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ACYCLICITY OF *BP*-RELATED HOMOLOGIES AND COHOMOLOGIES

Dedicated to Professor Itiro Tamura on his sixtieth birthday

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Introduction

BP is the Brown-Peterson spectrum at a fixed prime p. This spectrum is an associative and commutative ring spectrum whose homotopy is $BP_*=Z_{(p)}[v_1, \cdots, v_n, \cdots]$. For each $n\geqq 0$ there are associative BP-module spectra $P(n)$, $BP\langle n\rangle$, $k(n)$, L_nBP, M_nBP and N_nBP. If E is an associative BP-module spectrum, then we can form a weak associative BP-module spectrum $v_n^{-1}E$. When $E=P(n)$, $BP\langle n\rangle$ or $k(n)$, $v_n^{-1}E$ is written $B(n)$, $E(n)$ or $K(n)$ respectively.

For a CW-spectrum E we denote by $\langle E\rangle$ the Bousfield class of E [3]. Thus it is the equivalence class under the equivalence relation: $E\sim F$ when $E_*X=0$ if and only if $F_*X=0$. In [13] and [14] Ravenel has studied the Bousfield classes of the above BP-related spectra.

Theorem 0.1 ([13, Theorem 2.1] and [14, Lemma 3.1]).
i) $\langle B(n)\rangle=\langle K(n)\rangle=\langle M_nBP\rangle$,
ii) $\langle v_n^{-1}BP\rangle=\langle E(n)\rangle=\vee_{0\leqq i\leqq n}\langle K(i)\rangle=\langle L_nBP\rangle$,
iii) $\langle P(n)\rangle=\langle K(n)\rangle\vee\langle P(n+1)\rangle=\langle N_nBP\rangle$,
iv) $\langle k(n)\rangle=\langle K(n)\rangle\vee\langle HZ/p\rangle$, *and*
v) $\langle BP\langle n\rangle\rangle=\langle E(n)\rangle\vee\langle HZ/p\rangle$.

For a CW-spectrum E we denote by $\langle E\rangle^*$ the cohomological Bousfield class of E. Thus $\langle E\rangle^*=\langle F\rangle^*$ when $E^*X=0$ if and only if $F^*X=0$. Given a p-local CW-spectrum E there exists a p-local CW-spectrum ∇E related by a universal coefficient sequence

$$0\to \mathrm{Ext}(E_{*-1}X, Z_{(p)})\to \nabla E^*X\to \mathrm{Hom}(E_*X, Z_{(p)})\to 0$$

(see [5] or [16]). By using this sequence we can show that $\langle\nabla E\rangle^*=\langle E\rangle$, and moreover $\langle E\rangle^*=\langle\nabla E\rangle$ if E is of finite type. The BP-module spectrum $P(n)$, $BP\langle n\rangle$, $k(n)$ or $K(n)$ is of finite type, but $v_n^{-1}BP$, $B(n)$, $E(n)$, L_nBP, M_nBP or N_nBP is not of finite type. Nevertheless we obtain

Theorem 0.2. i) $\langle B(n)\rangle^*=\langle\nabla B(n)\rangle=\langle K(n)\rangle^*=\langle\nabla K(n)\rangle=\langle K(n)\rangle$,

ii) $\langle v_n^{-1}BP\rangle^*=\langle\nabla v_n^{-1}BP\rangle=\langle E(n)\rangle^*=\langle\nabla E(n)\rangle=\langle v_n^{-1}BP\rangle$,
iii) $\langle M_nBP\rangle^*=\langle\nabla M_nBP\rangle=\langle L_nBP\rangle^*=\langle\nabla L_nBP\rangle=\langle v_n^{-1}BP\rangle$,
iv) $\langle N_{n+1}BP\rangle^*=\langle\nabla N_{n+1}BP\rangle=\langle v_n^{-1}BP\rangle^\vee\langle\nabla P(1)\rangle$,
v) $\langle P(n)\rangle^*=\langle\nabla P(n)\rangle=\begin{cases}\langle\nabla P(1)\rangle^\vee\langle SQ\rangle & \text{if } n=0\\ \langle\nabla P(1)\rangle & \text{if } n\geqq 1,\end{cases}$
vi) $\langle BP\langle n\rangle\rangle^*=\langle\nabla BP\langle n\rangle\rangle=\langle HZ_{(p)}\rangle$ *for* $n\geqq 0$, *and*
vii) $\langle k(n)\rangle^*=\langle\nabla k(n)\rangle=\langle HZ/p\rangle$ *for* $n\geqq 1$.

However it is not valid that $\langle E\rangle^*=\langle\nabla E\rangle$ in general. As examples we have

Theorem 0.3. i) $\langle\Pi_{n\geqq m}B(n)\rangle^*=\langle\Pi_{n\geqq m}K(n)\rangle^*=\vee_{n\geqq m}\langle K(n)\rangle$, *but* $\langle\nabla(\Pi_{n\geqq m}B(n))\rangle=\langle\nabla(\Pi_{n\geqq m}K(n))\rangle=\vee_{n\geqq m}\langle K(n)\rangle^\vee\langle HZ/p\rangle$.
ii) $\langle\Pi_{n\geqq m}M_nBP\rangle^*=\langle\Pi_{n\geqq m}L_nBP\rangle^*=\langle\Pi_{n\geqq m}N_nBP\rangle^*=\vee_{n\geqq 0}\langle K(n)\rangle$, *but* $\langle\nabla(\Pi_{n\geqq m}M_nBP)\rangle=\langle\nabla(\Pi_{n\geqq m}L_nBP)\rangle=\langle\nabla(\Pi_{n\geqq m}N_nBP)\rangle=\langle BP\rangle$.
iii) $\langle\Pi_{n\geqq m}P(n)\rangle=\langle P(m)\rangle$ *and* $\langle\Pi_{n\geqq m}P(n)\rangle^*=\langle\nabla P(m)\rangle$, *but* $\langle\vee_{n\geqq m}\nabla P(n)\rangle^*=$ $\langle\nabla(\Pi_{n\geqq m}P(n))\rangle=\begin{cases}\langle HZ_{(p)}\rangle & \text{if } k=0\\ \langle HZ/p\rangle & \text{if } k\geqq 1.\end{cases}$

Let $\{E_n\}_{n\geqq m}$ and $\{F_n\}_{n\geqq m}$ be families of CW-spectra. If $\langle E_n\rangle=\langle F_n\rangle$ for all $n\geqq m$, then it is obvious that $\langle\vee_{n\geqq m}E_n\rangle=\langle\vee_{n\geqq m}F_n\rangle$. So it might be expected that $\langle\Pi_{n\geqq m}E_n\rangle=\langle\Pi_{n\geqq m}F_n\rangle$. But this equality doesn't hold in general. If a p-local CW-spectrum E is of finite type, then we have that $\langle\Pi_\Lambda E\rangle=\langle E\rangle$ for any indexed set Λ. But this is also false in general unless E is of finite type or Λ is finite. As examples we get

Theorem 0.4. i) $\langle\Pi_{n\geqq m}P(n)\rangle=\langle P(m)\rangle$ *but* $\langle\Pi_{n\geqq m}N_nBP\rangle=\langle BP\rangle$,
ii) $\langle\Pi_\Lambda K(n)\rangle=\langle K(n)\rangle$, $\langle\Pi_\Lambda P(n)\rangle=\langle P(n)\rangle$ *but* $\langle\Pi_\Lambda M_nBP\rangle=\langle v_n^{-1}BP\rangle$, $\langle\Pi_\Lambda N_nBP\rangle=\langle BP\rangle$ *if the indexed set* Λ *is infinite.*

In [14] Ravenel proved that the cofiber sequence $N_nS\to M_nS\to N_{n+1}S$ realizes the short exact sequence $0\to N_nBP_*\to M_nBP_*\to N_{n+1}BP_*\to 0$ of BP_*-modules defined inductively by $N_0BP_*=BP_*$ and $M_nBP_*=v_n^{-1}N_nBP_*$. His proof is established on the existence of certain finite CW-complexes X_n recently constructed by Mitchell [12]. By virtue of Ravenel's result we can investigate the localizations of homologies $P(n)_*(-)$, $BP\langle n\rangle_*(-)$ and $\nabla N_{n+1}BP_*(-)$.

Theorem 0.5. i) $L_{P(n)}X=\Sigma^nF(N_nS, L_{BP}X)$,
ii) $L_{BP\langle n\rangle}X=L_{(HZ/p,n)}X$ *and* $L_{\nabla N_{n+1}BP}X=L_{(\nabla P(1),n)}X$ *where* $L_{(F,n)}X$ *denotes the fiber of the composite map* $L_FX\to C_FX\to L_nC_FX$ *for* $F=HZ/p$ *or* $\nabla P(1)$.

In §1 we study the Bousfield classes $\langle E\rangle$ of well-known BP-related spectra and give a proof of Theorem 0.1 in the different way from Revanel's [13, 14]. We next discuss the Bousfield classes $\langle\nabla E\rangle$ of the Anderson dual spectra in

§2 and the cohomological Bousfield classes $\langle E\rangle^*$ in §3. As a result we obtain Theorem 0.2. In §4 we treat of wedge sums $\vee_{n\geq m}E_n$ and products $\Pi_{n\geq m}E_n$ of BP-related spectra and show Theorems 0.3 and 0.4. In §5 we recall Ravenel's result (Corollary 5.4) of the geometric realization and then discuss the $P(n)_*$-, $BP\langle n\rangle_*$- and $\nabla N_{n+1}BP_*$-localizations in order to prove Theorem 0.5.

1. Bousfield classes of BP-related spectra

1.1. Let E be a BP-module spectrum (with unit) having structure map μ, and v be an element of BP_* with dimension d. We can form a CW-spectrum $v^{-1}E$ defined to be the mapping telescope $\varinjlim \Sigma^{-id}E$ of the map $\mu(v_\wedge 1)\colon \Sigma^d E \to E$. If E is associative, then $v^{-1}E$ is a BP-module spectrum which is weak associative. Even if E is weak associative, the map $\mu(v_\wedge 1)$ induces multiplication by v in homotopy groups and hence $(v^{-1}E)_*X\cong v^{-1}E_*X$. In this case we write simply v in place of $\mu(v_\wedge 1)$.

