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HOMOTOPY REPRESENTATIONS AND SPHERES OF REPRESENTATIONS

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0. Introduction

T. tom Dieck and T. Petrie have introduced and studied homotopy representations in [7] and [8]. Let G be a finite group. A G-CW-complex X is called a homotopy representation of G if the H-fixed point set X^H is homotopy equivalent to a (dim X^H)-dimensional sphere for each subgroup H of G. (If X^H is empty, then we set $\dim X^H=-1$ and S^{-1} is empty.) A homotopy representation of G is called linear if it is G-homotopy equivalent to a unit sphere of a real representation of G. (See [7].) We denote the set of G-homotopy classes of homotopy representations [resp. linear homotopy representations] of G by $V^+(G, h^\infty)$ [resp. $V^+(G, \ell)$]. These sets are commutative semi-groups with addition induced by join. Let $V(G, \lambda)$ be the Grothendieck group associated to $V^+(G, \lambda)$ for $\lambda=\ell$ or h^∞. We call $V(G, h^\infty)$ the homotopy representation group of G and $V(G, \ell)$ the linear homotopy representation group of G. The group $V(G, h^\infty)$ has been studied by tom Dieck and Petrie ([7], [8]) and the group $V(G, \ell)$ has been studied by many authors. (See [1], [4], [10], [11], [13], [15], [18] and [19].)

Let $\phi(G)$ be the set of conjugacy classes of subgroups of G and $C(G)$ be the ring of all integer valued functions on $\phi(G)$. For any homotopy representation X, $\operatorname{Dim} X\in C(G)$ is defined by $(\operatorname{Dim} X)(H)=\dim X^H+1$, which is called the dimension function of X. Since $\operatorname{Dim} X*Y=\operatorname{Dim} X+\operatorname{Dim} Y$ ($*$ means the join), the homomorphism $\operatorname{Dim}\colon V(G, \lambda)\to C(G)$ is induced by the assignment $X\mapsto \operatorname{Dim} X$. This homomorphism is called the dimension homomorphism of $V(G, \lambda)$. The kernel of Dim is denoted by $v(G, \lambda)$. tom Dieck and Petrie have shown that $v(G, h^\infty)$ is isomorphic to the Picard group $\operatorname{Pic}(A(G))$ of the Burnside ring of G in [7].

We are interested in the difference between $V(G, \ell)$ and $V(G, h^\infty)$. We observe the homomorphisms which are induced from the inclusion $V^+(G, \ell)\to V^+(G, h^\infty)$:

$$I_G\colon V(G, \ell)\to V(G, h^\infty)$$
$$i_G\colon v(G, \ell)\to v(G, h^\infty)\,.$$

These homomorphisms are injective ([7]). We obtain the following results.

Theorem A. *The homomorphism I_G is isomorphic if and only if G is a cyclic group or a dihedral 2-group D_n ($n\geq 2$).*

Here $$D_n = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1} \rangle .$$

Theorem B. *Suppose that G is nilpotent. Then the homomorphism i_G is isomorphic if and only if G is a cyclic group or a dihedral 2-group D_n ($n\geq 2$).*

T. Petrie has already announced the analogous theorem of Theorem A for oriented homotopy representations in [16]. Our Theorems are the unoriented versions.

This paper is organized as follows. In section 1 we shall have the necessary conditions. In section 2 and section 3 we shall compute the order of $v(G, h^\infty)$. In section 4 the proofs of the main theorems will be completed.

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1. Necessary conditions

Using the results in [5] and [11], we shall show the following proposition.

Proposition 1.1. *Let G be an abelian group. Then the homomorphsim i_G is an isomorphism if and only if G is a cyclic group or $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$.*

Proof. It is known that the group $v(G, h^\infty)$ is isomorphic to

$$\prod_{H \subsetneq G} \boldsymbol{Z}_{|G/H|}{}^* / \{\pm 1\} \quad ([5])$$

and the group $v(G, \ell)$ is isomorphic to

$$\prod_{\substack{H \subsetneq G \\ G/H\,:\,\text{cyclic}}} \boldsymbol{Z}_{|G/H|}{}^* / \{\pm 1\} \quad ([11]) .$$

Here $\boldsymbol{Z}_{|G/H|}{}^*$ is the group of invertible elements in $\boldsymbol{Z}_{|G/H|}$. Hence it is easy to see this proposition since i_G is injective.

