b be given respectively in* $V\cap\mathring{K}$ *and H. Then there exists a solution u of* (2.4) *on some interval* $[0, T_1']$ *such that*

$$T_1'\leqq T \quad \textit{and}$$

$u(t)\in K$ *for* $0\leqq t<T_1'$, *and* $u(T_1')\in bdy(K)$ *else* $T_1'=T$, *and such that u belongs to* $W_\infty^1(0, T_1'; H)\cap C([0, T]; V)$ *and conserves the energy. Moreover u and* $\frac{d^-}{dt}u$ *are represented as in Lemma* 4.14. *with* $\rho=0$.

Proof. From the well-known result for linear hyperbolic equations and (4.33) the proof is easily obtained.

DEFINITION 4.1. We say a function $u\in C([0, T]; V)$ is a mild solution of (2.4) on $[0, T]$ if the following conditions are satisfied;

1) For any $t\in[0, T]$, $u(t)$ belongs to K,
2) u satisfies the equality 4) stated in Definition 2.2,
3) u and $\frac{d^-}{dt}u$ are represented as in Lemma 4.14, where ρ is a left continuous and nondecreasing function on $[0, T]$, $\rho(0)=0$, and $d\mathring{\rho}(t)=0$ provided $u(t)\in\mathring{K}$.

Since a mild solution is specified by a function ρ as above, we denote a mild solution by (u, ρ), where ρ is a function as mentioned in 3) of Definition 4.1.

The next lemma is redaily obtained from Lemma 4.14 and 4.15.

Lemma 4.16. *Let the initial values a and b be given respectively in* $V\cap K$ *and H. Then there exists a mild solution u of* (2.4) *on some interval* $[0, T_1]$ *where*

$$T_1=\begin{cases} T_1 & \textit{in Lemma } 4.14 \quad \textit{if} \quad a\in bdy(K)\,, \\ T' & \textit{in Lemma } 4.15 \quad \textit{if} \quad a\in\mathring{K}\,. \end{cases}$$

Lemma 4.17. *Let* (u_1, ρ_1) *be a mild solution of* (2.4) *on* $[0, T_1]$ *satisfying* $u(0)=a$, $\frac{d}{dt}u(0)=b$, (u_2, ρ_2) *a mild solution of* (2.4) *on* $[0, T_2]$ *with* $f(s, u)$ *replaced by* $f_2(s, u)\equiv f(s+T_1, u)$, *and suppose that* u_2 *satisfies* $u_2(0)=u_1(T_1)$, $\frac{d}{dt}u_2(0)=\frac{d^-}{dt}u_1(T_1)$. *Set*

$$u_3(t)=\begin{cases} u_1(t) & \textit{if} \quad 0\leqq t\leqq T_1 \\ u_2(t-T_1) & \textit{if} \quad T_1\leqq t\leqq T_2 \end{cases}$$

and

$$\rho_3(t)=\begin{cases} \rho_1(t) & \textit{if} \quad 0\leqq t\leqq T_1 \\ \rho_2(t-T_1)+\rho_1(T_1) & \textit{if} \quad T_1<t\leqq T_1+T_2\,. \end{cases}$$

Then (u_3, ρ_3) *is also a mild solution of* (2.4) *on* $[0, T_1+T_2]$ *satisfying* $u_3(0)=a$, $\frac{d}{dt}u_3(0)=b$. *Moreover* u_3 *enjoys the energy equality* (4.35) *on* $[0, T_1+T_2]$. *Thus this solution* u_3 *satisfies the energy inequality* (4.34) *on* $[0, T_1+T_2]$.

Proof. By the definition of mild solution we have

$$u_2(t-T_1) = C(t-T_1)\,u_1(T_1)+S(t-T_1)\frac{d}{dt}u_1(T_1)$$
$$+\int_0^{t-T_1} S(t-T_1-s)\,f_2(s, u_2(s))\,ds-\int_0^{t-T_1} S(t-T_1-s)\,\bar{n}(u_2(s))\,d\rho_2(s)$$

for $T_1\leqq t\leqq T_1+T_2$.
Using the integral representations of u_1 and $\frac{d^-}{dt}u_1$ and the group property of $\{U(t)\}$, we get

$$u_2(t-T_1) = C(t)a+S(t)b+\int_0^{T_1} S(t-s)\,f(s, u_1(s))\,ds$$
$$-\int_0^{T_1} S(t-s)\,\bar{n}(u_1(s))\,d\rho_1(s)+\int_{T_1}^{t} S(t-s)\,f(s, u_2(s-T_1))\,ds$$
$$-\int_{T_1}^{t} S(t-s)\,\bar{n}(u_2(s-T_1))\,d\rho_2(s-T_1)\,.$$

Hence by the definitions of u_3 and ρ_3 we get

$$u_3(t) = a(t)+W(t, u_3)-\int_0^t S(t-s)\,\bar{n}(u_3(s))\,d\rho_3(s)$$

for $t\in[0, T_1+T_2]$. Similary, we get

$$\frac{d^-}{dt}u_3(t) = \frac{d}{dt}a(t)+\frac{d}{dt}W(t, u_3)-\int_0^t C(t-s)\,\bar{n}(u_3(s))\,d\rho_3(s)\,.$$

for $t\in[0, T_1+T_2]$. Since u_1 and u_2 satisfy the energy equality it is easy to show that the energy equality is valid for u_3. Using this energy equality and applying the same method as in Lemma 4.4, we have the inequality (3.44).

Lemma 4.18. *Let* $a\in V\cap K$ *and* $b\in H$. *Then there exists a mild solution* (u, ρ) *of* (2.4) *on* $[0, T]$ *satisfying* $u(0)=a$, $\frac{d}{dt}u(0)=b$.