For a CW-spectrum E we denote by $\langle E\rangle$ the *Bousfield class* of E [3]. They are partially ordered by writing $\langle E\rangle\geqq\langle F\rangle$ when $E_*X=0$ implies $F_*X=0$. If $E\to F\to G$ is a cofiber sequence of BP-module spectra (and BP-module maps), then $\langle F\rangle\leqq\langle E\rangle\vee\langle G\rangle$ and more generally

(1.1) $\langle v^{-1}F\rangle\leqq\langle v^{-1}E\rangle\vee\langle v^{-1}G\rangle$

for any element v of BP_* (cf., [13, Proposition 1.23]). This is easily shown by making use of Five lemma (or Verdier's lemma [1]).

Lemma 1.1. *Let v and w be elements of BP_*, E be an associative BP-module spectrum and F be the cofiber of the map $w\colon \Sigma^d E\to E$ where $d=\dim w$. Then $\langle v^{-1}E\rangle=\langle w^{-1}v^{-1}E\rangle\vee\langle v^{-1}F\rangle$ and in particular $\langle E\rangle=\langle w^{-1}E\rangle\vee\langle F\rangle$* (cf., [13, Lemma 1.34]).

Proof. From (1.1) it follows immediately that $\langle v^{-1}E\rangle\geqq\langle v^{-1}F\rangle$, and so $\langle v^{-1}E\rangle\geqq\langle w^{-1}v^{-1}E\rangle\vee\langle v^{-1}F\rangle$. If $(v^{-1}F)_*X=0$, then the map w induces an isomorphism $(v^{-1}E)_*X\to(v^{-1}E)_*X$, and hence there is an isomorphism $(v^{-1}E)_*X\to(w^{-1}v^{-1}E)_*X$. This gives that $\langle v^{-1}E\rangle\leqq\langle w^{-1}v^{-1}E\rangle\vee\langle v^{-1}F\rangle$, and the result follows.

For $0\leqq k\leqq m+1\leqq\infty$, there are associative BP-module spectra $BP[k, m+1)$ whose homotopy are $BP[k, m+1)_*=BP_*/(p, v_1, \cdots, v_{k-1}, v_{m+1}, v_{m+2}, \cdots)$. In convention we write $BP[n, \infty)=P(n)$, $BP[0, n+1)=BP\langle n\rangle$ and $BP[n, n+1)=k(n)$ (see [6]). In particular, $P(0)=BP\langle\infty\rangle=BP$, $BP\langle 0\rangle=k(0)=HZ_{(p)}$ and $P(\infty)=BP\langle -1\rangle=k(-1)=HZ/p$. Multiplication by v_m gives cofiber sequences

$$
(1.2)\quad
\begin{aligned}
&\Sigma^{2(p^m-1)}BP[k, m+1)\xrightarrow{v_m} BP[k, m+1)\to BP[k, m)\\
&\Sigma^{2(p^m-1)}BP[m, n+1)\xrightarrow{v_m} BP[m, n+1)\to BP[m+1, n+1)
\end{aligned}
$$

of associative BP-module spectra.

When E is $P(n)$, $BP\langle n\rangle$ or $k(n)$, $v_n^{-1}E$ is denoted by $B(n)$, $E(n)$ or $K(n)$ respectively. Lemma 1.1 implies

(1.3) i) $\langle P(n)\rangle = \langle B(n)\rangle^{\vee}\langle P(n+1)\rangle$,

ii) $\langle BP\langle n\rangle\rangle = \vee_{0\leqq k\leqq n}\langle E(k)\rangle^{\vee}\langle HZ/p\rangle$, *and*

iii) $\langle k(n)\rangle = \langle K(n)\rangle^{\vee}\langle HZ/p\rangle$.

1.2. Let us denote by L_nE the localization of E with respect to the homology theory $(v_n^{-1}BP)_*(-)$, and by $\Sigma^{-n}N_{n+1}E$ the cofiber of the localization map $\eta_n\colon E\to L_nE$. Recall that

(1.4) $L_nE_\wedge X = L_n(E_\wedge X)$ *and* $N_{n+1}E_\wedge X = N_{n+1}(E_\wedge X)$

when E is an (associative) BP-module spectrum [18, Corollary 2.4]. The former gives

(1.5) $L_nE_*X = 0$ *if and only if* $v_n^{-1}BP_*(E_\wedge X) = 0$.

Lemma 1.2. *Let E and F be BP-module spectra.*

i) $\langle L_nE\rangle = \vee_{0\leqq k\leqq n}\langle v_k^{-1}E\rangle$ *when E is weak associative, and*

ii) $\langle L_nE\rangle \geqq \langle L_nF\rangle$ *if* $\langle E\rangle \geqq \langle F\rangle$.

Proof. i) Suppose that $L_nE_*X=0$, thus $v_n^{-1}BP_*(E_\wedge X)=0$. Then $v_{n-1}^{-1}BP_*(E_\wedge X)=0$ by means of [7, Theorem 0.1], and moreover $(v_n^{-1}E)_*X\cong v_n^{-1}E_*X=0$ since $\mu_*\colon BP_*(E_\wedge X)\to E_*X$ is epic. This shows that $\langle L_nE\rangle\geqq \langle L_{n-1}E\rangle^{\vee}\langle v_n^{-1}E\rangle$, and hence $\langle L_nE\rangle\geqq \vee_{0\leqq k\leqq n}\langle v_k^{-1}E\rangle$. For showing the opposite inequality we suppose that $v_k^{-1}E_*X=0$ for all k, $0\leqq k\leqq n$. Then it follows from [18, (2.3)] that $v_n^{-1}BP_*(E_\wedge X)\cong v_n^{-1}BP_*(v_n^{-1}E_\wedge X)=0$. So the equality holds.

ii) is immediate by use of (1.5).

Given an invariant regular ideal $J=(q_0, q_1, \cdots, q_{m-1})$ in BP_* of length m there is an associative BP-module spectrum BPJ whose homotopy is $BPJ_*=BP_*/(q_0, q_1, \cdots, q_{m-1})$. When J is $I_m=(p, v_1, \cdots, v_{m-1})$, BPJ is just $P(m)$. [7, Proposition 2.5] says that $\langle v_k^{-1}BPJ\rangle\geqq\langle v_{k-1}^{-1}BPJ\rangle$. So Lemma 1.2 i) implies

Corollary 1.3. $\langle L_nBPJ\rangle=\langle v_n^{-1}BPJ\rangle$ *for any invariant regular ideal J in* BP_*. *In particular,* $\langle L_nBP\rangle=\langle v_n^{-1}BP\rangle$ *and* $\langle L_nP(n)\rangle=\langle B(n)\rangle$.

Proposition 1.4. *Let J be an invariant regular ideal in* BP_* *of length m and* $m\leqq n$. *Then* $\langle BPJ\rangle=\langle N_mBP\rangle$ *and* $\langle L_nBPJ\rangle=\vee_{m\leqq k\leqq n}\langle B(k)\rangle$. *In particular,* $\langle P(n)\rangle=\langle N_nBP\rangle$ *and* $\langle L_nBP\rangle=\vee_{0\leqq k\leqq n}\langle B(k)\rangle$.

Proof. For $J=(q_0, q_1, \cdots, q_{m-1})$ we set $J_k=(q_0, q_1, \cdots, q_{k-1})$, $k\leqq m$. Consider the cofiber sequence $\Sigma^dN_mBPJ_{k-1}\xrightarrow{q_{k-1}}N_mBPJ_{k-1}\to N_mBPJ_k$ where $d=\dim$

q_{k-1}. N_mBPJ_{k-1*} is v_i-torsion for any i, $0\leqq i\leqq m-1$, and q_{k-1} is contained in the ideal $I_m=(p, v_1, \cdots, v_{m-1})$ which is just the radical of the ideal J [11, Theorem 1]. Therefore N_mBPJ_{k-1*} is q_{k-1}-torsion. Hence Lemma 1.1 implies that $\langle N_mBPJ_{k-1}\rangle=\langle N_mBPJ_k\rangle$ for each k, $1\leqq k\leqq m$, and so $\langle N_mBP\rangle=\langle N_mBPJ\rangle =\langle BPJ\rangle$.

Next, consider the cofiber sequence $\Sigma^{2(p^m-1)}L_nP(m)\xrightarrow{v_m}L_nP(m)\to L_nP(m+1)$. Applying Lemma 1.2 i) to $E=L_nP(m)$ we obtain that $\langle v_m^{-1}L_nP(m)\rangle=\langle L_mL_nP(m)\rangle=\langle L_mP(m)\rangle=\langle B(m)\rangle$. So Lemma 1.1 gives that $\langle L_nP(m)\rangle=\langle B(m)\rangle \vee\langle L_nP(m+1)\rangle$, and hence $\langle L_nP(m)\rangle=\bigvee_{m\leqq k\leqq n}\langle B(k)\rangle$. This result means that $\langle L_nBPJ\rangle=\bigvee_{m\leqq k\leqq n}\langle B(k)\rangle$ since $\langle BPJ\rangle=\langle P(m)\rangle$.

Setting $M_nE=L_nN_nE$ we have cofiber sequences

$$(1.6)\qquad \begin{array}{l} N_nE\to M_nE\to N_{n+1}E \\ \Sigma^{-n}M_nE\to L_nE\to L_{n-1}E \end{array}$$

(see [13]). Combining Proposition 1.4 with Lemma 1.2 ii) and Corollary 1.3 we get

Corollary 1.5. $\langle M_nBP\rangle=\langle B(n)\rangle$.