Let H be a normal subgroup of G and X a homotopy representation of G. Since X^H is a homotopy representation of G/H, the correspondence $X \mapsto X^H$ induces a homomorphism $f\colon v(G, \lambda) \to v(G/H, \lambda)$ for $\lambda = \ell$ or h^∞. The following lemma is elementary.

Lemma 1.2. *Under the above situation,*

i) *The homomorphism f is surjective.*
ii) *The following diagram is commutative.*

$$\begin{array}{ccc} v(G, \ell) & \xrightarrow{\dot{i}_G} & v(G, h^\infty) \\ f \downarrow & & \downarrow f \\ v(G/H, \ell) & \xrightarrow[\dot{i}_{G/H}]{} & v(G/H, h^\infty) \end{array}$$

iii) *If i_G is isomorphic, then $i_{G/H}$ is also isomorphic.*

Proof. i): Let ℓ: $v(G/H, \lambda) \to v(G, \lambda)$ be the homomorphism which is induced by considering G/H-spaces as G-spaces via the canonical projection $G \to G/H$. Then, the composition $f \circ \ell$ is the identity on $v(G, \lambda)$ and so f is surjective.
ii): This follows from the definitions of f and i_G.
iii): This follows directly from the injectivity of $i_{G/H}$, i) and ii).

Let us apply lemma 1.2 to the commutator subgroup $[G, G]$ of G. Then we have:

Proposition 1.3. *If i_G is an isomorphism, then $G/[G, G]$ is a cyclic group or $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. Furthermore suppose that G is nilpotent, then G is one of the following groups;*
(1) *cyclic group*
(2) *dihedral 2-group D_n $(n \geq 2)$*

$$D_n = \langle a, b \,|\, a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$$

(3) *quaternion 2-group Q_n $(n \geq 3)$*

$$Q_n = \langle a, b \,|\, a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle$$

(4) *semi-dihedral 2-group SD_n $(n \geq 4)$*

$$SD_n = \langle a, b \,|\, a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}} \rangle$$

Proof. The first half is clear from Proposition 1.1. Suppose that G is nilpotent. We may put $G = P_1 \times \cdots \times P_r$, where P_1 is a 2-group and P_i is a p_i-group (p_i: odd prime) for $i > 1$. If $G/[G, G]$ is cyclic, then G is also cyclic. In the case $G/[G, G] = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, $P_i/[P_i, P_i]$ is trivial for $i > 1$. Hence P_i must be trivial for $i > 1$, so that G is 2-group. Therefore, if G is abelian, then G is D_2 $(= \boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$. If G is not abelian, then G is D_n $(n \geq 3)$, Q_n $(n \geq 3)$ or SD_n $(n \geq 4)$ by [9, Chap. 5, Theorem 4.5].

2. The order of the Picard group of the Burnside ring

The group $v(G, h^\infty)$ is isomorphic to the Picard group $\mathrm{Pic}(A(G))$ of $A(G)$, where $A(G)$ is the Burnside ring of G. In this section we compute the order of $\mathrm{Pic}(A(G))$. We recall the Burnside ring $A(G)$. The set of G-isomorphism classes of finite G-sets becomes a commutative semi-ring with addition induced by disjoint union and multiplication induced by cartesian product. The Grothendieck ring of the semi-ring is called the Burnside ring. Let S be a finite G-set. The correspondence $S\mapsto|S^H|$ induces the ring homomorphism χ: $A(G)\to\prod_{(H)\in\phi(G)}\boldsymbol{Z}_H$, where $\boldsymbol{Z}_H=\boldsymbol{Z}$. As is well-known, the ring homomorphism χ is injective and if we consider $A(G)$ as a subring of $\prod_{(H)\in\phi(G)}\boldsymbol{Z}_H$ via χ, then

$$A(G)=\{(d_H)\in\prod_{(H)}\boldsymbol{Z}_H|\text{ congruences }(*)_H\quad\text{for }(H)\in\phi(G)\}\ ,$$

where

$$(*)_H\colon\quad\sum_{(K)}|NH/NH\cap NK|\varphi(|K/H|)d_K\equiv 0\mod|NH/H|\ .$$

