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ON NON-SINGULAR FPF-RINGS II

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In [2], we have proved that a right non-singular ring R is right FPF (=every finitely generated faithful right R-module generates the category of right R-modules) if and only if (1) R is right bounded, (2) The multiplication map $Q\otimes_R Q\to Q$ is an isomorphism and Q is flat as a right R-module, where Q means the maximal right quotient ring of R, (3) For any finitely generated right ideal I of R, $Tr_R(I)\oplus r_R(I)=R$ (as ideals), where $Tr_R(I)$ means the trace ideal of I and $r_R(I)$ means the right annihilator ideal of I. This characterization implies a following result of S. Page. "Let R be a right non-singular right FPF-ring and Q be the maximal right quotient ring of R. Then Q is also right FPF and is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings." However, as we can see from an example in section 1, not all non-singular right FPF-rings arise in this fashion.

Therefore, in this paper, we shall give a necessary and sufficient condition for a non-singular right FPF-ring to split into a finite direct product of full matrix rings over FPF-rings whose maximal right quotient rings are abelian regular self-injective rings. More precisely, we shall prove the following theorem.

Theorem 1. *Let R be a non-singular right FPF-ring. Then the following conditions are equivalent.*

(1) $R\simeq\prod_{i=1}^{t} M_{n(i)}(S_i)$, *where each S_i is a non-singular right FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring.*

(2) *R contains a faithful and reduced FPF idempotent and R satisfies general comparability.*

By Y. Utumi [5], non-singular (right) continuous rings are shown to be (Von Neumann) regular, and S. Page has determined the structure of regular (right) FPF-rings. Therefore we are interested in the structure on non-singular right quasi-continuous, right FPF-rings. In section 2, as an application of Theorem 1, we shall determine the structure of non-singular right quasi-continuons right FPF-rings.

0. Preliminaries

Throughout of this paper, we assume that a ring R has identity and all modules are unitary.

Let R be a ring. Then we say that R is right FPF if every finitely generated faithful right R-module is a generator in the category of right R-modules. If R is a right non-singular right FPF-ring, then R is also left non-singular by Theorem 3 of [4]. Therefore, we simply call R a non-singular right FPF-ring.

Let e be an element of $id(R)$ (=the set of all idempotents of R), where R is a non-singular right FPF-ring. Then we say that e is a faithful and reduced FPF idempotent if the right R-module eR is faithful and the ring eRe is a non-singular right FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring, where a regular ring R is said to be abelian if every idempotent of R is central.

A ring R satisfies general comparability provided that for any e, f in $id(R)$, there exists $h \in B(R)$ (=the set of all central idempotents of R) such that $h(eR) \lesssim h(fR)$ and $(1-h)(fR) \lesssim (1-h)(eR)$, where $h(eR) \lesssim h(fR)$ means that $h(eR)$ is isomorphic to a direct summand of $h(fR)$.

Let M be a right R-module. Then we use $r_R(M)$ to denote the right annihilator ideal of M, and we use $L_r(M)$ to denote the lattice of all submodules of M. M is said to have the extending property of modules for $L_r(M)$, provided that for any A in $L_r(M)$, there exists a direct summand A^* of M such that $A \subseteq_e A^*$, where $A \subseteq_e A^*$ means that A is an essential submodule of A^*.

We say a ring R is right quasi-continuous if (1) R has the extanding property of modules for $L_r(R)$, and (2) for A and B in $L_r(R)$, which are direct summand of R with $A \cap B = 0$, $A \oplus B$ is also a direct summand of R.

1. Proof of Theorem 1

In this section, we shall prove Theorem 1. First we show the following lemma.