By putting Corollary 1.3 and Proposition 1.4 together we obtain the equality $\langle v_n^{-1}BP\rangle=\bigvee_{0\leqq i\leqq n}\langle B(i)\rangle$. This shows especially that $v_n^{-1}BP_*X=0$ implies $B(n)_*X=0$. In [13, Theorem 2.11] Ravenel proved that the converse is true under the finiteness restriction on X. We here give a simple proof of this result.

Proposition 1.6 (Ravenel). *Assume that X is a finite CW-spectrum. If $B(n)_*X=0$, then $v_n^{-1}BP_*X=0$.*

Proof. It is sufficient to show that $B(n-1)_*X=0$ if $B(n)_*X=0$. By Landweber's invariant prime filtration theorem [9] (or [18]) there is a finite filtration $P(n-1)_*X=M_s\supset M_{s-1}\supset \cdots \supset M_1\supset M_0=\{0\}$ consisting of $P(n-1)_*P(n-1)$-comodules so that for $1\leqq k\leqq s$ each subquotient M_k/M_{k-1} is stably isomorphic to $P(m_k)_*$ for some $m_k\geqq n-1$. By induction on $k\leqq s$ we will show that $v_n^{-1}M_k \cong v_n^{-1}P(n-1)_*X$ under the hypothesis that $B(n)_*X=0$. The $k=s$ is trivial, so we assume that $v_n^{-1}M_{k+1}\cong v_n^{-1}P(n-1)_*X$. Our hypothesis implies that $v_n^{-1}P(n-1)_*X$ is uniquely v_{n-1}-divisible, and hence $v_n^{-1}(M_{k+1}/M_k)$ is v_{n-1}-divisible. Then we find that $m_{k+1}\geqq n+1$, and so $v_n^{-1}M_k\cong v_n^{-1}M_{k+1}$. Consequently we see that $v_n^{-1}P(n-1)_*X=0$, which implies that $B(n-1)_*X=0$ by use of [7, Proposition 2.5]. Thus $B(n)_*X=0$ implies $B(n-1)_*X=0$ as desired.

1.3. Let J be an invariant regular ideal in BP_*. A BPJ-module spectrum E is said to be (*weak*) *quasi-associative* if it admits a pairing $\mu: BPJ\wedge E\to E$

with unit making the diagram below (weak) homotopy commutative

$$\begin{tikzcd}
BP_\wedge BPJ_\wedge E \arrow[r, "1_\wedge\mu"] \arrow[dd, "\phi_\wedge 1"'] & BP_\wedge E \arrow[r, "j_\wedge 1"] & BPJ_\wedge E \arrow[dd, "\mu"'] & BPJ_\wedge BPJ_\wedge E \arrow[l, "1_\wedge\mu"'] & BPJ_\wedge BP_\wedge E \arrow[l, "1_\wedge j_\wedge 1"'] \arrow[d, "T_\wedge 1"] \\
 & & & & BP_\wedge BPJ_\wedge E \arrow[d, "\phi_\wedge 1"] \\
BPJ_\wedge E \arrow[rr, "\mu"'] & & E & & BPJ_\wedge E \arrow[ll, "\mu"]
\end{tikzcd}$$

where ϕ denotes the BP-module structure map of BPJ and $j=j_{m-1}\cdots j_0$: $BP=BPJ_0\to BPJ_1\to\cdots\to BPJ_m=BPJ$ (cf., [7, Remark 5.3]).

A BPJ_*-module M is said to be $\mathcal{BPJ}$-*flat* if the functor $M\underset{BP_*}{\otimes}BPJ_*(-)$ is exact (see [10] or [18]). Recall that a $P(m)_*$-module M is $\mathcal{P}(m)$-flat if and only if

(1.7) *multiplication by v_k is monic on $M\underset{BP_*}{\otimes}P(k)_*$ for every $k\geqq m$.*

The following result is a useful tool in determining Bousfield classes of BP-related spectra.

Lemma 1.7. *Let J be an invariant regular ideal in BP_* of length m, E be a (weak) quasi-associative BPJ-module spectrum and $n\geqq m$.*
i) *If $v_n^{-1}E_*$ is $\mathcal{BPJ}$-flat such that $v_n^{-1}E_*\underset{BP_*}{\otimes}P(n)_*\neq 0$, then $\langle v_n^{-1}E\rangle=\langle v_n^{-1}P(m)\rangle$.*
ii) *If E_* is $\mathcal{BPJ}$-flat such that $E_*\underset{BP_*}{\otimes}P(\infty)_*\neq 0$, then $\langle E\rangle=\langle P(m)\rangle$.*

Proof. i) The BPJ-module structure map of E gives an isomorphism $v_n^{-1}E_*\underset{BP_*}{\otimes}BPJ_*X\to(v_n^{-1}E)_*X$. By making use of [18, Proposition 2.6] we observe that $(v_n^{-1}E)_*X=0$ if and only if $v_n^{-1}BPJ_*X=0$, thus $\langle v_n^{-1}E\rangle=\langle v_n^{-1}BPJ\rangle$. On the other hand, $\langle v_n^{-1}BPJ\rangle=\langle v_n^{-1}P(m)\rangle$ by putting Corollary 1.3 and Proposition 1.4 together. So the result follows.

ii) Obviously $\langle BPJ\rangle\geqq\langle E\rangle$ since E is a BPJ-module spectrum. Suppose that $BPJ_*X\neq 0$ and choose a non-zero primitive element x in BPJ_*X. The annihilator ideal $\mathrm{Ann}(x)=\{\lambda\in BP_*;\ \lambda\cdot x=0\}$ is at least contained in the ideal $I_\infty=(p, v_1, \cdots, v_n, \cdots)$ because the radical $\sqrt{\mathrm{Ann}(x)}=\{\lambda\in BP_*;\ \lambda^k\cdot x=0$ for some $k\}$ is the ideal $I_n=(p, v_1, \cdots, v_{n-1})$ for a certain n, $m\leqq n\leqq\infty$ (see [11] or [18]). So our hypothesis implies that $E_*\underset{BP_*}{\otimes}BP_*/\mathrm{Ann}(x)\neq 0$. On the other hand, there is a monomorphism $E_*\underset{BP_*}{\otimes}BP_*/\mathrm{Ann}(x)\to E_*\underset{BP_*}{\otimes}BPJ_*X\cong E_*X$. Hence it is obvious that $E_*X\neq 0$. Consequently $\langle BPJ\rangle\leqq\langle E\rangle$. The result is now immediate from Proposition 1.4.

According to [15] $BP[k, m+1)$ is a quasi-associative $P(k)$-module spectrum, and so $v_n^{-1}BP[k, m+1)$ becomes a weak quasi-associative one. Note that $v_n^{-1}BP[k, m+1)_*$ is $P(k)$-flat. So Lemma 1.7 i) implies

Corollary 1.8. *$\langle v_n^{-1}BP[k, m+1)\rangle=\langle v_n^{-1}P(k)\rangle$ for $0\leqq k\leqq n\leqq m\leqq\infty$, and in particular $\langle v_n^{-1}BP\rangle=\langle E(n)\rangle$ and $\langle B(n)\rangle=\langle K(n)\rangle$.*

Theorem 0.1 is obtained as a summary of (1.3), Proposition 1.4 and Corollaries 1.3, 1.5 and 1.8.

2. Bousfield classes of Anderson dual spectra

2.1.. Given a p-local CW-spectrum E we can construct a universal coefficient sequence

(2.1) $0\to \mathrm{Ext}(E_{*-1}X, Z_{(p)})\to \nabla E^*X\to \mathrm{Hom}(E_*X, Z_{(p)})\to 0$

(see [5] or [16]). The p-local CW-spectrum ∇E has the same homotopy type as the function spectrum $F(E, \nabla SZ_{(p)})$. Therefore this Anderson duality functor ∇ is categorical and exact. Note that $HZ_{(p)}$, HZ/p and $K(n)$ are self-dual, i.e., $\nabla HZ_{(p)}=HZ_{(p)}$, $\nabla HZ/p=\Sigma^{-1}HZ/p$ and $\nabla K(n)=\Sigma^{-1}K(n)$ for every $n\geqq 0$. Moreover we notice that

(2.2) $E=\nabla\nabla E$ *if E is of finite type.*

Let E be a BP-module spectrum which is connective. The BP-module spectrum ∇E is then coconnective. By dimension reason ∇E_* is v-torsion for all v in BP_* with $\dim v>0$. This means

(2.3) $\langle v^{-1}\nabla E\rangle=0$ *if E is connective and* $\dim v>0$.

Apply the duality functor ∇ to the cofiber sequences (1.2) and use Lemma 1.1 and (2.3). Then we have

Proposition 2.1. i) *$\langle\nabla BP\rangle=\langle\nabla P(1)\rangle\vee\langle SQ\rangle$ and $\langle\nabla P(n)\rangle=\langle\nabla P(1)\rangle$ for each $n\geqq 1$,*
ii) *$\langle\nabla BP\langle n\rangle\rangle=\langle HZ_{(p)}\rangle=\langle HZ/p\rangle\vee\langle SQ\rangle$ for each $n\geqq 0$, and*
iii) *$\langle\nabla BP[k, m+1)\rangle=\langle HZ/p\rangle$ for $1\leqq k\leqq m<\infty$, and in particular $\langle\nabla k(n)\rangle=\langle HZ/p\rangle$ for each $n\geqq 1$.*

Let E be a coconnective CW-spectrum. It is represented as the direct limit of the Postnikov systems $E(-n, \infty)$. This fact gives

(2.4) $\langle E\rangle\leqq\langle HZ_{(p)}\rangle$, *and moreover $\langle E\rangle\leqq\langle HZ/p\rangle$ if $E_*\otimes Q=0$.*

Remark that $P(1)^*HZ/p=0$, because HZ/p is dissonant and $P(1)$ is harmonic [13]. This is equivalent to say that

(2.5) $\nabla P(1)_*HZ/p=0$

(use (3.2)). By use of (2.5) we notice

(2.6) i) $\langle HZ/p\rangle \gneqq \langle \nabla P(1)\rangle$, *and*

ii) $\langle P(n)\rangle \gneqq \vee_{i\geq n}\langle K(i)\rangle \vee \langle HZ/p\rangle$.

