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A NEW CLASS OF TRANSLATION PLANES OF ORDER q^3

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1. Introduction

Let q be an odd prime power, where 2 is a non-square in $GF(q)$. The aim of this paper is to construct a new class of translation planes of order q^3 and to determine their linear translation complements. Their kernels are isomorphic to $GF(q)$. If $q\neq 3$, then the linear translation complement of any plane of this class has exactly two orbits of length 2 and q^3-1 on the line at infinity and it is of order $3(q-1)(q^3-1)$. If $q=3$, then the plane is the Hering plane of order 27 and the translation complement is isomorphic to SL(2, 13).

The planes also differ from those which are generalized André planes [1] and semifield planes.

2. Preliminaries

We list some results that will be required in the calculations of the linear translation complements.

Let q be a prime power. For $\alpha\in GF(q^3)$ put $\bar{\alpha}=\alpha^q$ and $\bar{\bar{\alpha}}=\alpha^{q^2}$. Set

$$M(3, q^3) = \left\{\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix} \middle|\, \alpha_{ij}\in GF(q^3)\right\}$$

and

$$\mathfrak{A} = \left\{\begin{pmatrix}\alpha & \bar{\alpha} & \bar{\bar{\alpha}}\\ \beta & \bar{\beta} & \bar{\bar{\beta}}\\ \gamma & \bar{\gamma} & \bar{\bar{\gamma}}\end{pmatrix}\in GL(3, q^3)\right\}.$$

Then $\varepsilon\in\mathfrak{U}$ if and only if

$$\varepsilon = \begin{pmatrix}\alpha & \bar{\alpha} & \bar{\bar{\alpha}}\\ \beta & \bar{\beta} & \bar{\bar{\beta}}\\ \gamma & \bar{\gamma} & \bar{\bar{\gamma}}\end{pmatrix}$$

and α, β, γ are linearly independent over $GF(q)$. Set

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{pmatrix}\alpha & \bar{\gamma} & \bar{\bar{\beta}}\\ \beta & \bar{\alpha} & \bar{\bar{\gamma}}\\ \gamma & \bar{\beta} & \bar{\bar{\alpha}}\end{pmatrix}.$$

Lemma 2.1 (T. Oyama [3]). *If* $\varepsilon\in\mathfrak{A}$ *then*

$$M(3,q)^{\varepsilon}=\left\{\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}\in M(3,q^3)\right\}\ \textit{and}\ \ GL(3,q)^{\varepsilon}=\left\{\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}\in GL(3,q^3)\right\}.$$

Since ε is any element of $\mathfrak{A}$, we denote $M(3,q)^{\varepsilon}$ by $M(3,q)^*$ and $GL(3,q)^{\varepsilon}$ by $GL(3,q)^*$.

Set $GF(q^3)^*=GF(q^3)-\{0\}$. Let t be a generator of the multiplicative group $GF(q^3)^*$. Set $W=\begin{bmatrix}t\\ 0\\ 0\end{bmatrix}$ and $T=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$.

The following statements hold:

Lemma 2.2.
(i) $o(W)$ *(the order of* W*)*$=q^3-1$ *and* $o(T)=3$.
(ii) $T^{-1}WT=W^q$.

Lemma 2.3. *If* $o(W^i)\nmid q-1$, *then* $N_{GL(3,q)^*}(\langle W^i\rangle)=\langle W,T\rangle$ *and* $C_{GL(3,q)^*}(\langle W^i\rangle)=\langle W\rangle$.

Proof. Let $A\in N_{GL(3,q)^*}(\langle W^i\rangle)$. There exists an integer j with $W^iA=AW^{ij}$. Write $A=\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$. Then $\alpha t^i=\alpha t^{ij}$, $\beta\overline{t^i}=\beta t^{ij}$ and $\gamma\overline{\overline{t^i}}=\gamma t^{ij}$. Assume $\alpha\neq 0$. Then $t^i=t^{ij}$. Since $t^i\neq\overline{t^i}$, $\beta=0$ and $\gamma=0$. Thus $A\in\langle W\rangle$. Assume $\alpha=0$. If $\beta\neq 0$ and $\gamma\neq 0$, then $\overline{t^i}=\overline{\overline{t^i}}$. This is a contradiction. Thus $\beta=0$ or $\gamma=0$ and hence $A\in\langle W,T\rangle$. Therefore $N_{GL(3,q)^*}(\langle W^i\rangle)\subseteq\langle W,T\rangle$. On the other hand $N_{GL(3,q)^*}(\langle W^i\rangle)\supseteq\langle W,T\rangle$ by Lemma 2.2 (ii) and thus $N_{GL(3,q)^*}(\langle W^i\rangle)$ $\langle W,T\rangle$.

Similarly, using $\overline{t^i}\neq t^i$, we obtain $C_{GL(3,q)^*}(\langle W^i\rangle)=\langle W\rangle$.

Lemma 2.4. *Let* $A\in GL(3,q)^*$ *and* $o(A)=q^3-1$. *Then* $\langle A\rangle$ *is conjugate to* $\langle W\rangle$ *in* $GL(3,q)^*$.

Proof. Since $(q-1,q^2+q+1)=1$ or 3, there exists a prime r such that $r\nmid q-1$ and that $r\mid q^2+q+1$. From this $r\mid o(W)$ and $r\nmid q^3(q+1)(q-1)^3$ follow.