Lemma 1. *Let R be a non-singular right FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring. Then for any positive integer n, $M_n(R)$ satisfies general comparability.*

Proof. First we assume that $n=1$. Since Q, the maximal right quotient ring of R, is an abelian regular ring, $B(Q)=B(R)$, hence $id(Q)=id(R)$. Let e and f are idempotents of R. Then $eR \cap fR = efR$ and ef is idempotent since $id(R)=B(R)$. So $fR \cap eR \lesssim fR = f(fR)$ and $(1-f)(fR)\ (=0) \lesssim (1-f)(eR)$. Therefore R has general comparability. Next let $n>1$, and assume that the Theorem holds for $n-1$. Let e and f are idempotent of $M_n(R)$. Then we note that eR and fR are isomorphic to $e_1\oplus\cdots\oplus e_nR$ and $f_1R\oplus\cdots\oplus f_nR$ as right R-modules, where e_i and f_t $(i=1, 2, \cdots, n)$ are idempotents of R. We set $A=e_1R\oplus\cdots\oplus e_nR$ and $B=f_1R\oplus\cdots\oplus fR_n$, and set $A'=e_2R\oplus\cdots\oplus e_nR$ and $B'=f_2R\oplus\cdots\oplus f_nR$. By induction hypothesis, there exist central idempotents h_1, h_2 such that $h_1(e_1R)\lesssim h_1(f_1R)$, $(1-h_1)(f_1R)\lesssim(1-h_1)(e_1R)$, and $h_2A'\lesssim h_2B'$, $(1-h_2)B'\lesssim(1-h_2)A'$. Now we set that $t_1=h_1h_2$ and $t_2=(1-h_1)(1-h_2)$. Then t_1 and t_2 are in $B(R)$, and $t_1t_2=0$. Moreover we see that $t_1(e_1R)\lesssim t_1(f_1R)$ and $t_1A'\lesssim t_1B'$,. Hence $t_1A\lesssim t_1B$. Similarly, $t_2B\lesssim t_2A$. Further we set $g_1=h_1(1-h_2)$, $g_2=h_2(1-h_1)$ and $g=g_1+g_2$. Then since $h_1(e_1R)\lesssim h_1(f_1R)$, we have $g_1(e_1R)\lesssim g_1(f_1R)$ and $g_1(f_1R)\cong g_1(e_1R)\oplus D_1$ for some D_1. Similarly, we obtain that $g_2(e_1R)\cong g_2(f_1R)\oplus C_1$ for some C_1. Furthermore we have that $g_2B'=g_2A'\oplus D_2$ and $g_1A'=g_1B'\oplus C_2$ for D_2, C_2. We see that $gA\cong(g_1(e_1R)\oplus g_2(f_1R)\oplus g_1B'\oplus g_2A')\oplus(C_1\oplus C_2)$ and $gB\cong$ some $(g_1(e_1R)\oplus g_2(f_1R)\oplus g_1B'\oplus g_2A')\oplus(D_1\oplus D_2)$. On the other hand, since $C_1\lesssim g_2(e_1R)\lesssim g_2R$ and $C_2\lesssim g_1A'\lesssim(n-1)(g_1R)$, $C_1\oplus C_2\lesssim(n-1)(g_1+g_2)R\lesssim(n-1)gR$. Similarly, $D_1\oplus D_2\lesssim(n-1)gR$. Hence by induction hypothesis, there exists a central idempotent k of R such that $k(C_1\oplus C_2)\lesssim k(D_1\oplus D_2)$ and $(1-k)(D_1\oplus D_2)\lesssim(1-k)(C_1\oplus C_2)$. Set $t_3=gk$ and $t_4=g(1-k)$. Then t_3, t_4 are in $B(R)$ and $t_3t_4=0$ and $t_3+t_4=g$. We see that $t_3A\lesssim t_3B$ and $t_4B\lesssim t_4A$. Now we set $e=t_1+t_3$. Then $eA\lesssim eB$ and $(1-e)B\lesssim(1-e)A$. Therefore $M_n(R)$ has general comparability.

Proof of Theorem 1.