The sum is over the NH-conjugacy classes (K) such that H is a normal subgroup in K and K/H is cyclic. φ denotes the Euler function and NH denotes the normalizer of H in G. We can rewrite $(*)_H$ as the following:

$$(**)_H\colon\quad\sum_{(K)}n(H,K)d_K\equiv 0\mod|NH/H|\ ,$$

where each $n(H,K)$ is a certain integer and $n(H,H)=1$ for any (H), $(K)\in\phi(G)$ such that K/H is cyclic. The sum is over the G-conjugacy classes (K) such that H is normal in K and K/H is cyclic (see [6]).

We put $\phi(G)=\{(H_1),\cdots,(H_n)\}$ and assume that $(H_i)\le(H_j)$ implies $i\ge j$, where $(H_i)\le(H_j)$ means that H_i conjugates to a certain subgroup of H_j. We set

$$R_k=\{(d_{H_i})_{1\le i\le k}\in\prod_{i=1}^k\boldsymbol{Z}_{H_i}|(**)_{H_i},\ 1\le i\le k\}\ .$$

Note that $R_1=\boldsymbol{Z}$ and $R_n=A(G)$.

Lemma 2.1. *R_k is the subring of $\prod_{i=1}^k\boldsymbol{Z}_{H_i}$,*

Proof. It is trivial that R_k is an additive subgroup of $\prod_{i=1}^k\boldsymbol{Z}_{H_i}$. We note that there exists $(d'_{H_i})_{1\le i\le n}\in A(G)$ such that $d_{H_i}=d'_{H_i}$, $1\le i\le k$, for any $(d_{H_i})_{1\le i\le k}\in R_k$. Let $(d_{H_i})_{1\le i\le k}$ and $(e_{H_i})_{1\le i\le k}$ be any two elements in R_k. Since $A(G)$ is the ring, $(d'_{H_i}e'_{H_i})_{1\le i\le n}$ is in $A(G)$. Hence $(d_{H_i}e_{H_i})_{1\le i\le k}$ satisfies $(**)_{H_j}$, $1\le j\le k$, since $(d'_{H_i}e'_{H_i})_{1\le i\le n}$ satisfies $(**)_{H_j}$, $1\le j\le n$. Therefore $(d_{H_i}e_{H_i})_{1\le i\le k}$ is in R_k.

We define a map $p\colon R_{k-1}\to\boldsymbol{Z}_{|WH_k|}$, $WH_k=NH_k/H_k$, by

$$(d_{H_i})_{1\le i\le k-1}\mapsto-\sum_{j\atop 1\neq H_j/H_k\,:\,\text{cyclic}}n(H_k,H_j)d_{H_j}\mod|WH_k|\ .$$

Lemma 2.2. *p is the ring homomorphism.*

Proof. It is trivial to be an additive homomorphism. For $(d_{H_i})_{1\le i\le k}$ [resp. $(e_{H_i})_{1\le i\le k}$]$\in R_{k-1}$, we choose $(d'_{H_i})_{1\le i\le n}$ [resp. $(e'_{H_i})_{1\le i\le n}$]$\in A(G)$ like the proof of

(2.1). Then

$$d'_{H_k}e'_{H_k} \equiv -\sum_{1\neq H_j/H_k:\text{ cyclic}} {}_j\, n(H_k, H_j)d_{H_j}e_{H_j} \mod |WH_k|$$
$$d'_{H_k} \equiv -\sum_{1\neq H_j/H_k:\text{ cyclic}} {}_j\, n(H_k, H_j)d_{H_j} \mod |WH_k|$$
$$e'_{H_k} \equiv -\sum_{1\neq H_j/H_k:\text{ cyclic}} {}_j\, n(H_k, H_j)e_{H_j} \mod |WH_k| .$$

Hence $p((d_{H_i}e_{H_i})_{1\leq i\leq k})=p((d_{H_i})_{1\leq i\leq k})p((e_{H_i})_{1\leq i\leq k})$

Let $s: R_k \to R_{k-1}$ and $r: R_k \to \boldsymbol{Z}$ be the ring homomorphisms defined by

$$s: (d_{H_i})_{1\leq i\leq k} \mapsto (d_{H_i})_{1\leq i\leq k-1}$$
$$r: (d_{H_i})_{1\leq i\leq k} \mapsto d_{H_k} .$$

We have the following lemma. (See [5].)