Since $|GL(3, q)^*|=q^3(q+1)(q-1)^3(q^2+q+1)$, $\langle W\rangle$ includes a Sylow r-subgroup $\langle W^i\rangle$ of $GL(3, q)^*$. Thus there exists $B\in GL(3, q)^*$ with $B^{-1}\langle W^i\rangle B=\langle A^i\rangle$. Since $o(W^i)\nmid q-1$, $C_{GL(3,q)^*}(\langle W^i\rangle)=\langle W\rangle$ by Lemma 2.3. Therefore $\langle A\rangle\subseteq C_{GL(3,q)^*}(\langle A^i\rangle)=B^{-1}C_{GL(3,q)^*}(\langle W^i\rangle)B=B^{-1}\langle W\rangle B$. Considering the order of $\langle A\rangle$, we get $\langle A\rangle=B^{-1}\langle W\rangle B$.

Lemma 2.5. *Set* $I=\begin{bmatrix}1\\0\\0\end{bmatrix}$. *Let* $A\in GL(3, q)^*$ *and assume that* $det(A-xI)\neq 0$ *for any* $x\in GF(q)$. *The following statements hold:*

(i) $o(A)\mid q^3-1$ and $o(A)\nmid q-1$.

(ii) *There exists* $B\in GL(3, q)^*$ *with* $o(B)=q^3-1$ *and* $A\in\langle B\rangle$.

Proof. Let $\varepsilon\in\mathfrak{A}$. Set $V=V(3, q)$ and $C=A^{\varepsilon^{-1}}$. There exists a 2-dimensional subspace V_1 of V such that $V_1C\neq V_1$. Let $V_1\cap V_1C=\langle VC\rangle$. Since $\det(A-xI)=\det(C-xI)\neq 0$ for any $x\in GF(q)$, v, vC, vC^2 are linearly independent over $GF(q)$. Hence C is conjugate to

$$\begin{pmatrix}0&1&0\\0&0&1\\a_0&a_1&a_2\end{pmatrix}$$

in $GL(3, q)$, where a_0, a_1, $a_2\in GF(q)$ and $vC^3=a_0v+a_1vC+a_2vC^2$. Let λ be a root of the characteristic polynomial of C. Then $\lambda\in GF(q^3)$ and $\lambda^3=a_0+a_1\lambda+a_2\lambda^2$. Set

$$\mu=\begin{pmatrix}1&1&1\\\lambda&\bar{\lambda}&\bar{\bar{\lambda}}\\\lambda^2&\bar{\lambda}^2&\bar{\bar{\lambda}}^2\end{pmatrix}.$$

Since

$$\begin{pmatrix}0&1&0\\0&0&1\\a_0&a_1&a_2\end{pmatrix}^{\mu}=\begin{bmatrix}\lambda\\0\\0\end{bmatrix},$$

$A^{\varepsilon^{-1}\mu}$ is conjugate to $\begin{bmatrix}\lambda\\0\\0\end{bmatrix}$ in $GL(3, q)^*$. From this (i) and (ii) follow.

3. Description of the class of translation planes

Let q be an odd prime power where 2 is a non-square over $GF(q)$.

Any translation plane is defined by a spread. We define the spreads using the Oyama's Method (T. Oyama [3]). For $\alpha \in GF(q^3)$ put $((\alpha))=(\alpha, \bar{\alpha}, \bar{\bar{\alpha}})$. $X=\{((\alpha)) \mid \alpha \in GF(q^3)\}$ becomes a vector space of dimension 3 over $GF(q)$, when $+$ and scaler product are defined by $((\alpha))+((\beta))=((\alpha+\beta))$ and $a((\alpha))=((a\alpha))$. We may assume that $V=X \oplus X$ is the outer sum of two copies of X. Set $V(\infty)=\{(0, ((\alpha))) \mid ((\alpha)) \in X\}$. If there exists a subset Σ of $GL(3, q)^* \cup \{0\}$ such that $0 \in \Sigma$, $|\Sigma|=q^3$ and $\det(M_1-M_2) \neq 0$ for all $M_1 \neq M_2 \in \Sigma$, then we can costruct a translation plane $\pi(\Sigma)$ of order q^3 such that it's kernel contains $GF(q)$, as follows:

(a) The points of $\pi(\Sigma)$ are the vectors in V.

(b) The lines are all cosets of all the components of $\Pi=\{V(M) \mid M \in \Sigma \cup \{\infty\}\}$, where $V(M)=\{(((\alpha)), ((\alpha))M \mid ((\alpha)) \in X\}$ for $M \in \Sigma$.

(c) Incidence is inclusion.

We call Σ a spread set of degree 3 over $GF(q)$.

Set $S=\{\alpha^2 \mid \alpha \in GF(q^3)^*\}$. For $\alpha \in GF(q^3)$ put $n(\alpha)=\alpha\bar{\alpha}\bar{\bar{\alpha}}$ and $tr(\alpha)=\alpha+\bar{\alpha}+\bar{\bar{\alpha}}$. Clearly $n(\alpha)$, $tr(\alpha) \in GF(q)$ and $\det\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}=n(\alpha)+n(\beta)+n(\gamma)-tr(\alpha\bar{\beta}\bar{\bar{\gamma}})$.

For $\alpha \in S$ put $M(\alpha)=\begin{bmatrix}\bar{\alpha} \\ \alpha \\ 0\end{bmatrix}$ and $N(\alpha)=\begin{bmatrix}\bar{\alpha} \\ \alpha \\ -\alpha\bar{\alpha}\bar{\bar{\alpha}}^{-1}\end{bmatrix}$.

Theorem 3.1. $\Sigma=\{M(\alpha), N(\alpha) \mid \alpha \in S\} \cup \{0\}$ *is a spread set of degree* 3 *over* $GF(q)$.