(1)$\Rightarrow$(2); By Lemma 1, R has general comparability, and it is easily seen that R contains a faithful and reduced FPF imdempotent.

(2)$\Rightarrow$(1); Let g be a a faithful and reduced FPF idempotent of R. We claim that g is a faithful abelian idempotent of Q. Set $H=r_Q(gQ)$, then since Q is right self-injective regular, $H=eQ$ for some central idempotent e of R. Further by Theorem 1 of [3], $B(R)=B(Q)$, so e must be zero since gR is faithful. Hence gQ is a faithful right Q-module. We note that gQg is a maximal right quotient ring of gRg. Thus g is a faithful abelian idempotent of Q. Since Q is also right FPF by [4, Theorem 2], $Q\lesssim n(gQ)$ for some positive integer n. For each $t=1, 2, \cdots, n$ let e_t be the supremum of all $e\in B(R)$ for which $(eQ)_Q\lesssim t(egQ)$. By [2, Theorem 10, 15], $e_tQ\lesssim t(e_tgQ)$. Then $e_1\leqslant e_2\cdots\leqslant e_n=1$, and we define that $f_1=e_1$, and $f_t=e_t-e_{t-1}$ for all $t=2, \cdots, n$. Note that the f_t's are pairwise orthogonal, and $\vee f_t=\vee e_t=1$. Therefore $Q=\prod_{t=1}^{n} f_tQ$, and since $B(R)=B(Q)$, we see that $R=\prod_{t=1}^{n} f_tR$. Thus it only remains to show that each of the rings f_tR is isomorphic to a $t\times t$-matrix ring over a FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring. Since $f_1Q\lesssim f_1gQ\leqslant gQ$ and gQg is abelian, we see that the ring f_1Q is abelian. Furthermore, f_1Q is a maximal right quotient ring of f_1R. Now consider any integer $t\in\{2, \cdots, n\}$. Note that f_tQ is a regular ring of index t. We have that $f_t\leqslant e_t$

and $e_tQ \lesssim t(e_t gQ)$, whence $f_tQ \lesssim t(f_t gQ)$. Since R has general comparability, there exists a central idempotent e of R such that $(t-1)(egR) \lesssim eR$ and $(1-e)R \lesssim (t-1)(1-e)gR$. Then $(1-e)Q \lesssim (t-1)(1-e)gQ$, so $1-e \leqslant e_{t-1}$. Thus $f_t \leqslant 1-e_{t-1} \leqslant e$. Therefore $(t-1)(f_t gR) \lesssim f_tR$. We have that $f_tR = (t-1)(f_t gR) \oplus A$ for some A. Next by general comparability of R, there exists acentral idempotent h of R such that $h(f_t gR) \lesssim hA$, and $(1-h)A \lesssim (1-h)(f_t gR)$. Then $hA = h(f_t gR) \oplus B$ and $(1-h)(f_t gR) \cong (1-h)A \oplus C$ for some B and C. Now $f_tR \cong t(hf_t gR) \oplus B \oplus t((1-h)A) \oplus (t-1)C$. Hence there exist idempotents k_i ($i=1, 2, 3, 4$) of f_tR such that $k_1R \cong (t(hf_t gR)$, $k_2R \cong B$, $k_3 \cong t((1-h)A)$, and $k_4R \cong (t-1)C$. We claim that each k_i is central. If k_1 is not central, then $\mathrm{Hom}_Q(k_1Q, (f_t-k_1)Q) \cong (f_t-k_1)Qk_1 \neq 0$, whence $\mathrm{Hom}_Q(hf_t gQ, (f_t-k_1)Q) \neq 0$. Let φ be a nonzero element of $\mathrm{Hom}_Q(hf_t gQ, (f_t-k_1)Q)$. Then $\varphi(hf_t g)$ is nonzero and $\varphi(hf_t g)Q$ is a direct summand of $(f_t-k_1)Q$. In particular, $\varphi(hf_t g)Q$ is a projective and hence we have that $hf_t gQ = E \oplus \mathrm{Ker}\,\varphi$ for some E. Since $E \cong \varphi(hf_t g)Q$, we have that $E \lesssim hf_tQ$. But then hf_tQ contains a direct sums of $t+1$ nonzero pairwise isomorphic right ideals, which contradicts the index of f_tQ. Therefore k_1 is central. Likewise for k_2, k_3, k_4. Next we show that k_2 and k_4 are zero. It is easily seen that k_2 is zero since $f_t gR$ is a faithful right f_tR-module. Since $C \lesssim (1-h)(f_t gR)$, $R \cong (t-1)C \lesssim (t-1)(1-h)f_t gR$. Hence $(1-h)f_tk_4R \lesssim (t-1)\cdot(1-h)f_tk_4 gR$, so $(1-h)f_tk_4 \leqslant e_{t-1}$. But since $e_{t-1} = e_t - f_t$, e_{t-1} is orthogonal to f_{t-1}. Thus $k_4 = (1-h)f_tk_4 = 0$. Consequently, $f_tR = k_1R \oplus k_2R \cong t(hf_t gR) \oplus t((1-)h)A) \cong t(hf_t gR) \oplus t((1-h)f_t gR) \cong t(f_t gR)$. Thus $f_tR = M_t(f_t gRf_t g)$ as rings, since $f_t gQf_t g$ is a maximal right quotient ring of $f_t gRf_t g$ and is an abelian regular right self-injective ring. Thus the proof is complete.