Proof. First assume that $a\in bdy(K)$. We use the notation $(u, \rho, \alpha, \beta, g)$ to denote the mild solution of the problem

$$\begin{cases} \dfrac{d^2}{dt^2}u+Au+\partial I_K u\ni g(t, u)\,, \\ u(0) = \alpha,\ \dfrac{d}{dt}u(0) = \beta\,. \end{cases}$$

By Lemma 4.16 there exists a mild solution (u_1, ρ_1, a, b, f) on $[0, T_1]$, where

$$T_1 = \operatorname*{Sup}_t \{t \leqq T;\ \mathrm{dist}(u_1(s), bdy(K)) \leqq \delta N_{R+1}^{-1} \quad \text{and} \quad \|u_1(s)\| < R \quad \text{for} \quad 0 \leqq s < t\} .$$

If $T_1 = T$, then the proof is complete. Hence suppose that $T_1 \leqq T$. From the difinition of R' and (4.34) we have $u(t) \in B(R, 0)$ for $t \in [0, T_1]$. Then $\|u_1(s)\| < R$ for $0 \leqq s \leqq T_1$ and $\mathrm{dist}(u_1(T_1), bdy(K)) = \delta N_{R+1}^{-1}$ by the definition of T_1. Thus it follows from (4.34) that $T_1 \geqq \delta N_{R+1}^{-1} R'^{-1}$. Now Lemma 4.15 ensures that there exists a mild solution $(u_2, \rho_2, u_1(T_1), \frac{d}{dt} u_1(T_1), f(\cdot + T_1, \cdot))$ on $[0, T_2]$ where

$$T_2 = \mathrm{Min}\{T - T_1', T_1';\ T_1' \text{ in Lemma 4.15}\} .$$

Let u_3 and ρ_3 be the functions defined in Lemma 4.17. Then Lemma 4.17 implies that (u_3, ρ_3, a, b, f) gives a mild solution of (2.4) on $[0, T_1 + T_2]$. If $T_1 + T_2 = T$, then the proof is complete. Suppose then that $T_1 + T_2 < T$. From (4.35) and the definition of R it follows $\|u_3(s)\| < R$ for $0 \leqq s \leqq T_1 + T_2$. Since $dist(u_1(T_1), bdy(K)) = \delta N_{R+1}^{-1}$ we have $T_2 \geqq \delta N_{R+1}^{-1} R'^{-1}$. Lemma 4.16 again implies that there exists a mild solution

$$(u_4, \rho_4, u(T_1 + T_2), \frac{d^-}{dt} u(T_1 + T_2), f(\cdot + T_1 + T_2)) \text{ on } [0, T_3], \quad \text{where}$$

$$T_3 = \operatorname*{Sup}_t \{t \leqq T - (T_1 + T_2);\ dist(u_4(s), bdy(K)) < \delta N_{R+1} \quad \text{and} \quad \|u_4(s)\| < R \text{ for any } 0 \leqq s \leqq t\} .$$

We then put

$$u_5(t) = \begin{cases} u_3(t) & \text{if } 0 \leqq t \leqq T_1 + T_2 \\ u_4(t - T_1 + T_2) & \text{if } T_1 + T_2 \leqq t \leqq T_1 + T_2 + T_3 \end{cases}$$

and

$$\rho_5(t) = \begin{cases} \rho_3(t) & \text{if } 0 \leqq t \leqq T_1 + T_2 \\ \rho_4(t - T_1 - T_2) + \rho_3(T_1 + T_2) & \text{if } T_1 + T_2 < t \leqq T_1 + T_2 + T_3 \end{cases}$$

Then Lemma 4.17 states that (u_5, ρ_5, a, b, f) is a mild solution of (2.4) on $[0, T_1 + T_2 + T_3]$. If $T_1 + T_2 + T_3 = T$, then the proof is complete. Suppose then that $T_1 + T_2 + T_3 < T$. Then $T_3 \geqq \delta N_{R+1}^{-1} R'^{-1}$. Repeating this argument we get a sequence of mild solutions $(u_{2j-1}, \rho_{2j-1}, a, b, f)$ on $[0, T_1 + T_2 + \cdots + T_j]$, where $T_i \geqq \delta N_{R+1}^{-1} R'^{-1}$ for $1 \leqq i < j$.

Since each T_i is larger than $\delta N_{R+1}^{-1} R'^{-1}$ there must exist j_0 such that $T_1 + T_2 + \cdots + T_{j_0} = T$. In this case the assertion is proved.

Next let a belong to $\mathring{K}$. Using the similarly aboev method we can prove this lemma.

Lemma 4.19. *A mild solution (u, ρ) on $[0, T]$ is an energy conserving solution on $[0, T]$.*

Proof. Put

$$Y(t) = C(t)a + S(t)b + \int_0^t S(t-s) f(s, u(s))\, ds\,,$$

$$z(t) = \frac{d}{dt} Y(t) \qquad \text{for} \quad t \in [0, T]\,.$$

Then $\frac{d}{dt} z \in L_2(0, T; H)$, $Y \in W^1_\infty(0, T; V)$, and

$$\frac{d}{dt} z + AY = f(\cdot, u)\,. \tag{4.36}$$

Moreover, by Definition 4.1,

$$u(t) = Y(t) - \int_0^t S(t-s)\, \bar{n}(u(s))\, d\rho(s) \tag{4.37}$$

and

$$\frac{d^-}{dt} u(t) = z(t) - \int_0^t C(t-s)\, \bar{n}(u(s))\, d\rho(s)\,. \tag{4.38}$$

For any $v \in C^1([0, T]; H) \cap C([0, T]; V)$ we infer from (4.36) that

$$\begin{aligned} &\int_0^T \left(z(s), \frac{d}{ds} v(s)\right) ds - \int_0^T (AY(s) - f(s, u(s)), v(s))\, ds \\ &\quad = (z(T), v(T)) - (z(0), v(0))\,. \end{aligned} \tag{4.39}$$