Proof. Let $\alpha \in S$. Since $M(\alpha)=\begin{bmatrix}\alpha^{-1} \\ 0 \\ 0\end{bmatrix} M(1) \begin{bmatrix}\alpha\bar{\alpha} \\ 0 \\ 0\end{bmatrix}$ and $N(\alpha)=\begin{bmatrix}\alpha^{-1} \\ 0 \\ 0\end{bmatrix} N(1) \begin{bmatrix}\alpha\bar{\alpha} \\ 0 \\ 0\end{bmatrix}$, $\det(M(\alpha))=2 \cdot n(\alpha) \neq 0$ and $\det(N(\alpha))=4 \cdot n(\alpha) \neq 0$.

If $\alpha \neq 1$, then $\det(M(\alpha)-M(1))=2 \cdot n(\alpha-1) \neq 0$. Hence $\det(M(\alpha)-M(\beta)) \neq 0$ for any $\alpha \neq \beta \in S$.

If $\alpha \neq 1$, then $\det(N(\alpha)-N(1))=4(n(\alpha)+tr(\alpha)-tr(\alpha\bar{\alpha})-1)=4(\alpha-1) \cdot (\bar{\alpha}-1)(\bar{\bar{\alpha}}-1) \neq 0$. Hence $\det(N(\alpha)-N(\beta)) \neq 0$ for any $\alpha \neq \beta \in S$.

Suppose $\det(M(\alpha)-N(1))=0$. Since $\det(M(\alpha)-N(1))=2 \cdot n(\alpha)-(\mathrm{tr}(\alpha))^2+4 \cdot \mathrm{tr}(\alpha)-4$, $2=(tr(\alpha)-2)^2(n(t^{-i}))^2$, where $\alpha=t^{2i}$, a contradiction. Hence we have $\det(M(\alpha)-N(\beta)) \neq 0$ for any $\alpha, \beta \in S$.

Clearly $|\Sigma|=q^3$ and the results follow.

Let π be the translation plane which corresponds to the spread set Σ of Theorem 3.1 and G its linear translation complement. Set $\Sigma^*=\Sigma \cup \{\infty\}$ and $\Pi=\{V(M) \mid M \in \Sigma^*\}$.

4. The linear translation complement of π

In this section we show the linear translation complement G of π. We describe the Sherk's Theorem in the case $n=3$ using the Oyama's Method.

Lemma 4.1 (F.A. Sherk [2]). *Let $i\in\{1, 2\}$. Let Σ_i be a spread set of degree 3 over $GF(q)$ with $0\in\Sigma_i$ and π_i be the translation plane of order q^3 which corresponds to the spread set Σ_i. Set $\Pi_i=\{V(M)\mid M\in\Sigma_i\cup\{\infty\}\}$. Then π_1 and π_2 are isomorphic if and only if there exist A, B, C and D in $M(3, q)^*$ and θ in $Aut(GF(q^3))$ with the following properties.*

(a) $det\begin{pmatrix} A & C \\ B & D \end{pmatrix}\neq 0.$

(b) *One of the following holds*

(i) $B=0$, $det(A)\neq 0$ *and* $\Sigma_2=\{A^{-1}(C+M^\theta D)\mid M\in\Sigma_1\}$; *or*

(ii) $det(B)\neq 0$, $B^{-1}D\in\Sigma_2$, *there is* $M_0\in\Sigma_1$ *such that* $A+M_0^\theta B=0$ *and for any* $M\in\Sigma_1-\{M_0\}$, $det(A+M^\theta B)\neq 0$ *and* $(A+M^\theta B)^{-1}(C+M^\theta D)\in\Sigma_2$.

Each $\tau\in G$ induces a permutation on Π which we denote by $\tilde{\tau}$.

Theorem 4.2. *If $q=3$, then π is the Hering plane of order 27.*

Proof. Let $t\in GF(27)$ and $t^3=-1+t$. Then $GF(27)^*=\langle t\rangle$. Set

$$\varepsilon=\begin{pmatrix} 1 & 1 & 1 \\ t & \bar{t} & \bar{\bar{t}} \\ t^2 & \overline{t^2} & \overline{\overline{t^2}} \end{pmatrix}\in GL(3, 27).$$

F.A. Sherk [2] gave a spread set

$$S_H=\left\{\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}^{3i} R \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \,\middle|\, R=\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix},\ 1\leqq i\leqq 13\right\}\cup\{0\}$$

defining the Hering plane of order 27. Since

$$\varepsilon^{-1}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\varepsilon=\begin{bmatrix} t^7 & & \\ & t^8 & \\ & & t^{12} \end{bmatrix},\quad \varepsilon^{-1}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\varepsilon=\begin{bmatrix} & & t^9 \\ & -1 & \\ -t^7 & & \end{bmatrix}$$

$$\text{and } \varepsilon^{-1}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}\varepsilon=\begin{bmatrix} & & t^3 \\ & t^2 & \\ -t^2 & & \end{bmatrix},\quad \varepsilon^{-1}S_H\varepsilon=$$

$$\left\{\begin{bmatrix} t^7 & & \\ & t^8 & \\ & & t^{12}\end{bmatrix}^{3i} R \begin{bmatrix} t^7 & & \\ & t^8 & \\ & & t^{12}\end{bmatrix}^{i} \,\middle|\, R=\begin{bmatrix} & & t^9 \\ & -1 & \\ -t^7 & & \end{bmatrix} \text{ or } \begin{bmatrix} & & t^3 \\ & t^2 & \\ -t^2 & & \end{bmatrix},\ 1\leqq i\leqq 13\right\}\cup\{0\}$$

Set $\varepsilon^{-1}S_H\varepsilon=\Sigma_H$ and $\Pi_H=\{V(M)\,|\,M\in\Sigma_H\cup\{\infty\}\}$. Set $\varphi=\begin{pmatrix} A^{-1} & 0 \\ 0 & B\end{pmatrix}\in GL(V)$, where $A=\begin{bmatrix} -t & & \\ & t^6 & \\ & & -t^5\end{bmatrix}$ and $B=\begin{bmatrix} & & t \\ & -1 & \\ t^7 & & \end{bmatrix}$. By a computation, we get $\Pi^{\varphi}=\Pi_H$. Thus Thorem 4.2 is proved.