Lemma 2.3. *The following diagram is the pull-back of ring.*

$$\begin{array}{ccc} R_k & \xrightarrow{s} & R_{k-1} \\ r\downarrow & & \downarrow p \\ \boldsymbol{Z} & \xrightarrow[q]{} & \boldsymbol{Z}_{|WH_k|} \end{array} \qquad (2\leq k\leq n)$$

Here q is the canonical projection.

Proof. It is easy to show this lemma from the definitions of R_k, s, p and r. The next proposition is the main result in this section.

Proposition 2.4. *Let G be any finite group. Then*

$$|\mathrm{Pic}(A(G))| = 2^{-n}|A(G)^*| \prod_{(H)\in\phi(G)} \varphi(|NH/H|),$$

where $n=|\phi(G)|$ and $A(G)^*$ is the unit group of $A(G)$.

Proof. The pull-back diagram in Lemma 2.3 yields the Mayer-Vietoris exact sequence of the Picard group [6]. That is, the sequence:

$$0 \to R_k^* \to R_{k-1}^* \oplus \boldsymbol{Z}^* \to \boldsymbol{Z}_{|WH_k|}^*$$
$$\to \mathrm{Pic}\, R_k \to \mathrm{Pic}\, R_{k-1} \oplus \mathrm{Pic}\, \boldsymbol{Z} \to \mathrm{Pic}\, \boldsymbol{Z}_{|WH_k|}$$

is exact, $2\leq k\leq n$. Since $\mathrm{Pic}\, \boldsymbol{Z}=0$ and $\mathrm{Pic}\, \boldsymbol{Z}_{|WH_k|}=0$ by ([2], chap. 2, 5), we obtain the exact sequence:

$$0 \to R_k^* \to R_{k-1}^* \oplus \boldsymbol{Z}^* \to \boldsymbol{Z}_{|WH_k|}^*$$
$$\to \mathrm{Pic}\, R_k \to \mathrm{Pic}\, R_{k-1} \to 0 \qquad (2\leq k\leq n) .$$

Inductively, Pic R_k has a finite order and then we have

$$\frac{|\operatorname{Pic} R_k|}{|\operatorname{Pic} R_{k-1}|}=\frac{\varphi(|WH_k|)}{2}\frac{|R_k{}^*|}{|R_{k-1}{}^*|} \qquad (2\leq k\leq n).$$

Therefore,

$$\frac{|\operatorname{Pic} R_n|}{|\operatorname{Pic} R_1|}=\frac{1}{2^{n-1}}\frac{|R_n{}^*|}{|R_1{}^*|}\prod_{k=2}^{n}\varphi(|WH_k|).$$

Since $R_n=A(G)$, $R_1=\boldsymbol{Z}$ and $\varphi(|WH_1|)=1$, the desired result holds.

Corollary 2.5. $|v(G, h^\infty)|=2^{-n}\,|A(G)^*|\prod_{(H)}\varphi(|NH/H|)$, $n=|\phi(G)|$.

3. Examples

We indeed compute the order of Pic($A(G)$) (i.e. $v(G, h^\infty)$) for some groups.

EXAMPLE 3.1. Let G be D_n $(n\geq 2)$ in Proposition 1.3. Then $|\operatorname{Pic}(A(G))|=2^N$, where $N=(n-2)(n-3)/2$.