By switching the order of integration and integration by parts, we have

$$\begin{aligned} &\int_0^T \left(\int_0^t C(t-s)\, \bar{n}(u(s))\, d\rho(s), \frac{d}{dt} v(t) \right) dt \\ &\quad = \int_0^T \left(\int_s^T C(t-s)\, \bar{n}(u(s)), \frac{d}{dt} v(t) dt \right) d\rho(s) \\ &\quad = \left(\int_0^T C(T-s)\, \bar{n}(u(s))\, d\rho(s), v(T) \right) \\ &\quad - \int_0^T (\bar{n}(u(s)), v(s))\, d\rho(s) - \int_0^T \int_s^T (D^2 S(t-s)\, \bar{n}(u(s)), v(t))\, dt d\rho(s)\,, \end{aligned} \tag{4.40}$$

where the parenthesis of the integrand of the last term stands for the paring between V^* and V. The relation $D^2 = -A$ and Fubini's theorem together yield

$$(4.41)\qquad \int_0^T\int_s^T (D^2S(t-s)\bar{n}(u(s)),\ v(t))dtd\rho(s)$$
$$= \int_0^T\left(\int_0^t -AS(t-s)\bar{n}(u(s))d\rho(s),\ v(t)\right)dt\,.$$

Combining (4.37) through (4.41) we have

$$(4.42)\qquad \int_0^T\left(\frac{d}{ds}u(s),\ \frac{d}{ds}v(s)\right)ds-\int_0^T (Au(s)-f(s,\ u(s)),\ v(s))ds$$
$$=\left(\frac{d^-}{dt}(\bar{n}(T),\ v(T)\right)-(b,\ v(0))+\int_0^T(\bar{n}(u(s)),\ v(s))d\rho(s)\,.$$

Thus, putting

$$F(v)=\int_0^T(\bar{n}(u(s)),\ v(s))d\rho(s)\qquad\text{for}\quad v\in C([0,\ T];\ H)$$

we infer that u is the energy conserving solution on $[0,\ T]$.

Proof of Theorem 2. The proof is easily obtained from Lemma 4.18 and 4.19.

5. The representation of the linear functional F

Throughout this section we assume all of the conditions listed in the assumption A–2. In what follows we put

$$R=\{\text{the right side of (4.7)}\}^{1/2}\cdot T+||a||\,.$$

In this section we give the proof of Theorem 3.

We first list some notations which will be used throughout this section.

Let ε_0 be a positive number such that $0<2\varepsilon_0<N_{R+1}^{-1}$. For simplicity suppose that $dist(a,\ bdy(K))<\varepsilon_0$. Let $\{s_i\}_{1=i}^N$ be an increasing sequence satisfying the following conditions:

1) $s_0=0,\ S_n=T,$

2) For $j=1,\ 2,\ \cdots,\ N-1,\ dist(u(s),\ bdy(K))'=2\varepsilon_0$ and $dist(u(s_j),\ bdy(K))<2\varepsilon_0$ for $s_{j-1}\leqq s_j<s_j$ if j is odd; and $dist(u(s),\ bdy(K))=\varepsilon_0$ and $dist(u(s),\ bdy(K))>\varepsilon_0$ for $s_{j-1}\leqq s<s_j$ if j is even.

We put $I_j=[s_i,\ s_{i+1}]$ for $i=0,\ 1,\ \cdots,\ N-1$ and define

$$(5.1)\qquad n_0'(t)=\begin{cases} n(r(t)) & \text{if}\quad t\in I_{2j}\,,\\ \sigma_j n(r(s_{2j+1}))+(1-\sigma_j)n(r(s_{2j+2})) & \text{if}\quad t\in I_{2j+1}\,,\end{cases}$$

where $\sigma_j=(t-s_{2j+1})(s_{2j+2}-s_{2j+1})^{-1}$ and $r(\cdot)$ is the mapping defined before Lemma 4.3.

Further we define

$$n'(t) = n_0'(t)\|n_0'(t)\|^{-1}$$

and

$$\chi_{\varepsilon,t}(s) = \begin{cases} 1 & \text{if } 0 \leqq s < t-\varepsilon \\ (t-s)\varepsilon^{-1} & \text{if } t-\varepsilon \leqq s < t \\ 0 & \text{if } t \leqq s . \end{cases}$$

Lemma 5.1. *For any $v \in C([0, T]; H)$ there exists*

$$\lim_{\varepsilon \to 0} F(\chi_{\varepsilon,t} v) .$$

Proof. From condition 5) stated in Definition 2.1 it follows that for any $v \in C^1([0, T]; H) \cap C([0, T]; V)$

$$\int_0^{t-\varepsilon} \left\{ \left(\frac{d}{ds} u(s), \frac{d}{ds} v(s) \right) + (f(s, u(s)) - \partial\phi u(s), v(s)) \right\} ds$$
$$+ \int_{t-\varepsilon}^t (t-s)\varepsilon^{-1} \left\{ \left(\frac{d}{ds} u(s), \frac{d}{ds} v(s) \right) + (f(s, u(s)) - \partial\phi u(s), v(s)) \right\} ds$$
$$- \varepsilon^{-1} \int_{t-\varepsilon}^t \left(\frac{d}{ds} u(s), v(s) \right) ds + (b, v(0)) = F(\chi_{\varepsilon,t} v) .$$

From condition 3) of Definition 2.1 we infer that

$$\lim_{\varepsilon \to 0} F(\chi_{\varepsilon,t} v) = -\left(\frac{d^-}{dt} u(t), v(t) \right) + (b, v(0)) \tag{5.2}$$
$$+ \int_0^t \left\{ \left(\frac{d}{ds} u(s), \frac{d}{ds} v(s) \right) + (f(s, u(s)) - \partial\phi u(s), v(s)) \right\} ds .$$

Now let v be any element of $C([0, T]; H)$. Then there exists a sequence $T_j(v) \in C^1([0, T]; H) \cap C([0, T]; V)$ such that

$$\operatorname*{Sup}_{0 \leqq t \leqq T} \|T_j(v)(t) - v(t)\| \leqq j^{-1} \qquad \text{for any } j = 1, 2, \cdots.$$