The following statements hold:

Lemma 4.3.
(i) *If* $e\in GF(q)\cap S$, *then* $\begin{pmatrix} I & 0 \\ 0 & E\end{pmatrix}\in G_{V(\infty),V(I)}$, *where* $E=\begin{bmatrix} e & & \\ & 0 & \\ & & 0\end{bmatrix}$.

(ii) *Set* $\tau_0=\begin{pmatrix} 0 & 2T \\ I & 0\end{pmatrix}$, *where* $T=\begin{bmatrix} & & 0 \\ & 1 & \\ 0 & & \end{bmatrix}$. *Then* $\tau_0\in G$ *and* $\tilde{\tau}_0=$ $(V(\infty),\ V(0))(V(M(1)),\ V(N(1)))\cdots$.

(iii) *Set* $\Gamma=\{V(M(\alpha))\,|\,\alpha\in S\}$, $\Delta=\{V(N(\alpha))\,|\,\alpha\in S\}$ *and* $H=\{\varphi_\alpha\,|\,\alpha\in S\}$, *where* $\varphi_\alpha=\begin{pmatrix} A_\alpha & 0 \\ 0 & B_\alpha\end{pmatrix}$, $A_\alpha=\begin{bmatrix} \alpha & & \\ & 0 & \\ & & 0\end{bmatrix}$ *and* $B_\alpha=\begin{bmatrix} \alpha\bar{\alpha} & & \\ 0 & & \\ 0 & & \end{bmatrix}$. *Then* Γ *and* Δ *are H-orbits on* Π *and* $H\leq G_{V(0),V(\infty)}$.

Lemma 4.4. *Set* $K=\left\{\begin{pmatrix} aI & 0 \\ 0 & aI\end{pmatrix}\,\middle|\, a\in GF(q)-\{0\}\right\}$ *and* $L=G_{V(\infty),V(0),V(M(1))}$. *Then*

(i) $L=\left\langle\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\right\rangle\cdot K$, *where* $T=\begin{bmatrix} & & 0 \\ & 1 & \\ 0 & & \end{bmatrix}$.

(ii) *If* $\alpha\in S-GF(q)$, *then* $L_{V(M(\alpha))}=K$.

Proof.
Step 1. If $\alpha\in S$, then $\{V(M(\alpha)),\ V(M(\bar{\alpha})),\ V(M(\bar{\bar{\alpha}}))\}$ is an L-orbit.
Since $\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\in L$, there exists an L-orbit containing $\{V(M(\alpha)),\ V(M(\bar{\alpha})),\ V(M(\bar{\bar{\alpha}}))\}$.
Let $\tau\in L$. Since $V(\infty)^\tau=V(\infty)$, $V(0)^\tau=V(0)$ and $V(M(1))^\tau=V(M(1))$, $\tau=\begin{pmatrix} A & 0 \\ 0 & M(1)^{-1}AM(1)\end{pmatrix}$ for some $A\in GL(3,q)^*$. Set $B=M(1)^{-1}AM(1)$. Let $\alpha\in S$. Suppose that $A^{-1}M(\alpha)B=N(\beta)$ for some $\beta\in S$. Then $A^{-1}M(\alpha)M(1)^{-1}A=N(\beta)M(1)^{-1}$. Hence $\det(M(\alpha))=\det(N(\beta))$. From this $n(\alpha\beta^{-1})=2$ follows.

This is a contradiction since 2 is a nonsquare. Thus $A^{-1}M(\alpha)B=M(\beta)$ for some $\beta\in S$. Therefore $A^{-1}M(\alpha)M(1)^{-1}A=M(\beta)M(1)^{-1}$. Let $x\in GF(q)$. Since

$$\det(A^{-1}M(\alpha)M(1)^{-1}A-xI)=\det(M(\beta)M(1)^{-1}-xI),\ \det\begin{bmatrix}\overline{\alpha-x} \\ \alpha-x \\ 0\end{bmatrix}=\det\begin{bmatrix}\overline{\beta-x} \\ \beta-x \\ 0\end{bmatrix}.$$

Hence $n(\alpha)-x\cdot tr(\alpha\bar{\alpha})+x^2\cdot tr(\alpha)-x^3=n(\beta)-x\cdot tr(\beta\bar{\beta})+x^2\cdot tr(\beta)-x^3$. From this $n(\alpha)=n(\beta)$, $tr(\alpha\bar{\alpha})=tr(\beta\bar{\beta})$ and $tr(\alpha)=tr(\beta)$ follow. This implies that $(\beta-\alpha)\cdot(\beta-\bar{\alpha})(\beta-\bar{\bar{\alpha}})=0$. Thus $\beta\in\{\alpha,\ \bar{\alpha},\ \bar{\bar{\alpha}}\}$.

Step 2. $L=L_{V(N(1))}$.