2.2. The cofiber sequence $\Sigma^{2(p^k-1)}P(k) \xrightarrow{v_k} P(k) \to P(k+1)$ induces short exact sequences

$$0 \to \nabla M_n P(k)_* \xrightarrow{v_k} \nabla M_n P(k)_* \to \nabla M_n P(k+1)_* \to 0$$

$$0 \to \nabla N_{n+1} P(k)_* \xrightarrow{v_k} \nabla N_{n+1} P(k)_* \to \nabla N_{n+1} P(k+1)_* \to 0$$

for each k, $0\leq k\leq n$. Hence we observe

(2.7) $\nabla M_n BP_*$ *and* $v_n^{-1}\nabla N_{n+1}BP_*$ *are both* $\mathcal{BP}$*-flat.*

Proposition 2.2. i) $\langle \nabla M_n BP\rangle = \langle \nabla L_n BP\rangle = \langle v_n^{-1}BP\rangle$, *and*
ii) $\langle \nabla N_{n+1}BP\rangle = \langle v_n^{-1}BP\rangle \vee \langle \nabla P(1)\rangle$.

Proof. i) Use Lemma 1.7 i) and (2.7) to show that $\langle \nabla M_n BP\rangle = \langle v_n^{-1} \nabla N_{n+1}BP\rangle = \langle v_n^{-1}BP\rangle$. We here consider the cofiber sequence $\Sigma^n \nabla N_{n+1}BP \to \nabla L_n BP \to \nabla BP$. Then $\langle v_n^{-1}\nabla L_n BP\rangle = \langle v_n^{-1}\nabla N_{n+1}BP\rangle$ because by (2.3) $\langle v_n^{-1} \nabla BP\rangle = 0$ for all $n\geq 1$ and $\nabla N_1 BP = (\nabla BP)\hat{Z}_p$. So we get that $\langle \nabla L_n BP\rangle \geq \langle v_n^{-1}\nabla L_n BP\rangle = \langle v_n^{-1}BP\rangle$. The opposite inequality is shown by induction on n, the $n=0$ case being trivial. Assume that $\langle \nabla L_{n-1}BP\rangle \geq \langle v_{n-1}^{-1}BP\rangle$ and consider the cofiber sequence $\nabla L_{n-1}BP \to \nabla L_n BP \to \Sigma^n \nabla M_n BP$. Then it is immediate that $\langle \nabla L_n BP\rangle \leq \langle \nabla L_{n-1}BP\rangle \vee \langle \nabla M_n BP\rangle = \langle v_n^{-1}BP\rangle$, and so $\langle \nabla L_n BP\rangle = \langle v_n^{-1}BP\rangle$.

ii) Obviously $\langle \nabla N_{n+1}BP\rangle \geq \langle v_n^{-1}\nabla N_{n+1}BP\rangle = \langle v_n^{-1}BP\rangle$. On the other hand, an iterated use of (1.1) gives that $\langle \nabla N_{n+1}BP\rangle \geq \langle \nabla N_{n+1}P(n+1)\rangle = \langle \nabla P(n+1)\rangle$. Hence we obtain that $\langle \nabla N_{n+1}BP\rangle \geq \langle v_n^{-1}BP\rangle \vee \langle \nabla P(1)\rangle$ by means of Proposition 2.1 i). Conversely it is immediately seen that $\langle \nabla N_{n+1}BP\rangle \leq \langle \nabla L_n BP\rangle \vee \langle \nabla BP\rangle = \langle v_n^{-1}BP\rangle \vee \langle \nabla P(1)\rangle$. So the equality holds.

We don't know whether the sequence (1.2) after localized at v_n remains still a cofiber sequence. But we have

Lemma 2.3. *The sequence* $\nabla v_n^{-1}BP[k+1, m+1) \to \nabla v_n^{-1}BP[k, m+1) \xrightarrow{v_k} \Sigma^{-2(p^k-1)}\nabla v_n^{-1}BP[k, m+1)$ *is a cofiber sequence for each* k, $0\leq k\leq n\leq m\leq\infty$.

Proof. The $k=0$ case is trivial because $BP[1, m+1) = BP\langle m\rangle Z/p$ and so $v_n^{-1}BP[1, m+1) = (v_n^{-1}BP\langle m\rangle)Z/p$. We may assume that $k\geq 1$. Then $v_n^{-1}BP[k, m+1)_*X$ is always a torsion group, and hence $\nabla v_n^{-1}BP[k, m+1)^*X \cong \mathrm{Ext}(v_n^{-1}BP[k, m+1)_*X, Z_{(p)})$. As is easily checked, the triangle

$$\begin{tikzcd}[column sep=tiny] \nabla v_n^{-1}BP[k, m+1)^*X \arrow[rr, "v_k"] & & \nabla v_n^{-1}BP[k, m+1)^*X \arrow[dl, "\delta"] \\ & \nabla v_n^{-1}BP[k+1, m+1)^*X \arrow[ul] & \end{tikzcd}$$

is exact. Moreover the right diagonal map δ is trivial when X is the sphere spectrum S. By using these facts the $k\geqq 1$ cases follow immediately from [17, Lemma A].

Proposition 2.4. *$\langle \nabla v_n^{-1}BP[k, m+1)\rangle = \vee_{k\leqq i\leqq n}\langle K(i)\rangle$ for $0\leqq k\leqq n\leqq m\leqq \infty$. In particular, $\langle \nabla v_n^{-1}BP\rangle = \langle \nabla E(n)\rangle = \langle v_n^{-1}BP\rangle$ and $\langle \nabla B(n)\rangle = \langle \nabla K(n)\rangle = \langle K(n)\rangle$.*

Proof. Lemma 1.1 combined with Lemma 2.3 shows that $\langle \nabla v_n^{-1}BP[k, m+1)\rangle = \langle v_k^{-1}\nabla v_n^{-1}BP[k, m+1)\rangle \vee \langle \nabla v_n^{-1}BP[k+1, m+1)\rangle$. Notice that $\nabla v_n^{-1}BP[k, m+1)$ is a quasi-associative $P(k)$-module spectrum and $v_k^{-1}\nabla v_n^{-1}BP[k, m+1)_*$ is $P(k)$-flat. Use Lemma 1.7 i) to see that $\langle v_k^{-1}\nabla v_n^{-1}BP[k, m+1)\rangle = \langle K(k)\rangle$. The result is now shown by induction on $k\leqq n$.

Obviously $\langle BP\rangle = \langle L_n BP\rangle \vee \langle N_{n+1}BP\rangle$. We use Corollary 1.3 and Proposition 1.4 to replace this equality by

(2.8) $\langle BP\rangle = \langle v_n^{-1}BP\rangle \vee \langle P(n+1)\rangle\,.$

From [19, (2.3)] it follows that $v_n^{-1}BP_*P(n+1)=0$ (see [13, Lemma 2.3]). But $v_n^{-1}BP_*P(n)\cong BP_*B(n)\neq 0$. So we remark

(2.9) $\langle v_n^{-1}BP\rangle \gneqq \langle v_{n-1}^{-1}BP\rangle$ and $\langle P(n)\rangle \gneqq \langle P(n+1)\rangle\,.$

Lemma 2.5. *$HZ/p_*\nabla N_{n+1}P(n)=0$ and $K(m)_*\nabla N_{n+1}P(n)=0$ for all $m<n$, but $K(n)_*\nabla N_{n+1}P(n)\neq 0$.*

Proof. Consider the cofiber sequence $\Sigma^{n+1}\nabla P(n+1)\to \nabla N_{n+1}P(n)\xrightarrow{v_n}\Sigma^{-2(p^n-1)}\nabla N_{n+1}P(n)$. There is an isomorphism $HZ/p_*\nabla N_{n+1}P(n)\to HZ/p_*v_n^{-1}\nabla N_{n+1}P(n)$ because $HZ/p_*\nabla P(n+1)=0$ by (2.5). Note that $\nabla N_{n+1}P(n)_*$ is v_k-torsion for each $k<n$. Then [19, (2.3)] gives that $HZ/p_*v_n^{-1}\nabla N_{n+1}P(n)\cong v_n^{-1}HZ/p_*v_n^{-1}\nabla N_{n+1}P(n)=0$ and also $v_{n-1}^{-1}BP_*\nabla N_{n+1}P(n)\cong v_{n-1}^{-1}BP_*v_{n-1}^{-1}\nabla N_{n+1}P(n)=0$. Hence $HZ/p_*\nabla N_{n+1}P(n)=0$ and $K(m)_*\nabla N_{n+1}P(n)=0$ for all $m<n$. However $v_n^{-1}BP_*\nabla N_{n+1}P(n)\neq 0$ because $v_n^{-1}\nabla N_{n+1}P(n)_*\cong v_n^{-1}\mathrm{Ext}(N_{n+1}P(n)_*, Z_{(p)})\neq 0$. Therefore we observe that $K(n)_*\nabla N_{n+1}P(n)\neq 0$.

By use of (1.3), (2.6), Proposition 2.2 and Lemma 2.6 we here verify

(2.10) $\langle BP\langle n\rangle\rangle \gneqq \langle BP\langle n-1\rangle\rangle$ *and* $\langle \nabla N_{n+1}BP\rangle \gneqq \langle \nabla N_n BP\rangle\,.$

3. Cohomological Bousfield classes

3.1. Let us denote by $\langle E\rangle^*$ the *cohomological Bousfield class* of E, thus $\langle E\rangle^*\geqq\langle F\rangle^*$ when $E^*X=0$ implies $F^*X=0$. Recall that the Anderson dual spectrum ∇E is related by the universal coefficient sequence (2.1). Then [2, Proposition 2.3] implies that for a p-local E, $\nabla E^*X=0$ if and only if $E_*X=0$. This means

(3.1) $\langle \nabla E\rangle^* = \langle E\rangle$.