Since

$$|F(\chi_{\varepsilon_1,t} v) - F(\chi_{\varepsilon_2,t} v)| \leqq |F(\chi_{\varepsilon_1,t} v) - F(\chi_{\varepsilon_1,t} T_j(v))|$$
$$+ |F(\chi_{\varepsilon_1,t} T_j(v)) - F(\chi_{\varepsilon_2,t} T_j(v))| + |F(\chi_{\varepsilon_2,t} T_j(v)) - F(\chi_{\varepsilon_2,t} v)|$$

for small $0 < \varepsilon_1 < \varepsilon_2$ and

$$|F(\chi_{\varepsilon,t} v) - F(\chi_{\varepsilon,t} T_j(v))| \leqq \|F\|/j \qquad \text{for } \varepsilon > 0 ,$$

we get

$$\lim_{\varepsilon_1 \to 0,\, \varepsilon_2 \to 0} |F(\chi_{\varepsilon_1,t} v) - F(\chi_{\varepsilon_2,t} v)| \leqq 2\|F\|/j$$
$$+ \lim_{\varepsilon_1 \to 0,\, \varepsilon_2 \to 0} |F(\chi_{\varepsilon_1,t} T_j(v)) - F(\chi_{\varepsilon_2,t} T_j(v))| .$$

Therefore $\lim_{\varepsilon\to 0} F(\chi_{\varepsilon,t}v)$ exists by (5.2).

We then put $\rho(0)=0$ and

$$\rho(t) = \lim_{\varepsilon\to 0} F(\chi_{\varepsilon,t}n') \qquad \text{for any} \quad t\in(0, T]\,.$$

Lemma 5.2. *ρ is a left continuous nondecreasing function on* $[0, T]$. *Moreover if $u(t_0)\in\mathring{K}$ $d\mathring{\rho}(t_0)=0$.*

Proof. For any $0<t_1<t_2\leqq T$ we get

$$\begin{aligned}|\rho(t_2)-\rho(t_1)| \leqq &\lim_{\varepsilon\to 0} |F(\chi_{\varepsilon,t_2}n')-F(\chi_{\varepsilon,t_2}T_j(n'))| \\ &+\lim_{\varepsilon\to 0} |F(\chi_{\varepsilon,t_2}T_j(n'))-F(\chi_{\varepsilon,t_1}T_j(n'))| \\ &+\lim_{\varepsilon\to 0} |F(\chi_{\varepsilon,t_1}T_j(n'))-F(\chi_{\varepsilon,t_1}n')|\,.\end{aligned}$$

Hence condition 5) of Definition 2.1 yields

$$\begin{aligned}&|\rho(t_2)-\rho(t_1)| \leqq 2\|F\|/j \\ &+\int_{t_1}^{t_2}\left|\left(\frac{d}{ds}u(s), \frac{d}{ds}T_j(n')(s)\right)+(f(s, u(s))-\partial\phi u(s), T_j(n')(s))\right| ds \\ &+\left|\left(\frac{d^-}{dt}u(t_2)-\frac{d^-}{dt}u(t_1), T_j(n')(t_2)\right)\right| \\ &+\left\|\frac{d^-}{dt}u(t_1)\right\| \, \|T_j(n')(t_2)-T_j(n')(t_1)\|\,,\end{aligned}$$

and so condition 3) of Definition 2.1 implies that

$$\lim_{t_1\to t_2} |\rho(t_2)-\rho(t_1)| \leqq 2\|F\|/j\,.$$

This means that ρ is left continuous.
Next there exists a sufficiently small $\varepsilon_3>0$ and any $0\leqq t_1<t_2\leqq T$ $u(s)-\varepsilon_3(\chi_{\varepsilon,t_2}(s)-\chi_{\varepsilon,t_1}(s))n'(s)$ belongs to K for any $s\in(0, T)$ condition 5) of Definition 2.1 gives

$$F((\chi_{\varepsilon,t_2}-\chi_{\varepsilon,t_1})n')\geqq 0 \qquad \text{for} \quad 0<t_1<t_2\leqq T\,.$$

Letting $\varepsilon\to 0$, we see that ρ is nondecreasing over $[0, T]$. If $u(t_0)\in\mathring{K}$ for some $t_0>0$, there must exist $t_1, t_2\in[0, T]$ and $\varepsilon_0>0$ such that $t_1<t_0<t_2$ and $u(s)\pm\varepsilon_0(\chi_{\varepsilon,t_2}-\chi_{\varepsilon,t_1})(s)n'(s)\in K$ for $s\in[0, T]$. But $u(\cdot)\pm\varepsilon_0(\chi_{\varepsilon,t_2}-\chi_{\varepsilon,t_1})n'(\cdot)\in C([0, T]; K)$ by (5.1), and so condition of Definition 2.1 implies that

$$F((\chi_{\varepsilon,t_1}-\chi_{\varepsilon,t_2})n') = 0\,.$$

Thus

$$\rho(t_2)-\rho(t_1)=\lim_{\varepsilon\to 0}F((\chi_{\varepsilon,t_2}-\chi_{\varepsilon,t_1})n')=0\,.$$

This means that

(5.3) $\quad d\mathring{\rho}(t_0)=0 \qquad$ for any point $\quad t_0>0$ with $u(t_0)\in\mathring{K}$.

Proof of Theorem 3.

For a small $\varepsilon_3>0$ let $dist(u(t_0),\ bdy(K))>\varepsilon_3$. We assume that $r(t_0)-\varepsilon_3 n(r(t_0))+\sqrt{\varepsilon_3}(4N_{R+1})^{-1/2}e(t)$ is an exterior point of K, where $e(t_0)$ denotes a vector satisfying $(e(t_0),\ n(r(t_0)))=0$, $||e(t_0)||=1$. Since $r(t_0)-\varepsilon_3 n(r(t_0))$ is an interior point of K there exists a number $m_{\varepsilon_3}(t_0)$ such that

1) $r(t_0)-\varepsilon_3 n(r(t_0))+m_{\varepsilon_3}(t_0)e(t_0)\in bdy(K)$
2) $\sqrt{\varepsilon_3}(4N_{R+1})^{-1/2}\geqq m_{\varepsilon_3}>0$.