Let $\tau\in L$. Then $\tau=\begin{pmatrix}A & 0 \\ 0 & M(1)^{-1}AM(1)\end{pmatrix}$ for some $A\in GL(3,\ q)^*$. Set $B=M(1)^{-1}AM(1)$. By Step 1, $A^{-1}N(1)B=N(\alpha)$ for some $\alpha\in S$. Since

$$A^{-1}N(1)M(1)^{-1}A=N(\alpha)M(1)^{-1} \quad\text{and}\quad A^{-1}M(1)N(1)^{-1}A=M(1)N(\alpha)^{-1},$$
$$A^{-1}(N(1)M(1)^{-1}+M(1)N(1)^{-1})A=N(\alpha)M(1)^{-1}+M(1)N(\alpha)^{-1}.$$

From this, since

$$N(1)M(1)^{-1}+M(1)N(1)^{-1}=(5/2)I \text{ and } N(\alpha)M(1)^{-1}+M(1)N(\alpha)^{-1}$$
$$=1/2\begin{bmatrix}\bar{\alpha}+\bar{\bar{\alpha}}+\bar{\alpha}\bar{\bar{\alpha}}\alpha^{-1}+\overline{\alpha^{-1}}+\overline{\overline{\alpha^{-1}}} \\ \alpha-\bar{\bar{\alpha}}-\alpha\bar{\bar{\alpha}}\overline{\alpha^{-1}}+\overline{\alpha^{-1}} \\ \alpha-\bar{\alpha}-\alpha\bar{\alpha}\overline{\overline{\alpha^{-1}}}+\overline{\overline{\alpha^{-1}}}\end{bmatrix},$$

$$0=\alpha-\bar{\bar{\alpha}}-\alpha\bar{\bar{\alpha}}\overline{\alpha^{-1}}+\overline{\alpha^{-1}} \tag{4.1}$$

and

$$0=\alpha-\bar{\alpha}-\alpha\bar{\alpha}\overline{\overline{\alpha^{-1}}}+\overline{\overline{\alpha^{-1}}} \tag{4.2}$$

follow. From (4.1),

$$0=\bar{\alpha}-\alpha-\alpha\bar{\alpha}\overline{\overline{\alpha^{-1}}}+\overline{\overline{\alpha^{-1}}} \tag{4.3}$$

follows. Subtracting (4.3) from (4.2) we have $0=2\alpha-2\bar{\alpha}$. Therefore $\bar{\alpha}=\alpha$ and so $\alpha\in GF(q)$. From this and (4.3), $\alpha=1$ follows. Thus $V(N(1))^\tau=V(N(1))$.

Step 3. Proof of (ii).

Let $\alpha\in S-GF(q)$ and $\tau\in L_{V(M(\alpha))}$. Then $\tau=\begin{pmatrix}A & 0 \\ 0 & M(1)^{-1}AM(1)\end{pmatrix}$ for some $A\in GL(3,\ q)^*$. Since $V(M(\alpha))^\tau=V(M(\alpha))$, $A^{-1}M(\alpha)M(1)^{-1}A=M(\alpha)M(1)^{-1}$. Suppose that $\tau\notin K$. Since $\det(M(\alpha)M(1)^{-1}-xI)\neq0$ for any $x\in GF(q)$, there exists $W\in GL(3,\ q)^*$ such that $o(W)=q^3-1$ and $\langle W\rangle\ni M(\alpha)M(1)^{-1}$ by Lemma 2.5. By Lemma 2.3 and Lemma 2.5, $C_{GL(3,q)^*}(M(\alpha)M(1)^{-1})=\langle W\rangle$. Thus we get $A\in\langle W\rangle$ and $C_{GL(3,q)^*}(A)=\langle W\rangle$. By Step 2, $A^{-1}N(1)M(1)^{-1}A=N(1)M(1)^{-1}$, hence $N(1)M(1)^{-1}\in\langle W\rangle$. Clearly $2I\in\langle W\rangle$ and $N(1)M(1)^{-1}\neq2I$. Therefore $N(1)M(1)^{-1}-2I\in\langle W\rangle$ by Lemma 2.4 and so $\det(N(1)M(1)^{-1}-2I)\neq0$, a contradiction. Thus we get (ii).

Step 4. Proof of (i).

Let $\alpha \in S-GF(q)$. By Step 3, $L_{V(M(\alpha))}=K$. Furthermore $\{V(M(\alpha)), V(M(\bar{\alpha})), V(M(\bar{\bar{\alpha}}))\}$ is an orbit of both L and $\left\langle\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\right\rangle$ by Step 1. Hence $L=\left\langle\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\right\rangle \cdot K$.

Lemma 4.5.
(i) $G_{V(\infty),V(0),V(1)}=\left\langle\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\right\rangle \cdot K$, *where* $T=\begin{bmatrix} 0 \\ 1 \\ 0\end{bmatrix}$.

(ii) *If* $\alpha \in S-GF(q)$, *then* $G_{V(\infty),V(0),V(N(1)),V(N(\alpha))}=K$.

Proof.

(i) Since $\tau_0^{-1}\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\tau_0=\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}$, $G_{V(\infty),V(0),V(N(1))}=(G_{V(0),V(\infty),V(M(1))})^{\tau_0}=\left\langle\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\right\rangle \cdot K$ by Lemma 4.4 (i).

(ii) Let $\alpha \in S-GF(q)$. Since $V(M(\overline{\alpha^{-1}}))^{\tau_0}=V(N(\alpha))$, $G_{V(\infty),V(0),V(N(1)),V(N(\alpha))}=(G_{V(0),V(\infty),V(M(1)),V(M(\overline{\alpha^{-1}}))})^{\tau_0}=K$ by Lemma 4.4 (ii).

Lemma 4.6. $G_{V(\infty)}=G_{V(\infty),V(0)}$.

Proof.

Case (a). $q \neq 3$.