Moreover we remark

(3.2) $\langle E\rangle^* = \langle \nabla E\rangle$ *if E is of finite type,*

because of (2.2). As the above E we can take BP, $P(n)$, $BP\langle n\rangle$, $k(n)$, $K(n)$ and so on.

Let E be a connective CW-spectrum. It is represented as the inverse limit of the Postnikov systems $E(-\infty. n)$. This fact yields

(3.3) $\langle E\rangle^* \leqq \langle HZ_{(p)}\rangle$, *and moreover* $\langle E\rangle^* \leqq \langle HZ/p\rangle$ *if* $\mathrm{Hom}(Q, E_*) = 0 = \mathrm{Ext}(Q, E_*)$.

Let E be a BP-module spectrum and v be an element of BP_* with dimension d. We can form a CW-spectrum $\varprojlim_v E$ defined to be the mapping cotelescope of the map $\mu(v\wedge 1)\colon \Sigma^d E\to E$. If E is associative, then $\varprojlim_v E$ is a BP-module spectrum. By dimension reason we see

(3.4) $\langle \varprojlim_v E\rangle^* = 0$ *if E is connective and* $\dim v>0$.

Let M be a BP_*-module and v be an element of BP_*. Denote by K and C the kernel and the cokernel of multiplication by v on M respectively. As is easily seen, $\varprojlim_v K=0=\varprojlim_v^1 K$ and $\varprojlim_v C=0=\varprojlim_v^1 C$. An easy diagram chasing shows

(3.5) $\varprojlim_v M$ *and* $\varprojlim_v^1 M$ *are both uniquely v-divisible.*

This gives

Lemma 3.1. *Let E be an associative BP-module spectrum and v be an element of BP_*. Then the BP_*-module $(\varprojlim_v E)_*$ is uniquely v-divisible.*

Similarly to (1.1) we have

(3.6) $\langle v^{-1}F\rangle^* \leqq \langle v^{-1}E\rangle^* \vee \langle v^{-1}G\rangle^*$

for any element v of BP_*, if $E\to F\to G$ is a cofiber sequence of BP-module spectra. By a parallel argument to Lemma 1.1 we can show

Lemma 3.2. *Let v and w be elements of BP_*, E be an associative BP-module spectrum and F be the cofiber of the map $w\colon \Sigma^d E\to E$ where $d=\dim w$. Then $\langle v^{-1}E\rangle^* = \langle \varprojlim_w v^{-1}E\rangle^* \vee \langle v^{-1}F\rangle^*$. In particular, $\langle E\rangle^* = \langle \varprojlim_w E\rangle^* \vee \langle F\rangle^*$.*

A BPJ_*-module M is said to be $\mathcal{BPJ}$-*injective* if the functor $\mathrm{Hom}_{BP_*}(BPJ_*(-), M)$ is exact (see [8] or [18]). Recall that a $P(m)_*$-module M is $\mathcal{P}(m)$-injective if

(3.7) *multiplication by v_k is epic on* $\mathrm{Hom}_{BP_*}(P(k)_*, M)$ *for every $k \geqq m$, and in addition* w. $\dim_{\mathcal{BP}(m)} M$ *is finite.*

Note that w. $\dim_{\mathcal{BP}(m)} v_n^{-1}M \leqq n-m$. As a dual of Lemma 1.7 we have a useful tool in determining some cohomological Bousfield classes.

Lemma 3.3. *Let J be an invariant regular ideal in BP_* of length m, E be a (weak) quasi-associative BPJ-module spectrum and $n \geqq m$.*
i) *If $v_n^{-1}E_*$ is $\mathcal{BPJ}$-injective such that* $\mathrm{Hom}_{BP_*}(P(n)_*, v_n^{-1}E_*) \neq 0$, *then* $\langle v_n^{-1}E \rangle^* = \langle v_n^{-1}P(m) \rangle$.
ii) *If E_* is $\mathcal{BPJ}$-injective such that* $\mathrm{Hom}_{BP_*}(P(\infty)_*, E_*) \neq 0$, *then* $\langle E \rangle^* = \langle P(m) \rangle$.

Proof. i) The BPJ-module structure map of E gives an isomorphism $(v_n^{-1}E)^*X \to \mathrm{Hom}_{BP_*}(BPJ_*X, v_n^{-1}E_*)$. So we use [18, Proposition 3.4] to show that $\langle v_n^{-1}E \rangle^* = \langle v_n^{-1}BPJ \rangle$. Now the result follows from the fact that $\langle v_n^{-1}BPJ \rangle = \langle v_n^{-1}P(m) \rangle$.

ii) is also proved by a parallel argument to the proof of Lemma 1.7 ii), so we omit it.

REMARK. In proving Lemma 3.3 i) the (weak) quasi-associativity of E is only needed to show that there is an isomorphism $(v_n^{-1}E)^*X \to \mathrm{Hom}_{BP_*}(BPJ_*X, v_n^{-1}E_*)$. We may assume instead the existence of such a natural isomorphism, if it is not easy to check whether E becomes a (weak) quasi-associative BPJ-module spectrum.

3.2. Consider the short exact sequences

$$0 \to M_nP(k+1)_* \to M_nP(k)_* \xrightarrow{v_k} M_nP(k)_* \to 0$$

$$0 \to N_{n+1}P(k+1)_* \to N_{n+1}P(k)_* \xrightarrow{v_k} N_{n+1}P(k)_* \to 0$$

for each k, $0 \leqq k \leqq n$. Since $\varprojlim^1_{v_n} N_{n+1}P(k+1)_* = 0$, we see easily

(3.8) *M_nBP_* and $\varprojlim_{v_n} N_{n+1}BP_*$ are both $\mathcal{BP}$-injective.*

As an analogous result to Proposition 2.2 we have

Proposition 3.4. i) $\langle M_nBP \rangle^* = \langle L_nBP \rangle^* = \langle v_n^{-1}BP \rangle$, *and*
ii) $\langle N_{n+1}BP \rangle^* = \langle v_n^{-1}BP \rangle \vee \langle \nabla P(1) \rangle$.

Proof. i) First apply Lemma 3.3 i) to $E = M_nBP$ to obtain that $\langle M_nBP \rangle^* = \langle v_n^{-1}BP \rangle$. Note that $(\varprojlim_{v_n} N_{n+1}BP)_* \cong \varprojlim_{v_n} N_{n+1}BP_*$. The BP-module structure map of $N_{n+1}BP$ gives a natural homomorphism $(\varprojlim_{v_n} N_{n+1}BP)^*X \to \mathrm{Hom}_{BP_*}(BP_*X, \varprojlim_{v_n} N_{n+1}BP_*)$, which becomes an isomorphism. So $\langle \varprojlim_{v_n} N_{n+1}BP \rangle^* = \langle v_n^{-1}BP \rangle$ by using Remark following Lemma 3.3. We then observe that $\langle L_nBP\rangle^* \geqq \langle \varprojlim_{v_n} L_nBP\rangle^* = \langle \varprojlim_{v_n} N_{n+1}BP\rangle^* = \langle v_n^{-1}BP\rangle$ by use of (3.4). The opposite inequality is easily shown by induction on n.

ii) Obviously $\langle N_{n+1}BP\rangle^* \geqq \langle N_{n+1}P(n+1)\rangle^* = \langle P(n+1)\rangle^*$. Then it follows from (3.2) and Proposition 2.1 i) that $\langle N_{n+1}BP\rangle^* \geqq \langle \varprojlim_{v_n} N_{n+1}BP\rangle^{*\vee} \langle P(n+1)\rangle^* = \langle v_n^{-1}BP\rangle^\vee \langle \nabla P(1)\rangle$. The converse is easily seen because $\langle N_{n+1}BP\rangle^* \leqq \langle L_nBP\rangle^{*\vee}\langle BP\rangle^*$.

As an analogous result to Proposition 2.4 we have

Proposition 3.5. *$\langle v_n^{-1}BP[k,m+1)\rangle^* = \vee_{k\leqq i\leqq n}\langle K(i)\rangle$ for $0\leqq k\leqq n\leqq m\leqq\infty$. In particular, $\langle v_n^{-1}BP\rangle^* = \langle E(n)\rangle^* = \langle v_n^{-1}BP\rangle$ and $\langle B(n)\rangle^* = \langle K(n)\rangle^* = \langle K(n)\rangle$.*

Proof. Assume that $k<n$. Note that $(\varprojlim_{v_k} v_n^{-1}BP[k,\ m+1))_* \cong \varprojlim^1_{v_k} v_n^{-1}BP[k,\ m+1)_*$ and it is $P(k)$-flat and $P(k)$-injective because of (3.5). Since the $P(k)$-module spectrum $v_n^{-1}BP[k,\ m+1)$ is weak quasi-associative, its structure map gives a natural homomorphism $(\varprojlim^1_{v_k} v_n^{-1}BP[k,\ m+1)_*)\underset{BP_*}{\otimes} P(k)_*X \to (\varprojlim_{v_k} v_n^{-1}BP[k,\ m+1))_*X$ which is an isomorphism. Therefore we can define a natural homomorphism $(\varprojlim_{v_k} v_n^{-1}BP[k,\ m+1))^*X \to \mathrm{Hom}_{BP_*}(P(k)_*X,\ \varprojlim^1_{v_k} v_n^{-1}BP[k,\ m+1)_*)$ in the canonical way. This is an isomorphism, too. Hence $\langle \varprojlim_{v_k} v_n^{-1}BP[k,\ m+1)\rangle^* = \langle B(k)\rangle$ by means of Remark following Lemma 3.3. Now induction on $k\leqq n$ shows the desired equality, the $k=n$ case being trivial by use of Lemma 3.3 i).

Theorem 0.2 follows from Propositions 2.1, 2.2, 2.4, 3.4 and 3.5, and (3.2).