Since $r(t_0)\in B(0,\ R+1)$, Lemma 4.1 implies that

$$|(n(r(t_0)),\ \varepsilon_3 n(r(t_0))-m_{\varepsilon_3}(t_0)e(t_0))|\leqq N_{R+1}\{\varepsilon_3^2+m_{\varepsilon_3}(t_0)^2\}\,,$$

which gives

$$\varepsilon_3\leqq N_{R+1}\{\varepsilon_3^2+m_{\varepsilon_3}(t_0)^2\}\,.$$

If $N_{R+1}\varepsilon_3\leqq 1/2$, then we have

$$2^{-1}\varepsilon_3\leqq N_{R+1}m_{\varepsilon_3}(t_0)^2\leqq(\varepsilon_3 4N_{R+1})^{-1}N_{R+1}\leqq\varepsilon_3/4\,.$$

This is a contadiction. Hence it is concluded $\{r(t_0)-\varepsilon_3 n(r(t_0))+\sqrt{\varepsilon_3}(4N_{R+1})^{-1/2}\cdot e(t_0)\}$ belongs to K. Therefore there exists an $\varepsilon_4>0$ such that for any $\varepsilon\in(0,\ \varepsilon_4)$ and any $t\in[0,\ T]$,

$$\{u(t)-\varepsilon n'(t)\pm\sqrt{\varepsilon}(4N_{R+1})^{-1/2}e(t)\} \qquad \text{belongs to} \quad K\,,$$

where $e(\cdot)$ is a function in $C([0,\ T];\ H)$ with $(e(t),\ n'(t))=0$ and $||e(t)||=1$. Thus from condition 5) of Definition 2.1 it follows that

$$F(-\sqrt{\varepsilon}\,n'\pm(4N_{R+1})^{-1/2}e)\leqq 0\,.$$

Letting $\varepsilon\to 0$ we get $F(e)=0$.

Let $e'\in C([0,\ T];\ H)$ and let $(e'(t),\ n'(t))=0$. Then

(5.4) $\quad F(e')=||e'||F(e'/||e'||)=0\,.$

For any $v\in C([0,\ T];\ H)$ we write

$$v(t)=(v(t),\ n'(t))n'(t)+e'(t)\equiv\alpha_v(t)n'(t)+e'(t)\,.$$

Then (5.4) yields

(5.5) $\quad F(v)=F(\alpha_v n')+(F(e')=F(\alpha_v n')\,.$

Let $\{t_i^m\}_{i=0}^m$ be any sequence satisfying

1) $0 = t_0^m < t_1^m < t_2^m < \cdots < t_m^m = T$,
2) $|t_{j+1}^m - t_j^m| \leqq 2T/m$ for $j = 1, 2, 3, \cdots, m-1$.

Then $F(v)$ can be decomposed as

$$(5.6) \qquad F(v) = F(\chi_{\varepsilon, t_1^m} v) + \sum_{i=1}^{m-1} F((\chi_{\varepsilon, t_{i+1}^m} - \chi_{\varepsilon, t_i^m})v) + F((1-\chi_{\varepsilon, t_m^m})v)$$
$$\equiv I_1 + I_2 + I_3 .$$

First we consider I_2. Since

$$F((\chi_{\varepsilon, t_{i+1}^m} - \chi_{\varepsilon, t_i^m})v) = F((\chi_{\varepsilon, t_{i+1}^m} - \chi_{\varepsilon, t_i^m})\alpha_v n')$$
$$= \alpha_v(t_i^m)\{F(\chi_{\varepsilon, t_{i+1}^m} n') - F(\chi_{\varepsilon, t_{i=1}^m} n')\}$$
$$+ F((\chi_{\varepsilon, t_{i+1}^m} - \chi_{\varepsilon, t_i^m})(\alpha_v - \alpha_v(t_i^m))n'),$$

we have

$$|I_2 - \sum_{i=1}^{m-1} \alpha_v(t_i^m)\{F(\chi_{\varepsilon, t_{i+1}^m} n') - F(\chi_{\varepsilon, t_i^m} n')\}|$$
$$\leqq \omega_m \|F\| \operatorname{Sup}_t \sum_{i=1}^{m-1} (\chi_{\varepsilon, t_{i=1}^m} - \chi_{\varepsilon, t_i^m})(t) \leqq 2\omega_m \|F\| ,$$

where $\omega_m = \operatorname{Sup}_{|t-s| \leqq 2T/m} |\alpha_v(t) - \alpha_v(s)|$. Thus it follows

$$(5.7) \qquad \overline{\lim_{\varepsilon \to 0}} |I_2 - \sum_{i=1}^{m-1} \alpha_v(t_i^m)(\rho(t_{i+1}^m) - \rho(t_i^m))| \leqq 2\omega_m \|F\| .$$

Next, (5.5) implies that

$$(5.8) \qquad \overline{\lim_{\varepsilon \to 0}} |I_1 - \alpha_v(0)\rho(t_1^m)| \leqq \omega_m \|F\| .$$

Finally, using condition 5) of Definition 2.1 and noting that $\frac{d^-}{dt} u(\cdot)$ is left continuous we have for $j=1, 2, \cdots$

$$\lim_{m \to \infty} \lim_{\varepsilon \to 0} (F(1-\chi_{\varepsilon, t_m^m})T_j(v)) = 0 .$$

Combining $\|T_j(v) - v\| \leqq 1/j$ and the above we see

$$(5.9) \qquad \lim_{m \to \infty} \lim_{\varepsilon \to 0} F((1-\chi_{\varepsilon, t_m^m})v) = 0 .$$

Noting $\lim_{m \to \infty} \omega_m = 0$, Combining (5.6), (5.7), (5.8) and (5.9) and then letting m go to ∞, we get the desired integarl representation of the functional F:

$$F(v) = \int_0^T (v(s), n'(s)) d\rho(s) .$$

This, togher with (5.3), implies that for any $v \in C([0, T]; H)$,

$$F(v) = \int_0^T (v(s), \bar{n}(u(s)))d\rho(s)\,. \tag{5.10}$$

In aprticular, for any $\sigma \in C([0, T])$, we obtain an integral representation of the type

$$F(\sigma n') = \int_0^T \sigma(s)d\rho(s)\,.$$

On the other hand if θ is left continuous, non decreasing, of bounded variation, $\theta(0)=0$, and

$$\int_0^T \sigma(s)d\theta(s) = 0 \qquad \text{for any} \quad \sigma \in C([0, T])\,,$$

then it follows that $\theta(t)=0$ for any $t \in [0, T]$. This means that the function ρ is uniquely determined by the solution u.