EXAMPLE. There exists a non-singular two-sided FPF-ring R, which is not isomorphic to a full matrix ring of non-singular FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring.

Proof. Let D be a *Prüfer* domain which is not a principal ideal domain and let I be a non-principal finitely generated ideal of D. We set $R = \begin{pmatrix} D & I \\ I^{-1} & D \end{pmatrix}$.

It is easy to see that R is non-singular FPF-ring, and R does not satisfy general comparabillity. Hence by Thoerem 1, R is not isomorphic to a full matrix ring over a non-singular FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring.

2. Non-singular quasi-continuous FPF-rings

In this section, as an application of Theorem 1, we shall determine the structure of non-singular quasi-continuous FPF-rings. First we recall that a ring R is right quasi-continuous if (1) R has the extending property of modules for $L_r(R)$, and (2) For any A, B in $L_r(R)$, which are direct summand of R with

$A \cap B=0$, $A \oplus B$ is also a direct summand of R.

In [3], K. Oshiro has studied qusai-continuous modules, and proved the following.

Proposition 1 ([3, Propositions 1.5 and 3.1]). *Let M be a right R-module. Then the following conditions are equivalent.*

(1) *M is right quasi-continuous.*

(2) (i) *M has the extending property of modules for $L_r(M)$.*

(ii) *For any A and B in $L_r(M)$ such that $B \lessdot \oplus M$ i.e. B is a direct summand of M, and $A \cap B=0$, every homomorphism of A to B is extended to a homomorphism of M to B.*

(3) *Every decomposition $E(M)=E_1 \oplus E_2 \oplus \cdots \oplus E_n$ implies that $M=(E_1 \cap M) \oplus (E_2 \cap M) \oplus \cdots \oplus (E_n \cap M)$, where $E(M)$ denotes the injective hull of M.*

In order to determine the structure of non-singular quasi-continuous FPF-rings, we shall need the following lemma.