4. Bousfield classes of sums and products

4.1. Fix $m\geqq 0$ and take a family $\{E_n\}_{n\geqq m}$ of CW-spectra. Trivially we have

(4.1) $\langle \vee_{n\geqq m}E_n\rangle = \vee_{n\geqq m}\langle E_n\rangle$ *and* $\langle \Pi_{n\geqq m}E_n\rangle^* = \vee_{n\geqq m}\langle E_n\rangle^*$.

Denote by C, D and F the cofibers of the maps $\vee_{n\geqq m}E_n \to \Pi_{n\geqq m}E_n$, $\vee_{n\geqq m}\nabla E_n \to \Pi_{n\geqq m}\nabla E_n = \nabla(\vee_{n\geqq m}E_n)$ and $\vee_{n\geqq m}\nabla E_n \to \nabla(\Pi_{n\geqq m}E_n)$ respectively. By Verdier's lemma we have a cofiber sequence $\nabla C\to F\to D$. As is easily seen,

(4.2) i) $\langle \Pi_{n\geqq m}E_n\rangle = \langle \nabla(\Pi_{n\geqq m}E_n)\rangle^* = \vee_{n\geqq m}\langle E_n\rangle^\vee\langle C\rangle$,
ii) $\langle \vee_{n\geqq m}E_n\rangle^* = \vee_{n\geqq m}\langle E_n\rangle^{*\vee}\langle C\rangle^*$,
iii) $\langle \Pi_{n\geqq m}\nabla E_n\rangle = \vee_{n\geqq m}\langle \nabla E_n\rangle^\vee\langle D\rangle$,
iv) $\langle \vee_{n\geqq m}\nabla E_n\rangle^* = \vee_{n\geqq m}\langle E_n\rangle^\vee\langle D\rangle^*$, *and*
v) $\langle \nabla(\Pi_{n\geqq m}E_n)\rangle = \vee_{n\geqq m}\langle \nabla E_n\rangle^\vee\langle F\rangle$.

Consider the families $\{P(n)\}_{n\geqq m}$, $\{k(n)\}_{n\geqq m}$ and $\{K(n)\}_{n\geqq m}$ of BP-module spectra. Then $C=\Pi_{n\geq m}P(n)/\vee_{n\geq m}P(n)$, $\Pi_{n\geq m}k(n)/\vee_{n\geq m}k(n)$ or $\Pi_{n\geq m}K(n)/\vee_{n\geq m}K(n)$ is the Eilenberg-MacLane spectrum HA of type A where $A=\Pi_{\chi_0}Z/p/\oplus_{\chi_0}Z/p$, and hence $\nabla C=\Sigma^{-1}H\nabla A$ where $\nabla A=\mathrm{Ext}(A, Z_{(p)})$. Moreover $D=\Sigma^{-1}H\mathcal{A}$ and so $F=\Sigma^{-1}H(A\oplus\nabla A)$.

Proposition 4.1. i) $\langle\Pi_{n\geq m}P(n)\rangle=\langle P(m)\rangle$,
ii) $\langle\vee_{n\geq m}P(n)\rangle^*=\langle\Pi_{n\geq m}\nabla P(n)\rangle=\langle\vee_{n\geq m}\nabla P(n)\rangle^*=\langle\nabla(\Pi_{n\geq m}P(n))\rangle=\langle\vee_{n\geq m}k(n)\rangle^*=\langle\Pi_{n\geq m}\nabla k(n)\rangle=\langle\vee_{n\geq m}\nabla k(n)\rangle^*=\langle\nabla(\Pi_{n\geq m}k(n))\rangle=\begin{cases}\langle HZ_{(p)}\rangle & \textit{if } m=0\\ \langle HZ/p\rangle & \textit{if } m\geq 1,\end{cases}$ *and*
iii) $\langle\Pi_{n\geq m}k(n)\rangle=\langle\Pi_{n\geq m}K(n)\rangle=\langle\vee_{n\geq m}K(n)\rangle^*=\langle\nabla(\Pi_{n\geq m}K(n))\rangle=\vee_{n\geq m}\langle K(n)\rangle\vee\langle HZ/p\rangle$.

Proof. i) is easy.

ii) and iii): From [2, Theorem 4.5] it follows that $\langle HA\rangle=\langle H(A\oplus\nabla A)\rangle=\langle HA\rangle^*=\langle H(A\oplus\nabla A)\rangle^*=\langle HZ/p\rangle$ because A is a Z/p-module. So the results are shown easily by (2.6) and (4.2).

Next, consider the family $\{BP\langle n\rangle\}_{n\geq m}$ of BP-module spectra. $C=\Pi_{n\geq m}B\langle n\rangle/\vee_{n\geq m}B\langle n\rangle$ is connective, and hence ∇C is coconnective. Also, D and F are coconnective.

Proposition 4.2. i) $\langle\Pi_{n\geq m}BP\langle n\rangle\rangle=\langle\Pi_{n\geq m}BP\langle n\rangle/\vee_{n\geq m}BP\langle n\rangle\rangle=\langle\vee_{n\geq m}\nabla BP\langle n\rangle\rangle^*=\langle BP\rangle$, *and*
ii) $\langle\vee_{n\geq m}BP\langle n\rangle\rangle^*=\langle\Pi_{n\geq m}\nabla BP\langle n\rangle\rangle=\langle\nabla(\Pi_{n\geq m}BP\langle n\rangle)\rangle=\langle HZ_{(p)}\rangle$.

Proof. i) Since $BP=\varprojlim_{n\geq m}BP\langle n\rangle$ and $\nabla BP=\varinjlim_{n\geq m}\nabla BP\langle n\rangle$, it is immediate that $\langle\Pi_{n\geq m}BP\langle n\rangle\rangle=\langle\vee_{n\geq m}\nabla BP\langle n\rangle\rangle^*=\langle BP\rangle$. On the other hand, we remark that $\Pi_{n\geq m}BP\langle n\rangle_*/\oplus_{n\geq m}BP\langle n\rangle_*$ is $\mathcal{BP}$-flat. Use Lemma 1.7 ii) to show the remaining equality.

ii) Since C is connective and both D and F are coconnective, $\langle C\rangle^*\vee\langle D\rangle\vee\langle F\rangle\leq\langle HZ_{(p)}\rangle$ by use of (2.4) and (3.3). So the desired equalities follow from Proposition 2.1 ii) and (4.2).

4.2. Let E be a p-local CW-spectrum of finite type and $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of $Z_{(p)}$-modules. Then [16, Lemma 4] gives that $E(\Pi_{\alpha\in\Lambda}A_\alpha)=\Pi_{\alpha\in\Lambda}EA_\alpha$. By use of [2, Proposition 2.3] we observe

$$(4.3)\quad \langle\Pi_{\alpha\in\Lambda}EA_\alpha\rangle=\begin{cases}\langle EZ/p\rangle & \textit{if } (\Pi_{\alpha\in\Lambda}A_\alpha)\otimes Q=0\\ \langle E\rangle & \textit{if not so}\,.\end{cases}$$

In particular,

(4.4) $\langle\Pi_\Lambda E\rangle=\langle E\rangle$ *for any indexed set* Λ.

Lemma 4.3. *Let E be a p-local CW-spectrum of finite type and A be a*

$Z_{(p)}$-module. Then

$$\langle\nabla(EA)\rangle=\begin{cases}\langle(\nabla E)Z/p\rangle & \text{if } \mathrm{Hom}(A, Z_{(p)})=0=\mathrm{Ext}(A, Z_{(p)})\otimes Q\\ \langle\nabla E\rangle & \text{if not so}.\end{cases}$$

In particular, $\langle\nabla(\Pi_\Lambda E)\rangle=\langle\nabla E\rangle$ for any indexed set Λ.

Proof. Take a free resolution $0\rightarrow F_1\rightarrow F_0\rightarrow A\rightarrow 0$ of $Z_{(p)}$-modules and put $P_i=\mathrm{Hom}(F_i, Z_{(p)})$ for $i=0, 1$. From [16, Lemma 4] it follows that $\nabla(EF_i)=(\nabla E)P_i$ for $i=0, 1$. Let T denote the cofiber of the map $SP_0\rightarrow SP_1$ of Moore spectra. Applying Five lemma we obtain immediately that $\nabla(EA)=\Sigma^{-1}\nabla E_\wedge T$. As is easily seen, $B\otimes P_0\cong B\otimes P_1$ if and only if $B\otimes A_H=0$ and $B\otimes A_E=0=\mathrm{Tor}(B, A_E)$ for a $Z_{(p)}$-module B where $A_H=\mathrm{Hom}(A, Z_{(p)})$ and $A_E=\mathrm{Ext}(A, Z_{(p)})$. This shows that $\nabla(EA)_*X=0$ if and only if $(\nabla E)A_{H*}X=0=(\nabla E)A_{E*}X$, thus $\langle\nabla(EA)\rangle=\langle(\nabla E)A_H\rangle^\vee\langle(\nabla E)A_E\rangle$. Now the result follows from [2, Proposition 2.3].

To $E=P(n)$, $\nabla P(n)$, $BP\langle n\rangle$, $\nabla BP\langle n\rangle$, $k(n)$, $\nabla k(n)$ and $K(n)$ we may apply (4.4) and Lemma 4.3.