In view of this, we denote by ρ_u the function ρ associated with u in the following.

6. $\{t_i\}$-energy conserving solutions

In this section we discuss the relation of the energy conserving solution to the mild solution and study the existence and uniqueness of $\{t_i\}$-energy conserving solutions. Throughout this section we assume all of the conditions listed in the assumption A–2.

Lemma 6.1. *An energy conserving solution of* (2.4) *is a mild solution of* (2.4). *More precisely, if u is an energy conserving solution, (u, ρ_u) is a mild solution.*

Proof. Let u be an energy conserving solution of (2.4) and set

$$\bar{Y}(t) = Y(t) - \int_0^t S(t-s)\bar{n}(u(s))d\rho_u(s)\,,$$

where $Y(\cdot)$ is the function defined in the proof of Lemma 4.19 and ρ_u the function provided by Therem 3. Using (4.36) and applying the same method as in the verifications of (4.40) and (4.41), we have

$$\int_0^T \left\{\left(\frac{d}{ds}\bar{Y}(s), \frac{d}{ds}v(s)\right) + (f(s, u(s)) - A\bar{Y}(s), v(s))\right\}ds$$
$$+(b, v(0)) - \left(\frac{d^-}{dt}\bar{Y}(T), v(T)\right) = \int_0^T (\bar{n}(u(s)), v(s))d\rho_u(s)$$

for any $v\in C^1([0, T]; H)\cap C[([0, T]; V)$. Put $\bar{Y}-u=w$. The above relation and (4.42) together yield

$$\int_0^T\left\{\left(\frac{d}{ds}w(s), \frac{d}{ds}c(s)\right)-(Aw(s), v(s))\right\}ds-\left(\frac{d}{dt}w(T), v(T)\right)=0. \tag{6.1}$$

For each $g\in C([0, T]; H)$ we denote by v the solution of the problem

$$\begin{cases} \dfrac{d^2}{dt^2}v+Av=g, & 0<t<T, \\ v(T)=0, \quad \dfrac{d}{dt}v(T)=0. \end{cases} \tag{6.2}$$

From (6.1) and (6.2) it follows that

$$\int_0^T\left(w(s), \frac{d^2}{dt^2}v(s)+Av(s)\right)ds=0$$

and

$$\int_0^T(w(s), g(s))ds=0.$$

Since $C^1([0, T]0; H))$ is dense in $L_2(0, T; H)$, we infer that $w(s)=0$ for $a\cdot e$ $s\in[0, T]$. Since $\bar{Y}$ and u are continuous, the proof is complete.

We are now in a position to give the proof of Theorem 4.

Let M_j, $j=0, 1, \cdots$, be the sets as mentioned in Definition 2.3. For each energy conserving solution u let ρ_u be the associated function provided by Theorem 3.

Lemma 6.2. *All of M_j, $j=1, 2, \cdots$, are not empty.*

Proof. We put

$$\inf_{w\in M_0}\rho_w(t_1)=\alpha_1. \tag{6.3}$$

Then one can choose a sequence $\{u_j\}$ of energy conserving solutions such that

$$\lim_{j\to\infty}\rho_{u_j}(t_1)=\alpha_1.$$

The application of Helly's theorem to $\{\rho_{u_j}\}$ implis that there exists a convergent subsequence $\{\rho_{u_{j_i}}\}$ such that

$$\lim_{j_i\to\infty}\rho_{u_{j_i}}(t)=\rho_1(t) \qquad \text{for} \quad t\in[0, T]-Q_1$$

where ρ_1 is an increasing left continuous function and $\rho_1(0)=0$ and Q_1 is some countable set in $[0, T]$. Let Q_i be some countable sets in $[0, T]$ for $i=1, 2, \cdots$.

For simplicity in notation we denote the subsequence $\{u_{j_i}\}$ by $\{u_j\}$. Applying lemma 6.1 to u_j and using the same method as in Section 4, we infer that there exists a subsequence $\{u_{j_i}\}$ such that

$$\lim_{j_i\to\infty} u_{j_i}(t) = u(t) \qquad \text{uniformly on} \quad [0, T],$$

$$\lim_{j_i\to\infty} \frac{d}{dt}u_{j_i}(t) = \frac{d}{dt}u(t) \qquad \text{for} \quad t\in[0, T]-Q_1,$$

$$\lim_{j_i\to\infty} A^{1/2}u_{u_{j_i}}(t) = A^{1/2}u(t) \qquad \text{for any} \quad t\in[0, T].$$

It is easy to show if $u(t)\in \mathring{K}$ then $\rho_1\equiv\text{Const}$ near t. It is also clear that u satisfies the enregy equality (4.35) as well a as the energy inequality (4.34). Hence Lemma 4.19 states that u is an energy conserving solution. Since $\rho_{u_{j_i}}(t)\leqq\rho_{u_{j_i}}(t_1)$ for $t<t_1$ and so $\rho_1(t)\leqq\alpha_1$ for $a\cdot e$ $t\in[0, t_1)$. Now the left continuity of ρ_1 yields $\rho_1(t_1)\leqq\alpha_1$. Combining this with the fact that u is an energy conserving solution, we get $\rho_1(t_1)=\alpha_1$. Thus M_1 is not an empty set. Suppose then that M_i, $1\leqq i\leqq j$, are not empty, and put

$$\inf_{w\in M_j}\rho_w(t_{j+1}) = \alpha_{j+1}. \tag{6.4}$$

Using the same method as in the case $j=1$, we can show that there exists a sequence $\{u_k\}$ in M_j such that

$$\lim_{k\to\infty} u_k(t) = u(t) \qquad \text{uniformly for} \quad t\in[0, T],$$

$$\lim_{k\to\infty} \rho_{u_k}(t_{j+1}) = \alpha_{j+1}, \quad \text{and}$$

$$\lim_{k\to\infty} \rho_{u_k}(t) = \rho_u(t) \qquad \text{for} \quad t\in[0, T]-Q_{j+1},$$