Proof. Conjugacy classes of subgroups of D_n are the following:

$$\begin{aligned}&(D_n)\\&(\langle a^{2^i}\rangle),\ i=0,1,\cdots,n-1\\&(\langle a^{2^i}, b\rangle),\ i=1,2,\cdots,n-1\\&(\langle a^{2^i}, ab\rangle),\ i=1,2,\cdots,n-1,\end{aligned}$$

and WH of these subgroups in G are the following:

$$\begin{aligned}&WD_n=1\\&W\langle a^{2^i}\rangle=D_{i+1},\ i=0,1,\cdots,n-1\\&W\langle a^{2^i}, b\rangle=\boldsymbol{Z}_2,\ i=1,2,\cdots,n-1\\&W\langle a^{2^i}, ab\rangle=\boldsymbol{Z}_2,\ i=1,2,\cdots,n-1.\end{aligned}$$

Hence $|\phi(G)|=3n-1$ and $\prod_{(H)}\varphi(|WH|)=2^{n(n-1)/2}$. We see that $|A(G)^*|$ is 2^{n+2} by Theorem 4.1 or Example 4.8 in [14]. By Proposition 2.4,

$$\begin{aligned}|\operatorname{Pic}(A(G))|&=2^{-(3n-1)}\times 2^{n(n-1)/2}\times 2^{n+2}\\&=2^N.\end{aligned}$$

One can show the following Examples 3.2 and 3.3 by the same argument.

EXAMPLE 3.2. Let G be Q_n $(n\geq 3)$ in Proposition 1.3. Then $|\operatorname{Pic}(A(G))|=2^{N+1}$, where $N=(n-2)(n-3)/2$.

EXAMPLE 3.3. Let G be SD_n $(n\geq 4)$ in Proposition 1.3. Then $|\operatorname{Pic}(A(G))|$

$=2^N$, where $N=(n-2)(n-3)/2$.

EXAMPLE 3.4. If $v(G, h^\infty)=0$, then $|G|=2^n$ or $2^n p_1 p_2 \cdots p_r$, where p_i, $1\leq i\leq r$, are distinct odd primes of forms 2^e+1.

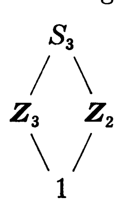
Proof. Put $|G|=2^n p_1^{e_1} \cdots p_r^{e_r}$, where p_i, $1\leq i\leq r$, are distinct odd primes. Then $\varphi(|G|)=2^{n-1}(p_1-1)p_1^{e_1-1} \cdots (p_r-1)p_r^{e_r-1}$. If $v(G, h^\infty)=0$, then $\varphi(|G|)$ must be 2-power by Corollary 2.5. Hence $e_i=1$ and p_i-1 is 2-power for any i.

EXAMPLE 3.5. Let S_r be the symmetric group on r letters, where $r=3$, 4 or 5. We have the following table.

G	$V(G, \ell)$	$V(G, h^\infty)$
S_3	$\boldsymbol{Z}^3$	$\boldsymbol{Z}^4$
S_4	$\boldsymbol{Z}^5$	$\boldsymbol{Z}^8$
S_5	$\boldsymbol{Z}^7$	$\boldsymbol{Z}^{15}\oplus\boldsymbol{Z}_2$

$\boldsymbol{Z}^n=\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus \cdots \oplus\boldsymbol{Z}$ (n-times)

Proof. We note that $V(G, \lambda)=\text{Dim } V(G, \lambda)\oplus v(G, \lambda)$ and Dim $V(G, \lambda)$ is a free abelian group, $\lambda=\ell$ or h^∞. The rank of Dim $V(G, \ell)$ is equal to the number of conjugacy classes of cyclic subgroups of G by [6]. The rank of Dim $V(G, h^\infty)$ is equal to the number of conjugacy classes (H) of subgroups H of G such that $H/[H, H]$ is cyclic by [7]. By these facts Dim $V(G, \lambda)$ is computable. For the symmetric group on n letters S_n, $v(S_n, \ell)=0$ by [12] or [6]. Now we compute $v(S_r, h^\infty)$ for $r=3$, 4 or 5. In case $r=3$, the diagram of conjugate subgroups of S_3 is the following:



and $WS_3=1$, $W\boldsymbol{Z}_3=\boldsymbol{Z}_3$, $W\boldsymbol{Z}_2=1$ and $W1=S_3$. Hence $|\phi(S_3)|=4$ and $\prod_{(H)} \varphi(|WH|)=2$. The order of $A(S_3)^*$ is 8 by using the congruences $(*)_H$ in section 2. Therefore $V(S_3, h^\infty)=0$. In cases $r=4$ and $r=5$, using the diagrams of conjugate subgroups of S_4 and S_5 in [14], one can show $|v(S_4, h^\infty)|=1$ and $|v(S_5, h^\infty)|=2$ respectively. Hence $v(S_4, h^\infty)=0$ and $v(S_5, h^\infty)=\boldsymbol{Z}_2$.

EXAMPLE 3.6. Let G be the symmetric group on n letters. Then i_G is an isomorphism if $n\leq 4$ and is not an isomorphism if $n>4$.

Proof. By [12], $v(S_n, \ell)=0$ for any positive integer n. If $n\geq 6$, then $v(S_n, h^\infty)\neq 0$ by Example 3.4. Since $v(S_r, h^\infty)=0$ $(r\leq 4)$ and $v(S_5, h^\infty)=\boldsymbol{Z}_2$ by Example 3.5, the desired result follows.

REMARK. $v(G, h^\infty)=0$ implies that the stable G-homotopy class of the homotopy representation X is decided by its dimension function Dim X, where homotopy representations X and Y are stable G-homotopy equivalent if there exists a homotopy representation Z such that $X*Z$ and $Y*Z$ are G-homotopy equivalent.

4. Proofs

We shall prove Theorem B by compairing orders of $v(G, \ell)$ and $v(G, h^\infty)$. Let m be the exponent of G and u_m a primitive m-th root of unity. Let Γ be the Galois group $\mathrm{Gal}(\boldsymbol{Q}(u_m)/\boldsymbol{Q})$. Γ acts on the set $\mathrm{Irr}(G, \boldsymbol{R})$ of real irreducible characters of G by $(\gamma\cdot\chi)(g)=\gamma(\chi(g))$ for $\chi\in\mathrm{Irr}(G, \boldsymbol{R})$, $\gamma\in\Gamma$ and $g\in G$. We need the following theorem in [4] and [6] in order to compute the order of $v(G, \ell)$.

Theorem 4.1 (T. tom Dieck). *Let G be a p-group. Then $v(G, \ell)$ is isomorphic to $\bigoplus_{A\in X}\Gamma/\Gamma_A$, where $X=\mathrm{Irr}(G, \boldsymbol{R})/\Gamma$ and $\Gamma_A=\{\gamma\in\Gamma\mid\gamma\cdot\chi=\chi\}$, $(\chi\in A)$.*

REMARK. For any group G, $v(G, \ell)$ is isomorphic to a quotient group of $\bigoplus_{A\in X}\Gamma/\Gamma_A$. (See [4] or [6].)

Lemma 4.2. *For D_n, Q_n and SD_n in Proposition 1.3, we have $|v(D_n, \ell)|=2^N$, $|v(Q_n, \ell)|=2^N$ and $|v(SD_n, \ell)|=2^{N-1}$. Here $N=(n-2)(n-3)/2$.*

Proof. We need the real irreducible character tables of D_n, Q_n and SD_n. By [17, Chap. 13, 2.], we have the next tables.

In the D_n case:

	a^k	a^kb
θ_1	1	1
θ_2	1	-1
θ_3	$(-1)^k$	$(-1)^k$
θ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$u_m^{hk}+u_m^{-hk}$	0

Here $1\leq h<m/2$, $m=2^{n-1}$.

In the Q_n case:

	a^k	$a^k b$
θ_1	1	1
θ_2	1	-1
θ_3	$(-1)^k$	$(-1)^k$
θ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$u_m^{2hk}+u_m^{-2hk}$	0
ψ_s	$\psi_s(a^k)$	0

Here $\psi_s(a^k)=2(u_m^{(2s-1)k}+u_m^{-(2s-1)k})$
$1\leq h<m/4,\ 1\leq s\leq m/4,\ m=2^{n-1}$.