By Corollary 1.8 and Proposition 2.4 $\langle B(n)\rangle=\langle\nabla B(n)\rangle=\langle K(n)\rangle$. Corresponding to this result we have

Proposition 4.4. i) *$\langle\Pi_{n\geq m}B(n)\rangle=\langle\Pi_{n\geq m}\nabla B(n)\rangle=\langle\nabla(\Pi_{n\geq m}B(n))\rangle=\vee_{n\geq m}\langle K(n)\rangle^\vee\langle HZ/p\rangle$, and*
ii) *$\langle\Pi_\Lambda B(n)\rangle=\langle\Pi_\Lambda\nabla B(n)\rangle=\langle\nabla(\Pi_\Lambda B(n))\rangle=\langle K(n)\rangle$.*

Proof. i) $\langle\Pi_{n\geq m}B(n)\rangle\geqq\langle\Pi_{n\geq m}K(n)\rangle$ since $\Pi_{n\geq m}K(n)$ is a $\Pi_{n\geq m}B(n)$-module spectrum. According to [7, Theorem 4.10] the Boardmann map $B(n)\rightarrow K(n)_\wedge B(n)$ induces a Hurewicz monomorphism $B(n)_*X\rightarrow K(n)_*(B(n)_\wedge X)$. This implies that $(\Pi_{n\geq m}B(n))_*Y\rightarrow(\Pi_{n\geq m}K(n)_\wedge B(n))_*Y$ is a monomorphism for any finite Y, and for a general Y when passing to the direct limit. Hence we get a monomorphism $(\Pi_{n\geq m}B(n))_*X\rightarrow(\Pi_{n\geq m}K(n))_*((\Pi_{n\geq m}B(n))_\wedge X)$ and so $\langle\Pi_{n\geq m}B(n)\rangle\leqq\langle\Pi_{n\geq m}K(n)\rangle$. Thus the equality $\langle\Pi_{n\geq m}B(n)\rangle=\langle\Pi_{n\geq m}K(n)\rangle$ holds.

Since $\nabla(\Pi_{n\geq m}B(n))$ and $\Pi_{n\geq m}\nabla B(n)$ are both $\Pi_{n\geq m}B(n)$-module spectra, we see that $\langle\nabla(\Pi_{n\geq m}B(n))\rangle\leqq\langle\Pi_{n\geq m}B(n)\rangle$ and $\langle\Pi_{n\geq m}\nabla B(n)\rangle\leqq\langle\Pi_{n\geq m}B(n)\rangle$. Note that $K(n)_*$, $B(n)_*$ and $\nabla B(n)_*$ are all $\rho(n)$-flat. Obviously $B(n)_*X\rightarrow K(n)_*X$ is epic and $K(n)_*X\rightarrow\nabla B(n)_*X$ is monic. Using the preceding argument we can show that $\nabla(\Pi_{n\geq m}K(n))_*X\rightarrow\nabla(\Pi_{n\geq m}B(n))_*X$ and $(\Pi_{n\geq m}K(n))_*X\rightarrow(\Pi_{n\geq m}\nabla B(n))_*X$ are both monic for a general X. This implies that $\langle\nabla(\Pi_{n\geq m}B(n))\rangle\geqq\langle\nabla(\Pi_{n\geq m}K(n))\rangle=\langle\Pi_{n\geq m}K(n)\rangle$ and $\langle\Pi_{n\geq m}\nabla B(n)\rangle\geqq\langle\Pi_{n\geq m}K(n)\rangle$. The desired result is now immediate from Proposition 4.1 iii).

ii) Use Lemma 1.7 i).

4.3. By Corollary 1.8 $\langle v_n^{-1}BP\rangle=\langle E(n)\rangle$. Corresponding to this result we have

Proposition 4.5. i) $\langle\Pi_{n\geq m}v_n^{-1}BP\rangle=\langle\Pi_{n\geq m}E(n)\rangle=\langle BP\rangle$, *and*
ii) $\langle\Pi_\Lambda v_n^{-1}BP\rangle=\langle\Pi_\Lambda E(n)\rangle=\vee_{0\leq i\leq n}\langle K(i)\rangle$.

Proof. i) $\Pi_{n\geq m}v_n^{-1}BP_*$ and $\Pi_{n\geq m}E(n)_*$ are both $\mathcal{BP}$-flat. We have an isomorphism $(\Pi_{n\geq m}E(n)_*)\underset{BP_*}{\otimes}P(k)_*\to\Pi_{n\geq k}v_n^{-1}BP[k,n+1)_*$ since $P(k)_*$ is finitely presented as a BP_*-module for each k, $m\leqq k<\infty$. So it is clear that $(\Pi_{n\geq m}E(n)_*)\underset{BP_*}{\otimes}P(\infty)_*\neq 0$ and $(\Pi_{n\geq m}v_n^{-1}BP_*)\underset{BP_*}{\otimes}P(\infty)_*\neq 0$. Apply Lemma 1.7 ii) to obtain the desired equalities.

ii) Apply Lemma 1.7 i).

Proposition 4.6. i) $\langle\Pi_{n\geq m}L_nBP\rangle=\langle\Pi_{n\geq m}N_nBP\rangle=\langle BP\rangle$, *and*
ii) $\langle\Pi_\Lambda M_nBP\rangle=\langle\Pi_\Lambda L_nBP\rangle=\vee_{0\leq i\leq n}\langle K(i)\rangle$ *and* $\langle\Pi_\Lambda N_nBP\rangle=\langle BP\rangle$ *if the indexed set* Λ *is infinite.*

Proof. i) [19, Theorem 4.8] says that BP is s-harmonic, thus $BP=\varprojlim_{n\geq m}L_nBP$. Hence it is obvious that $\langle\Pi_{n\geq m}L_nBP\rangle=\langle BP\rangle$. Next, consider the cofiber sequence $\Pi_{n\geq m}N_nP(k+1)\to\Pi_{n\geq m}N_nP(k)\xrightarrow{v_k}\Pi_{n\geq m}\Sigma^{-2(p^k-1)}N_nP(k)$ for each k, $0\leqq k\leqq m-1$. Since $\Pi_{n\geq m}N_nP(k)_*$ is not v_k-torsion, $\langle v_k^{-1}\Pi_{n\geq m}N_nP(k)\rangle=\langle B(k)\rangle$ by means of Lemma 1.7 i). An iterated use of Lemma 1.1 shows that $\langle\Pi_{n\geq m}N_nBP\rangle=\vee_{0\leq k\leq m-1}\langle B(k)\rangle^\vee\langle\Pi_{n\geq m}N_nP(m)\rangle$. So we use (2.8) to obtain that $\langle\Pi_{n\geq m}N_nBP\rangle=\langle v_{m-1}^{-1}BP\rangle^\vee\langle P(m)\rangle=\langle BP\rangle$.

ii) is obtained by a similar argument to the latter part of i).

Proposition 4.7. i) $\langle\Pi_{n\geq m}\nabla M_nBP\rangle=\langle\nabla(\Pi_{n\geq m}M_nBP)\rangle=\langle\Pi_{n\geq m}\nabla L_nBP\rangle=\langle\nabla(\Pi_{n\geq m}L_nBP)\rangle=\langle\Pi_{n\geq m}\nabla N_nBP\rangle=\langle\nabla(\Pi_{n\geq m}N_nBP)\rangle=\langle BP\rangle$,
ii) $\langle\Pi_\Lambda\nabla M_nBP\rangle=\langle\nabla(\Pi_\Lambda M_nBP)\rangle=\langle\Pi_\Lambda\nabla L_nBP\rangle=\langle\nabla(\Pi_\Lambda L_nBP)\rangle=\langle v_n^{-1}BP\rangle$ *and* $\langle\Pi_\Lambda\nabla N_{n+1}BP\rangle=\langle\nabla(\Pi_\Lambda N_{n+1}BP)\rangle=\langle v_n^{-1}BP\rangle^\vee\langle\nabla P(1)\rangle$.

Proof. i) $\Pi_{n\geq m}\nabla M_nBP_*$ and $\nabla(\Pi_{n\geq m}M_nBP)_*$ are both $\mathcal{BP}$-flat. Since $(\Pi_{n\geq m}\nabla P(n)_*)\underset{BP_*}{\otimes}P(k)_*\simeq\Pi_{n\geq k}\nabla P(n)_*$ for each $k\geqq m$, it is easily seen that $(\Pi_{n\geq m}\nabla P(n)_*)\underset{BP_*}{\otimes}P(\infty)_*\neq 0$ and hence $(\Pi_{n\geq m}\nabla M_nBP_*)\underset{BP_*}{\otimes}P(\infty)_*\neq 0$. Apply Lemma 1.7 ii) to show that $\langle\Pi_{n\geq m}\nabla M_nBP\rangle=\langle BP\rangle=\langle\nabla(\Pi_{n\geq m}M_nBP)\rangle$.

Using the cofiber sequence $\Pi_{n\geq m}\Sigma^n\nabla M_nBP\to\Pi_{n\geq m}\nabla L_{n-1}BP\to\Pi_{n\geq m}\nabla L_nBP$, it is immediate that $\langle\Pi_{n\geq m}\nabla L_{n-1}BP\rangle\geqq\langle\Pi_{n\geq m}\Sigma^n\nabla M_nBP\rangle=\langle BP\rangle$, and hence $\langle\Pi_{n\geq m}\nabla L_{n-1}BP\rangle=\langle BP\rangle$. The remaining equalities are similarly shown.

ii) is obtained by the same argument as in the proof of Proposition 2.2.

Theorems 0.3 and 0.4 follow from (4.1), Propositions 4.1, 4.4, 4.6 and 4.7.

5. $P(n)_*$- and $BP\langle n\rangle_*$-localizations

5.1. In [12] Mitchell constructed a finite p-local CW-spectrum X_n such that

(5.1) $v_{n-1}^{-1}BP_*X_n=0$ *and* $v_n^{-1}BP_*X_n\neq 0$.

We call such a finite CW-spectrum X_n *a Mitchell complex of type n.* Ravenel [14, Lemma 4] proved the following useful result.

Proposition 5.1. *Let X_n be a Mitchell complex of type n. Then $\langle P(n)\rangle=\langle P(n)_\wedge X_n\rangle$.*

By making use of Proposition 5.1 we have

Lemma 5.2. *Let X_{n+1} be a Mitchell complex of type $n+1$ and E be a CW-spectrum. Then*
 i) $N_{n+1}BP_\wedge E=pt$ *if* $X_{n+1\wedge}E=pt$,
 ii) $X_{n+1\wedge}L_nE=pt=X_{n+1\wedge}F(L_nS, E)$, *and*
 iii) $N_{n+1}BP_\wedge L_nE=pt=N_{n+1}BP_\wedge F(L_nS, E)$.