First we see in the same as way as the above that u belongs to M_0. Since $M_j\subset M_1$, we have $\rho_{u_k}(t_1)=\alpha_1$ for $k=1, 2, 3,\cdots$. Now the left continuity of ρ_u yields $\alpha_1\geqq\rho_u(t_1)$. Hence $\alpha_1=\rho_u(t_1)$ by the definition of M_1. Thus $u\in M_1$. We next assume $u\in M_i$ $(0\leqq i<j)$. Since $M_j\subset M_{i+1}$, we have $\rho_{u_k}(t_{i+1})=\alpha_{i+1}$ for all k. Hence, in the same way as the above, we see from the left continuity of ρ_u and the defini ion of M_{i+1} that $\alpha_{i+1}=\rho_u(t_{i+1})$ and $u\in M_{i+1}$. By induction we conclude that $u\in M_j$. Therefore we can apply the same method as in the case $j=1$ to get $u\in M_{j+1}$, and the proof is complete.

Lemma 6.3. *For $u\in M_j$, $j\geqq 2$, we have*

$$\rho_u(t_k) = \alpha_k = \min_{w\in M_{k-1}}\rho_w(t_k) \qquad \textit{for} \quad k = 1, 2, \cdots, j.$$

The proof follows directly from the definition of M_j.

Proof of Theorem 4.

First we show that $\bigcap_{j=1}^{\infty} M_j$ is nonempty. We can choose a sequence $\{u_k\}_{k=1}^{\infty}$ such that $u_k \in M_k$ for $k=1, 2, \cdots$. For simplicity in notation we denote ρ_{u_k} by ρ_k. Then Lemma 6.3 yields

$$\lim_{k\to\infty} \rho_k(t_j) = \alpha_j .$$

Applying Helly's theorem to $\{\rho_k\}$, we get a $\{\rho_{k_i}\}$ subsequence such that

$$\lim_{k_i\to\infty} \rho_{k_i}(t) = \rho(t) \qquad \text{for} \quad t \in [0, T]-Q_\infty$$

where ρ is left continuous and Q_∞ is some countable set in $[0, T]$. For brevity in notation we write ρ_i for ρ_{k_i}. Following the argument of Section 4 we see that

$$\lim_{i\to\infty} u_i(t) = u(t) \qquad \text{uniformly on} \quad [0, T]$$

$$\lim_{i\to\infty} A^{1/2} u_i(t) = A^{1/2} u(t) \qquad \text{for any} \quad t \in [0, T],$$

$$\lim_{i\to\infty} \frac{d}{dt} u_i(t) = \frac{d}{dt} u(t) \qquad a{\cdot}e \ \ t \in [0, T],$$

and that the limit function u is the energy conserving solution. Further, by the method employed in the proof of Lemma 6.2 we can show $u \in M_j$ for $j=1, 2, \cdots$. Thus $\bigcap_{j=1}^{\infty} M_i$ is nonempty.

Second we demonstrate that $\bigcap_{j=1}^{\infty} M_i$ is a singletion set. Let $u, w \in \bigcap_{j=1}^{\infty} M_j$. Then we have $\rho_u(t_i) = \rho_w(t_i)$ for any $i=1, 2, \cdots$. Therefore it follows from the left continuity of ρ_u, ρ_w and the desnseness of $\{t_i\}$ that $\rho_u(t) = \rho_w(t)$ for any $t \in [0, T]$. We then put $\rho_u = \rho_w = \rho$.

We now assume that there exists a number τ, $\bar{T}$ and a subse subset $\{\xi_k\}_{k=1}^{\infty}$ of $(\tau, \bar{T}]$ such that

$$u(t) = w(t) \qquad \text{for any} \quad 0 \leqq t \leqq \tau ,$$

$$u(\xi_k) \neq w(\xi_k) \qquad \text{and} \quad \lim_{k\to\infty} \xi_k = \tau .$$

If $\tau > 0$, then we have

$$\frac{d^-}{dt} u(\tau) = \frac{d^-}{dt} w(\tau);$$

and if $\tau=0$, then we understand as $\frac{d^-}{dt} u(0) = \frac{d^-}{dt} w(0) = b$. Recalling $\{U(t)\}$ is a group, we have

$$(6.5) \qquad u(t+\tau) = C(t)u(\tau) + S(t)\frac{d^-}{dt} u(\tau)$$
$$- \int_0^t S(t-s)\bar{n}(u(s+\tau))\, d\rho(s+\tau) + \int_0^t S(t-s) f(s+\tau, u(s+\tau)) ds$$

for $0 \leqq \tau < \tau + t \leqq \bar{T}$.

1) Case of $u(\tau) \in \mathring{K}$. The third term on the right side of (6.5) vanishes for t small. Hence $u(t+\tau) = w(t+\tau)$ for those values of t, which contradicts the definition of τ.

2) Case of $u(\tau) \in bdy(K)$. From (6.5) we have

$$||u(t+\tau) - w(t+\tau)|| \leqq N_{R+1} \int_0^t ||u(s+\tau) - w(s+\tau)|| d\rho(s+\tau)$$
$$+ \int_0^t h(s+\tau) ||u(s+\tau) - w(s+\tau)|| ds$$

for positive sufficiently small t. Applying Gronwall's inequality, we obtain

$$(6.6) \qquad ||u(t+\tau) - w(t+\tau)||$$
$$\leqq N_{R+1}(1 + C \cdot \exp C) \int_0^t ||u(s+\tau) - w(s+\tau)|| d\rho(s+\tau),$$

where $C = \int_0^T h(s) ds$.