In the SD_n case:

	a^k	$a^k b$
θ_1	1	1
θ_2	1	-1
θ_3	$(-1)^k$	$(-1)^k$
θ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$u_m^{2hk}+u_m^{-2hk}$	0
ψ_s	$\psi_s(a^k)$	0

Here $\psi_s(a^k)=u_m^{(2s-1)k}+u_m^{-(2s-1)k}$
$+u_m^{(m/2-(2s-1))k}+u_m^{-(m/2-(2s-1))k}$
$1\leq h<m/4,\ 1\leq s<m/8,\ m=2^{n-1}$.

By the irreducible character tables of D_n, Q_n and SD_n, we have

$$X=\operatorname{Irr}(G,\boldsymbol{R})/\Gamma$$
$$=\begin{cases}\{\{\theta_i\},\ A_t\,|\,1\leq i\leq 4,\ 1\leq t\leq n-2\} & \text{if } G=D_n\\ \{\{\theta_i\},\ A_t,\ B\,|\,1\leq i\leq 4,\ 1\leq t\leq n-3\} & \text{if } G=Q_n\\ \{\{\theta_i\},\ A_t,\ C\,|\,1\leq i\leq 4,\ 1\leq t\leq n-3\} & \text{if } G=SD_n\,.\end{cases}$$

Here $A_t=\{\chi_h\,|\,h\equiv 2^{t-1} \mod 2^t\}$

$B=\{\psi_s \mid 1\leq s\leq m/4\}$
$C=\{\psi_s \mid 1\leq s\leq m/8\}$.

Hence,

$$|\Gamma/\Gamma_{A_t}| = |A_t| = \begin{cases} 2^{n-2-t} & \text{if } G=D_n \\ 2^{n-3-t} & \text{if } G=SD_n \text{ or } Q_n, \end{cases}$$

$$|\Gamma/\Gamma_B| = 2^{n-3}$$

and

$$|\Gamma/\Gamma_C| = 2^{n-4}.$$

Therefore, by Theorem 4.1, we have

$$|v(G, \ell)| = \begin{cases} \prod_{t=1}^{n-1} |A_t| = 2^N & \text{if } G=D_n \\ |B|\prod_{t=1}^{n-3} |A_t| = 2^N & \text{if } G=Q_n \\ |C|\prod_{t=1}^{n-3} |A_t| = 2^{N-1} & \text{if } G=SD_n. \end{cases}$$

Proposition 4.3. *The homomorphism i_G is an isomorphism if $G=D_n$ $(n\geq 2)$ and is not an isomorphism if $G=Q_n$ $(n\geq 3)$ or SD_n $(n\geq 4)$.*

Proof. The desired result follows from Examples 3.1, 3.2 and 3.3 and Lemma 4.2.

Corollary 4.4. *In the case $G=Q_n$ or SD_n, $v(G, \ell)$ is a subgroup of index 2 of $v(G, h^\infty)$.*

Proof of Theorem B. This follows directly from Propositions 1.1, 1.3 and 4.5.

Proof of Theorem A. Theorem A follows from Theorem B and the theorem of tom Dieck and Petrie ([3], [7]), that is, Dim $V(G, \ell)$=Dim $V(G, h^\infty)$ if and only if G is nilpotent. Indeed, if I_G is an isomorphism, then G is nilpotent since Dim $V(G, \ell)$=Dim $V(G, h^\infty)$. Since i_G is also isomorphic, G is a cyclic group or D_n by Theorem B.

Conversely, suppose that G is a cyclic group or D_n, then i_G is an isomorphism and Dim $V(G, \ell)$=Dim $V(G, h^\infty)$. It is sufficient to show that I_G is surjective. Let x be any element of $V(G, h^\infty)$. Then there exists an element $u\in v(G, \ell)$ such that Dim x=Dim u. Hence $x-I_G(u)$ is in $v(G, h^\infty)$. Since i_G is an isomorphism, there exists an element $v\in v(G, \ell)$ such that $x-I_G(u)$ $=i_G(v)$. Hence $x=I_G(u+v)$. Therefore I_G is surjective.

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