Lemma 3. *Let R be a non-singular right quasi-continuous right FPF-ring. Then R satisfies general comparability.*

Proof. Let e and f are idempotents of R, and consider an exact sequence $0 \to eR \cap fR \to fR \to (1-e)fR \to 0$. Since R is right quasi-continuous, R has the extending property of modules for $L_r(R)$, $(1-e)fR$ is projective. Thus $eR \cap fR$ is a dierct summand of fR. Similarly, $eR \cap fR \lessdot \oplus eR$. Hence $eR=(eR \cap fR) \oplus e_1R$ and $fR=(eR \cap fR) \oplus f_1R$. It follows that if there exists a central idempotent h of R such that $h(e_1R) \lesssim h(f_1R)$ and $(1-h)(f_1R) \lesssim (1-h)(e_1R)$, then $h(eR) \lesssim h(fR)$ and $(1-h)(fR) \lesssim (1-h)(eR)$. Hence R satisfies general comparability. Therefore, we may assume that $eR \cap fR=0$. Let X denote the collection of all triples (A, B, φ) such that $A \subseteq eR$, $B \subseteq fR$, and φ: $A \to B$ is an isomorphism. Define a partial order on X by setting $(A'', B'', \varphi'') \leqslant (A', B', \varphi')$ whenever $A'' \leqslant A'$, $B'' \leqslant B'$, and φ' is an extention of φ''. By Zorn's lemma, there exists a maximal element (A', B', φ') in X. Since R has the extending property of modules for $L_r(R)$, there exist direct summands A^*, B^* of R such that $A' \subseteq_e A^*$, $B' \subseteq_e A^*$. Then since R is right quasi-continuous, by the condition (2) of Proposition 1, we have homomorphisms φ: $A^* \to B^*$, and ψ: $B^* \to A^*$, whih which are extensions of the isomorphisms φ' and $(\varphi')^{-1}$ respectively. We show that φ is an isomorphism. Let m be an element of A^* such that $\varphi(m)=0$. Then since $A' \subseteq_e A^*$, $J=\{r \in R \mid mr \in A'\}$ is an essential right ideal of R and $\varphi'(mr)=0$ for any $r \in J$. Thus $mJ=0$ since φ' is an isomorphism, and $m=0$. Hence φ is a monomorphism. Next let m be any element of B^*. Then for any element r of $H=\{r \in R \mid mr \in B'\}$, $\varphi \cdot \psi(mr)=\varphi'(\varphi')^{-1}(mr)=mr$, where φ is an epimorphism. Therefore φ is an isomorphism. On the other hand, by the maximality of (A', B', φ'), $(A', B', \varphi')=(A^*, B^*, \varphi)$, hence A' and B' are direct summand of

R. Thus $eR=A'\oplus e_1R$ and $fR=B'\oplus f_1R$. Set $I=r_R(e_1R)$. Since R is right FPF, there exists a central idempotent h of R such that $I=hR$ by [3, Theorem 1]. In this case, we have that $h(eR)=h(A')\cong h(B')\lesssim h(fR)$. Next we show that $f_1R(1-h)=0$. If $f_1R(1-h)\neq 0$, then there is a nonzero element $x\in f_1R(1-h)$. Since xR is non-singular, xR is projective, so that there exists an idempotent t of R such that $xR\cong tR$ by the extending property to R. We note that $tRh\cong xRh=0$. Since $t\neq 0$, it follows that $t\notin hR$. Then $yt\neq 0$ for some $y\in e_1R$. We have an exact sequence $xR\to tR\to ytR\to 0$, and that ytR is projective. Thus there exists a monomorphism $g\colon ytR\to xR$. Since $yt\in e_1R$ and $x\in f_1R$, we obtain that $(A'\oplus ytR, B'\oplus g(ytR), \varphi'\oplus g)\in X$. But this contradicts the maximality of (A', B', φ'). Therefore $f_1R(1-h)=0$. Now $(1-h)(fR)=(1-h)(B')\cong(1-h)(A')\lesssim(1-h)(eR)$. Therefore R satisfies general comparability.