Proof. i) Suppose that $X_{n+1\wedge}E=pt$. Then $P(n+1)_\wedge X_{n+1\wedge}E=pt$, which implies that $P(n+1)_\wedge E=pt$ by Proposition 5.1. Now we obtain the desired result since $\langle P(n+1)\rangle=\langle N_{n+1}BP\rangle$.

ii) X_{n+1} is $v_n^{-1}BP^*$-acyclic because $\langle v_n^{-1}BP\rangle^*=\langle v_n^{-1}BP\rangle$ by Proposition 3.5. Hence the Spanier-Whitehead dual DX_{n+1} of X_{n+1} becomes $v_n^{-1}BP_*$-acyclic, too (or use [3, Proposition 2.10]). Therefore $L_nE^*DX_{n+1}=0$ and $DX_{n+1\wedge}L_nS=L_nDX_{n+1}=pt$. These show that $L_nE_*X_{n+1}=0=F(L_nS, E)_*X_{n+1}$.

iii) is immediate from i) and ii).

Proposition 5.3. *Let X_{n+1} be a Mitchell complex of type $n+1$. The following conditions are all equivalent*:
 i) *a CW-spectrum E is $v_n^{-1}BP$-local,*
 ii) *E is BP-local and $X_{n+1\wedge}E=pt$, and*
 iii) *E is BP-local and $N_{n+1}BP_\wedge E=pt$.*

Proof. The implications i)$\to$ii) and ii)$\to$iii) follow from Lemma 5.2 i) and ii).

iii)$\to$i): Note that $L_nBP_\wedge N_{n+1}E=pt=N_{n+1}BP_\wedge L_nE$ because of Corollary 1.3 and Lemma 5.2 iii). Then the localization map $\eta_n\colon E\to L_nE$ is a BP_*-equivalence under our hypothesis that $N_{n+1}BP_\wedge E=pt$. Hence it becomes an equivalence since E and L_nE are both BP-local. Thus E is $v_n^{-1}BP$-local.

If E is a BP-module spectrum, then so is $E_\wedge X$ for any CW-spectrum X. However $E_\wedge X$ is not necessarily BP-local even if E is so. Bousfield [4] introduced *BP-nilpotent* spectra E, which have the property that $E_\wedge X$ are also BP-nilpotent for any X. Each BP-module spectrum is BP-nilpotent, and each BP-nilpotent spectrum is BP-local.

Corollary 5.4 (Ravenel). *If E is a BP-nilpotent spectrum, then $E_\wedge L_n X = L_n(E_\wedge X) = L_n E_\wedge X$ and $E_\wedge N_{n+1} X = N_{n+1}(E_\wedge X) = N_{n+1} E_\wedge X$ for any CW-spectrum X.* (See [14, Theorem 1]).

Proof. $E_\wedge L_n X$ and $L_n E_\wedge X$ are both BP-nilpotent, and hence they are BP-local. Moreover $X_{n+1\wedge} E_\wedge L_n X = pt = X_{n+1\wedge} L_n E_\wedge X$ by Lemma 5.2 ii). So Proposition 5.3 shows that $E_\wedge L_n X$ and $L_n E_\wedge X$ are both $v_n^{-1}BP$-local. Now the result follows immediately.

The above corollary gives easily

(5.2) $\langle L_n E\rangle \geqq \langle L_n F\rangle$ *and* $\langle N_{n+1} E\rangle \geqq \langle N_{n+1} F\rangle$

if $\langle E\rangle \geqq \langle F\rangle$ for BP-nilpotent spectra E and F. (Cf., Lemma 1.2 ii)).

5.2. We here describe the $P(n)_*$-localization in terms of the BP_*- and $v_n^{-1}BP_*$-localizations.

Lemma 5.5. *Let E be a BP-nilpotent spectrum. If a CW-spectrum X is E-local, then the function spectrum $F(N_n S, X)$ is $N_n E$-local.*

Proof. If Y is $N_n E_*$-acyclic, then $E_*(Y_\wedge N_n S) \cong N_n E_* Y = 0$ by use of Corollary 5.4. Thus $Y_\wedge N_n S$ is E_*-acyclic. Hence $F(N_n S, X)^* Y \cong X^*(Y_\wedge N_n S) = 0$ when X is E-local. So we obtain the desired result.

Theorem 5.6. *Given a CW-spectrum X, the composite map $X \to L_{BP} X \to \Sigma^n F(N_n S, L_{BP} X)$ is the $P(n)_*$-localization of X. Thus $L_{P(n)} = \Sigma^n F(N_n S, L_{BP})$, where L_E denotes the E_*-localization functor for $E = BP$ or $P(n)$.*

Proof. From Lemma 5.5 it follows that $F(N_n S, L_{BP} X)$ is $P(n)$-local because $\langle P(n)\rangle = \langle N_n BP\rangle$. Moreover $F(L_{n-1} S, L_{BP} X)$ is $P(n)_*$-acyclic by means of Lemma 5.2 iii). Hence the composite map $X \to L_{BP} X \to \Sigma^n F(N_n S, L_{BP} X)$ becomes a $P(n)_*$-equivalence. So we observe that $L_{P(n)} X = \Sigma^n F(N_n S, L_{BP} X)$.

We next study the $BP\langle n\rangle_*$- and $\nabla N_{n+1} BP_*$-localizations. Recall that $\langle BP\langle n\rangle\rangle = \langle v_n^{-1}BP\rangle \vee \langle HZ/p\rangle$ and $\langle \nabla N_{n+1} BP\rangle = \langle v_n^{-1}BP\rangle \vee \langle \nabla P(1)\rangle$.

Proposition 5.7. *Let E be a BP-nilpotent spectrum with $\langle E\rangle \geqq \langle v_n^{-1}BP\rangle$. Then a CW-spectrum X is E-local if and only if X is BP-local and $F(N_{n+1} S, X)$ is $N_{n+1} E$-local.*

Proof. The "only if" part: Note that $BP_* Y = 0$ implies $E_* Y = 0$ when E is BP-nilpotent. Thus X is BP-local if it is E-local. The latter part follows from Lemma 5.5.

The "if" part: Suppose that $E_*Y=0$. By making use of Corollary 5.4 we see that $BP_*(Y_\wedge L_nS)\cong L_nBP_*Y=0$ since $\langle E\rangle\geqq\langle v_n^{-1}BP\rangle=\langle L_nBP\rangle$, and $N_{n+1}E_*Y\cong E_*(Y_\wedge N_{n+1}S)=0$. So the localities of X and $F(N_{n+1}S, X)$ give that $X^*(Y_\wedge L_nS)=0$ and $X^*(Y_\wedge N_{n+1}S)\cong F(N_{n+1}S, X)^*Y=0$, which imply $X^*Y=0$. Thus X is E-local.

Since $\langle N_{n+1}BP\langle n\rangle\rangle=\langle HZ/p\rangle$ and $\langle N_{n+1}\nabla N_{n+1}BP\rangle=\langle\nabla P(1)\rangle$ we have

Corollary 5.8. i) *A CW-spectrum X is $BP\langle n\rangle$-local if and only if X is BP-local and $L_{P(n+1)}X$ is HZ/p-local.*
ii) *A CW-spectrum X is $\nabla N_{n+1}BP$-local if and only if X is BP-local and $L_{P(n+1)}X$ is $\nabla P(1)$-local.*

When a CW-spectrum X is connective, it is $HZ_{(p)}$-local. So Corollary 5.8 i) implies

(5.3) $L_{P(1)}X=\Sigma^1F(N_1S, X)$ *is HZ/p-local if X is connective.* (Cf., [4, Theorem 3.1]).

Given a CW-spectrum F we denote by C_FX the cofiber of the localization map $\eta_F\colon X\to L_FX$. When $F=v_n^{-1}BP$, C_FX is written $\Sigma^{-n}N_{n+1}X$. Consider the composite map $L_FX\to C_FX\to L_nC_FX$, whose cofiber is denoted by $\Sigma^1L_{(F,n)}X$. Then we have a commutative diagram

$$\begin{array}{ccccc}
X & \to & L_{(F,n)}X & \to & \Sigma^{-n-1}N_{n+1}C_FX \\
\| & & \downarrow & & \downarrow \\
X & \xrightarrow{\eta_F} & L_FX & \longrightarrow & C_FX \\
 & & \downarrow & & \downarrow \eta_n \\
 & & L_nC_FX & = & L_nC_FX
\end{array}$$

involving four cofiber sequences.

Proposition 5.9. *Let F be a BP-nilpotent spectrum and E be a CW-spectrum with $\langle E\rangle=\langle v_n^{-1}BP\rangle\vee\langle F\rangle$. Then the map $X\to L_{(F,n)}X$ is the E_*-localization of X for any CW-spectrum X. Thus $L_E=L_{(F,n)}$.*

Proof. Since $F_*N_{n+1}C_FX\cong N_{n+1}(F_\wedge C_FX)_*=0$ by Corollary 5.4, we see that $N_{n+1}C_FX$ is in fact E_*-acyclic. Moreover L_FX and L_nC_FX are both E-local, so $L_{(F,n)}X$ is E-local, too. Hence we verify that $L_EX=L_{(F,n)}X$.

The above proposition states the $BP\langle n\rangle_*$- and $\nabla N_{n+1}BP_*$-localizations in terms of the $v_n^{-1}BP_*$-, HZ/p_*- and $\nabla P(1)_*$-localizations.

Corollary 5.10. $L_{BP\langle n\rangle}=L_{(HZ/p,n)}$ *and* $L_{\nabla N_{n+1}BP}=L_{(\nabla P(1),n)}$.

Theorem 5.6 and Corollary 5.10 give Theorem 0.5.

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