Now we consider the case such that $N_{R+1}(1 + C \cdot \exp C)(\rho(\tau+0) - \rho(\tau)) < 1/2$. We put

$$T_2 = \text{Min}\,\{T_1 \text{ as in lemma 4.18, the Maximum of number of } t \text{ satisfying } N_{R+1}(1 + C \cdot \exp C)(\rho(t+\tau) - \rho(\tau)) \leqq 1/2\}.$$

Let $t_2 \in (0, T_2]$ be such that $\max_{0 \leqq t \leqq T_2} ||u(t+\tau) - w(t+\tau||) = ||u(t_2+\tau) - w(t_2+\tau)||$. Then we see from (6.6) that

$$||u(t_2+\tau) - w(t_2+\tau)||$$
$$\leqq N_{R+1}(1 + C \cdot \exp C)(\rho(T_2+\tau) - \rho(\tau)) ||u(t_2+\tau) - w(t_2+\tau)||$$
$$\leqq 2^{-1} ||u(t_2+\tau) - w(t_2+\tau)||.$$

Thus we have $u(t+\tau) = w(t+\tau)$ for any $0 \leqq t \leqq T_2$. This is also a contradiction. Next suppose $N_{R+1}(1 + C \cdot \exp C)(\rho(\tau+0) - \rho(\tau)) \geqq 1/2$. Since u is a mild solution of (2.4) by Lemma 6.1 it follows from (6.5) that

$$\frac{d^+}{dt} u(\tau) = \frac{d^-}{dt} u(\tau) - (\rho(\tau+0) - \rho(\tau)) \bar{n}(u(\tau)).$$

In view of the energy equality stated in Definition 2.2 we have

$$||\frac{d^+}{dt} u(\tau)|| = ||\frac{d^-}{dt} u(\tau)||.$$

This equality and the relation $\rho(\tau+0) - \rho(\tau) > 0$ together yield

$$\rho(\tau+0) - \rho(\tau) = 2\left(\frac{d^-}{dt} u(\tau), \bar{n}(u(\tau))\right) < 0.$$

Hence

$$(6.7)\qquad \left(\frac{d^+}{dt}u(\tau),\ \bar{n}(u(\tau))\right) = -\left(\frac{d^-}{dt}u(\tau),\ \bar{n}(u(\tau))\right) < 0\,.$$

Further, assume that there exists a sequence $\{s_i\}_{i=1}^{\infty}$ in $(\tau, \bar{T}]$ such that

$$\lim_{i\to\infty} s_i = \tau \quad \text{and} \quad u(s_i) \in bdy(K) \qquad \text{for any} \quad i\,.$$

Then Lemma 3.1 implies that

$$\left(\frac{d^+}{dt}u(\tau),\ \bar{n}(u(\tau))\right) = 0\,,$$

which contradicts (6.7). Hence there would exist $t_3 > 0$ such that

$$u(t+\tau) \in \mathring{K} \qquad \text{for any} \quad 0 \leqq t \leqq t_3\,.$$

But we see with the aid of the result of linear hyperbolic equation that

$$u(t+\tau) = w(t+\tau) \qquad \text{for} \quad 0 \leqq t \leqq t_3\,.$$

This contradictions the definition of τ.
Thus $u(t) = w(t)$ for any $0 \leqq t \leqq T$.
It is concluded that $\bigcap_{j=1}^{\infty} M_j$ is a singleton set.

7. Examples

EXAMPLE 1. Let $H = L_2(0, 1)$, $V = \mathring{W}_2^1(0, 1) = \{u \in W_2^1(0, 1);\ u(0) = u(1) = 0\}$ and define the function $\phi\colon V \to [0, \infty]$ by

$$\phi(u) = \int_0^1 \left\{2^{-1}\left|\frac{d}{dx}u(x)\right|^2 + 4^{-1}|u(x)|^4\right\}dx\,.$$

We then introduce the closed linear subspace of H

$$L = \{f \in L_2(0, 1);\ \int_0^1 f(x)\sin(2m\pi x)dx = 0 \quad \text{for any} \quad m = 0, 1, 2, \cdots, N (N < \infty)\},$$

and the closed convex subset of H

$$K = \{f \in L;\ \int_0^1 |f(x)|\,dx \leqq 1\}\,.$$

Then

$$L^{\perp} = \{f \in L_2(0, 1);\ f(x) = \sum_{m=0}^{N} \sigma_m \sin(2m\pi x) \qquad \text{for} \quad \sigma_m \in (-\infty, \infty)\}$$

and conditions 1), 2) and 4) of the assumption A–1 are easily verified. Moreover the application of Sobolev's imbedding theorem implies that for any $u, v\in V$

$$\operatorname*{Sup}_{0\leqq x\leqq 1} |u(x)-v(x)| \leqq \text{Const}\,(||u-v||_V||u-v||)^{1/2} .$$

On the other hand it is seen that

$$\partial\phi u = -\frac{d^2}{dx^2}u+u^3 .$$

Combining the above two facts we conclused that the operator satisfies condition 3) of the assumption A–1, too. Thus all of the conditions listed in the assumption A–1 are satisfied.

EXAMPLE 2. Let $\Omega\subset R^n$ be a domain with smooth boundary and consider the Hilbert space $H=L_2(\Omega)$. Let $\{p_j\}_{j=1}^{\infty}$ be an orthonomal base of H and $\{\alpha_j\}_{j=1}^{\infty}$ a set of positive numbers such that

$$0<\delta_0\leqq\alpha_j\leqq\delta_0^{-1} \qquad \text{for any} \quad j=1, 2, \cdots .$$

We then define the closed convex set K by

$$K= \{x\in H; \sum_{j=1}^{\infty} \alpha_j(x, p_j)^2\leqq 1\} .$$

The set K may be regarded as an "infinit dimensional elliptic"
Then, defining

$$L(x) = \sum_{j=1}^{\infty} 2\alpha_j(x, p_j)p_j \qquad \text{for any} \quad x\in bdy(K) ,$$

we have

$$n(x) = ||L(x)||^{-1}L(x) \qquad \text{for any} \quad x\in bdy(K) .$$

Moreover we infer that

$$||n(x)-n(y)||\leqq N_R||x-y|| \qquad \text{for} \quad x, y\in bdy(K)\cap B(0, R) .$$

Thus it is concluded that all of the conditions given in the assumption A–2 are satisfied.

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