Suppose false. By Lemma 4.3 (iii), there exists $\tau \in G_{V(\infty)}$ with $V(0)^{\tau}=V(M(1))$ or $V(N(1))$. As $V(\infty)^{\tau}=V(\infty)$, there exist $A, B, C \in M(3, q)^*$ such that $\tau=\begin{pmatrix} A & C \\ 0 & B\end{pmatrix}$. Since $|GF(q)\cap S|=(q-1)/2\geq 2$, there exists $e \in S\cap GF(q)$ with $e\neq 1$. Since $\begin{pmatrix} I & 0 \\ 0 & E\end{pmatrix}\in G$ where $E=\begin{bmatrix} e \\ 0 \\ 0\end{bmatrix}$, $\Sigma=\{A^{-1}MB+A^{-1}C \mid M\in\Sigma\}=\{EA^{-1}MB+A^{-1}C \mid M\in\Sigma\}$. Assume that $V(0)^{\tau}=V(M(1))$. Then $A^{-1}C=M(1)$. There exists $M_0\in\Sigma$ with $A^{-1}M_0B+M(1)=N(1)$, Therefore $A^{-1}M_0B=\begin{bmatrix} 0 \\ 0 \\ -1\end{bmatrix}$ and $EA^{-1}M_0B+M(1)=\begin{bmatrix} 1 \\ 1 \\ -e\end{bmatrix}\in\Sigma$. From this $e=1$ follows, a contradiction.

Next assume that $V(0)^{\tau}=V(N(1))$. Then by the similar argument above, we have a contradiction again.

Case (b). $q=3$.

Assume that $G_{V(\infty)}$ is transitive on $\Pi-\{V(\infty)\}$. Then G is 2-transitive on Π. Since π has not a Baer subplane, π is a desarguesian plane by Theorem 39.3 of [4]. This is a contradiction.

Assume that $G_{V(\infty)} \neq G_{V(\infty),V(0)}$. Then $\{V(0)\} \cup \Gamma$ or $\{V(0)\} \cup \Delta$ is $G_{V(\infty)}$-orbit on Π. Write this orbit by Ω. $G_{V(\infty)}$ induces the permutation group $G_{V(\infty)}/K$ on Ω by Lemma 4.4 (ii) and Lemma 4.5 (ii). Since $|\Omega|=14$ and $G_{V(\infty)}/K$ is a 2-transitive permutation group on Ω, $G_{V(\infty)}/K \geq PSL(2, 13)$. Since $|\Omega|=14$, the permutation group $PSL(2,13)$ on Ω contains an involution g which fixes exactly two points of Ω. There exists $\tau \in G_{V(\infty)}$ with $g=\tau K$. Since $|K|=2$, $o(\tau)=2$ or 4. Suppose that $0(\tau)=2$. As π has not a Baer subplane, τ is a $((0, 0), l_\infty)$-perspectivity. Therefore τ fixes any component of Ω, a contradiction. Suppose that $o(\tau)=4$. As τ^2 is a $((0, 0), l_\infty)$-perspectivity, any cycle of τ on $V(\infty)-\{(0, 0), V(\infty) \cap l_\infty\}$ is 4-cycle. Therefore $4|26$, a contradiction.

Lemma 4.7. $G_{V(0)}=G_{V(0),V(\infty)}$.

Proof. $G_{V(0)}=(G_{V(\infty)})^{\tau_0}=(G_{V(\infty),V(0)})^{\tau_0}=G_{V(0),V(\infty)}$.

Lemma 4.8. *Set* $\Psi=\{V(\infty), V(0)\}$. *Then* Ψ *is a G-block on* Π.

Proof. Suppose $\varphi \in G$ and $\Psi^\varphi \cap \Psi \neq \phi$. We may assume that $V(\infty)^\varphi = V(\infty)$ or $V(0)^\varphi=V(\infty)$. Assume $V(\infty)^\varphi=V(\infty)$. Then $\varphi \in G_{V(\infty)}=G_{V(\infty),V(0)}$ and so $V(0)^\varphi=V(0)$. Assume $V(0)^\varphi=V(\infty)$. Then $G_{V(0),V(\infty)}=G_{V(\infty)}=G_{V(0)^\varphi}= G_{V(\infty)^\varphi, V(0)^\varphi}=G_{V(\infty)^\varphi V(\infty)}$. From this and Lemma 4.3 (iii), $V(\infty)^\varphi=V(0)$ follows. Therefore $\Psi^\varphi=\Psi$.

Lemma 4.9. Γ *and* Δ *are* $G_{V(\infty)}$*-orbits on* Π.

Proof. Suppose false. By Lemma 4.3 (iii) there exists $\tau \in G_{V(\infty)}$ with $V(M(1))^\tau=V(N(1))$. Since $V(0)^\tau=V(0)$, $\tau=\begin{pmatrix} A & 0 \\ 0 & M(1)^{-1}AN(1)\end{pmatrix}$ for some $A \in GL(3, q)^*$. Set $B=M(1)^{-1}AN(1)$. Assume $A^{-1}N(1)B=N(\alpha)$ for some $\alpha \in S$. From $\det(A^{-1}N(1)B)=\det(N(\alpha))$, $n(\alpha)=2$ follows, a contradiction. Thus $A^{-1}N(1)B=M(\alpha)$ for some $\alpha \in S$. Let $q=p^n$ with p a prime.

Step 1. $p=3$ or $p=5$. If $p=3$, then $A^{-1}N(1)B=M(1)$. If $p=5$, then $A^{-1}N(1)B=M(-1)$.