Theorem 2. *Let R be a non-singular right FPF-ring and Q be the maximal right quotient ring of R. Then the following conditions are equivalent.*

(1) *R is right quasi-continuous.*

(2) *$id(R)=id(Q)$.*

(3) *$R\cong R_1\times\prod_{i=1}^{t}M_{n(i)}(S_i)$, where R_1 is a non-singular right FPF-ring whose maximal right quotient ring is an abelian regular self-injective ring, and each S_i is an abelian regular self-injective ring and $n(i)\geqslant 2$.*

Proof. (1)$\Rightarrow$(3); Since Q is a regular self-injective ring of bounded index, Q has an faithful and abelian idempotent e, i.e. the right Q-module eQ is faithful and the ring eQe is abelian regular. Then $Q=eQ\oplus(1-e)Q$. Thus by the condition (3) of Proposition 1, $R=e'R\oplus e''R$ for some idempotents e', e'' of R. We show that e' is a faithful and reduced FPF-idempotent. Let $I=r_R(e'R)$. Then since R is right FPF, by Theorem 1 of [3], there exists a central idempotent f of R such that $I=fR$. Set $J=\{r\in R\mid er\in R\}$. Then we obtain that $eJ\subseteq eQ\cap R=e'R$ and $eJf=0$, so $ef=0$ since J is an essential right ideal of R. While since eQ is faithful, f must be zero. Hence $e'R$ is faithful. Furthermore, eQe is clearly, a maximal right quotient ring of $e'Re'$. Therefore e' is a faithful and reduced FPF idempotent. Moreover, by Lemma 2, R satisfies general comparability. Therefore it follows from Theorem 1 that $R\cong\prod_{i=1}^{t}M_{n(i)}(S_i)$, where each S_i is a non-singular right FPF-ring whose the maximal right quotient ring is an abelian regular self-injective ring. We claim that if $n(i)\geqslant 2$, then each S_i is a self-injective regular ring. To prove this, we set $R=M_n(S)$ $(n\geqslant 2)$, where S is a non-singular right FPF-ring whose the maximal right quotient ring $Q(S)$ is abelian regular. Assume that $Q(S)\neq S$ and let w be any element of $Q(S)-S$. Set $\bar{e}=\begin{pmatrix}1, 0, \cdots, 0, w\\ 0\end{pmatrix}$. Then clearly, $\bar{e}Q=\begin{pmatrix}Q(S), Q(S), \cdots, Q(S)\\ 0\end{pmatrix}$ and

$(1-e)Q = \left\{\begin{pmatrix} wx_1, wx_2, \cdots\cdots, wx_n \\ Q(S), Q(S), \cdots, Q(S) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1, x_2, \cdots\cdots, x_n \end{pmatrix} \,\middle|\, x, x, \cdots, x_n \in Q(S)\right\}$. In this case

$eQ \cap R = \begin{pmatrix} S, S, \cdots, S \\ 0 \end{pmatrix}$ and $(1-e)Q \cap R = \left\{\begin{pmatrix} wy_1, wy_2, \cdots, wy_n \\ S, S, \cdots\cdots, S \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_2, y_1, \cdots\cdots, y_n \end{pmatrix} \,\middle|\, y_1, y_2, \cdots, y_n \in J\right.$

$= \{r \in S \mid wr \in S\}\}$. Hence $(eQ \cap R) \oplus ((1-e)Q \cap R) = \begin{pmatrix} S, \cdots, S \\ S, \cdots, S \\ J, \cdots, J \end{pmatrix} \neq R$. But this contradicts that R is right quasi-continuous. Therefore $Q(S)=S$, so S is regular self-injective.

(3)$\Rightarrow$(2); Since idempotent of R_1 are central, $id(R_1)=id(Q_{\max}(R_1))$. Therefore $id(R)=id(Q)$.

(2)$\Rightarrow$(1); By the condition (3) of Proposition 1, it suffices to show that if $Q=e_1Q \oplus e_2Q \oplus \cdots \oplus e_nQ$ for some idempotents e_i of Q, then $R=(e_1Q \cap R) \oplus (e_2Q \cap R) \oplus \cdots (e_nQR)$. But since $id(R)=id(Q)$, this is clear.

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