Set $\rho=\tau_0^2=\begin{pmatrix} M(1)N(1) & 0 \\ 0 & M(1)N(1)\end{pmatrix}$. Since $\rho^\tau \in G_{V(\infty),V(0),V(N(1))}=\langle\rho\rangle \cdot K$, $\rho^\tau=b\rho$ or $b\rho^2$ for some $b \in GF(q)$. Therefore τ fixes $\{V(M) \mid V(M)^\rho=V(M)\} - \{V(\infty), V(0)\}=\{V(M(x)), V(N(x)) \mid x \in S \cap GF(q)\}$ as a set. Thus $\alpha \in S \cap GF(q)$. Set $\alpha=a$. Clearly $V(N(1))^\tau=V(M(a))$.

Let $x \in GF(q)$. Clearly $A^{-1}N(1)M(1)^{-1}A=M(a)N(1)^{-1}$. Since $\det(A^{-1}N(1)M(1)^{-1}A-xI)=\det(M(a)N(1)^{-1}-xI)$, $\det((N(1)-M(x))N(1))= \det((M(a)-N(x))M(1))$. From this $(12a-18)x^2+(24-9a^2)x+2a^3-8=0$. Therefore $3a=9/2$ and $a^3=4$. If $p=3$, then $a=1$. If $p \neq 3$, then $p=5$ and

$a=-1$.

Step 2. $3\not| q-1$.

If $p=3$, then $3\not| q-1$.

Assume $p=5$. Suppose that n is even. Then 2 is a square in $GF(5^n)$, a contradiction. Therefore n is odd. Then $5^n-1\equiv(-1)^n-1\equiv-2 \pmod 3$. Thus $3\not| q-1$.

Step 3. Set $T=\begin{bmatrix}0\\1\\0\end{bmatrix}$. Then $\begin{pmatrix}T&0\\0&T\end{pmatrix}^\tau=\begin{pmatrix}T&0\\0&T\end{pmatrix}$. If $p=3$ then $\tau^2\in\left\langle\begin{pmatrix}T&0\\0&T\end{pmatrix}\right\rangle\cdot K$. If $p=5$, then $\tau^2\in\left\langle\begin{pmatrix}T&0\\0&-T\end{pmatrix}\right\rangle\cdot K$.

If $p=3$, then $\tau^2\in G_{V(\infty),V(M(1))}=\left\langle\begin{pmatrix}T&0\\0&T\end{pmatrix}\right\rangle\cdot K$. If $p=5$, then $\tau^2\in\left\langle\begin{pmatrix}T&0\\0&-T\end{pmatrix}\right\rangle\cdot K$ as $V(\infty)^{\tau^2}=V(\infty)$ and $V(M(1))^{\tau^2}=V(M(-1))$. Since $(G_{V(\infty),V(M(1))})^\tau=G_{V(\infty),V(N(1))}=\left\langle\begin{pmatrix}T&0\\0&T\end{pmatrix}\right\rangle\cdot K$, $\begin{pmatrix}T&0\\0&T\end{pmatrix}^\tau=a\begin{pmatrix}T&0\\0&T\end{pmatrix}$ or $a\begin{pmatrix}T^2&0\\0&T^2\end{pmatrix}$ for some $a\in GF(q)$. From this $a^3=1$ follows as $o\left(\begin{pmatrix}T&0\\0&T\end{pmatrix}^\tau\right)=3$. Thus since $3\not| q-1$, we have $a=1$.

Suppose $\begin{pmatrix}T&0\\0&T\end{pmatrix}^\tau=\begin{pmatrix}T^2&0\\0&T^2\end{pmatrix}$. Then $\begin{pmatrix}A^{-1}TA&0\\0&B^{-1}TB\end{pmatrix}=\begin{pmatrix}T^2&0\\0&T^2\end{pmatrix}$. Since $A^{-1}TA=T^2$, $A=\begin{bmatrix}\alpha\\\bar{\bar{\alpha}}\\\bar{\alpha}\end{bmatrix}$ for some $\alpha\in GF(q^3)-GF(q)$. If $\alpha\notin S$, then we take $e\tau$ instead of τ with $e\in GF(q)-GF(q)^2$. Thus we may assume that $\alpha\in S$. Since $A^2=\begin{bmatrix}tr(\alpha^2)\\tr(\alpha\bar{\alpha})\\tr(\alpha\bar{\alpha})\end{bmatrix}$ and $\begin{pmatrix}A^2&0\\0&B^2\end{pmatrix}\in\left\langle\begin{pmatrix}T&0\\0&\pm T\end{pmatrix}\right\rangle\cdot K$, $tr(\alpha\bar{\alpha})=0$ and $tr(\alpha^2)\neq 0$. Set $tr(\alpha)=b$. Since $b^2=tr(\alpha^2)+2\cdot tr(\alpha\bar{\alpha})=tr(\alpha^2)$, $b\neq 0$. Set $V(M(\alpha))^\tau=V(\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix})$. By an easy computation, $B=M(1)^{-1}AN(1)=-1/2\begin{bmatrix}b-4\bar{\bar{\alpha}}\\b-4\bar{\alpha}\\b-4\alpha\end{bmatrix}$ follows. Since $A^{-1}M(\alpha)B=-1/2b^{-2}\begin{bmatrix}\alpha\\\bar{\bar{\alpha}}\\\bar{\alpha}\end{bmatrix}\begin{bmatrix}\bar{\alpha}\\\alpha\\0\end{bmatrix}\begin{bmatrix}b-4\bar{\bar{\alpha}}\\b-4\bar{\alpha}\\b-4\alpha\end{bmatrix}$, $-2b^2\alpha_1=b(\bar{\alpha}^2+\bar{\bar{\alpha}}^2+2\alpha\bar{\alpha}+2\alpha\bar{\bar{\alpha}})-4(n(\alpha)+\bar{\alpha}^3+\alpha\bar{\bar{\alpha}}^2+\bar{\alpha}\bar{\bar{\alpha}}^2+\bar{\alpha}\alpha^2+\bar{\bar{\alpha}}\alpha^2)$ and $-2b^2\alpha_2=b(\alpha^2+\bar{\bar{\alpha}}^2+2\alpha\bar{\alpha}+2\bar{\alpha}\bar{\bar{\alpha}})-4(n(\alpha)+\alpha^3+\bar{\alpha}\bar{\bar{\alpha}}^2+\bar{\bar{\alpha}}\bar{\alpha}^2+\alpha\bar{\alpha}^2+\alpha\bar{\bar{\alpha}}^2)$. Since $-2b^2\alpha_1=-\overline{2b^2\alpha_2}$, we get $b(\bar{\bar{\alpha}}^2+2\alpha\bar{\alpha})=b(\alpha^2+2\bar{\alpha}\bar{\bar{\alpha}})$. Since $b\neq 0$, $\bar{\bar{\alpha}}^2+2\alpha\bar{\alpha}=\alpha^2+2\bar{\alpha}\bar{\bar{\alpha}}$. Therefore $(\bar{\bar{\alpha}}-\alpha)(\bar{\bar{\alpha}}+\alpha-2\bar{\alpha})=0$ and so $\bar{\bar{\alpha}}+\alpha=2\bar{\alpha}$. From this $b=3\bar{\alpha}$ follows. Thus $\alpha\in GF(q)$, a contradiction.

Step 4. Contradiction.

Since $A^{-1}TA=T$, $A=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some $a, b, c \in GF(q)$. Assume $p=3$. Since $A^{-1}N(1)B=M(1)$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} A=A\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. From this $c=-(a+b)$ follows. But $\det(A)=\det\begin{bmatrix} a \\ b \\ -(a+b) \end{bmatrix}=a^3+b^3-(a+b)^3=0$, a contradiction. Assume $p=5$. Since $A^2=\begin{bmatrix} a^2+2bc \\ c^2+2ab \\ b^2+2ac \end{bmatrix}$ and $\begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \in \left\langle \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \right\rangle \cdot K$, $A^2=\begin{bmatrix} a^2+2bc \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ c^2+2ab \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ b^2+2ac \end{bmatrix}$. Let $\lambda \in S-GF(q)$. We consider the case $A^2=\begin{bmatrix} a^2+2bc \\ 0 \\ 0 \end{bmatrix}$. Then $c^2+2ab=b^2+2ac=0$. Suppose $b=0$. Then $c=0$, $A=aI$ and $B=M(1)^{-1}AN(1)$ $=a\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. Therefore $V(M(\lambda))^\tau=V(\begin{bmatrix} \bar{\lambda} \\ \lambda \\ 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix})=V(\begin{bmatrix} -\bar{\lambda}+2\bar{\bar{\lambda}} \\ -\lambda+2\bar{\lambda} \\ \lambda+2\bar{\lambda} \end{bmatrix})$. Thus $-\bar{\lambda}+2\bar{\bar{\lambda}}=\overline{(-\lambda+2\bar{\lambda})}$. This implies $\lambda \in GF(q)$, a contradicion. Suppose $b\neq 0$. Substituting $a=-c^2/2b$ in $b^2+2ac=0$, we get $b=c$ and $a=2b$. From this $A=b\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $B=M(1)^{-1}AN(1)=bI$ follow. Therefore $V(M(\lambda))^\tau=$ $V(\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} M(\lambda))=V(\begin{bmatrix} 2\lambda-\bar{\lambda} \\ -\lambda+2\bar{\lambda} \\ 2\lambda+2\bar{\lambda} \end{bmatrix})$. Thus $2\lambda-\bar{\lambda}=\overline{(-\lambda+2\bar{\lambda})}$. This implies $\lambda \in GF(q)$, a contradiction. Also when $A^2=\begin{bmatrix} 0 \\ c^2+2ab \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ b^2+2ac \end{bmatrix}$, similarly we have a contradiction.

Theorem 4.10. *If $q\neq 3$, then G has two orbits of length 2 and length q^3-1 on Π.*

Proof. Let $q\neq 3$. Suppose false. Then G is transitive on Π. Since $\{V(\infty), V(0)\}$ is a G-block by Lemma 4.8, there exists $V(M)\in\Pi-\{V(\infty), V(0), V(M(1))\}$ such that $\Lambda=\{V(M(1)), V(M)\}$ is a G-block. Since $V(M(1))^{\tau_0^2}=V(M(1))$, $\Lambda^{\tau_0^2}=\Lambda$ and so $V(M)^{\tau_0^2}=V(M)$. Therefore $M=M(a)$ or $N(a)$ for some $a\in S\cap GF(q)$.

Assume that $\Lambda=\{V(M(1)),\ V(M(a))\}$ is a G-block. Set $\rho=\begin{pmatrix} I & 0 \\ 0 & aI \end{pmatrix}\in G$. Then $\Lambda^{\rho}=\{V(M(a)),\ V(M(a^2))\}=\Lambda$. Therefore $M(a^2)=M(1)$ and so $a=-1$ as $a\neq 1$. Since $\langle\{\varphi_\alpha|\alpha\in S\},\ \tau_0\rangle\leq G$ by Lemma 4.3, $\{V(M),\ V(-M)\}$ is a G-block for any $V(M)\in\Pi-\{V(\infty),\ V(0)\}$. Set $\sigma_1=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\in G$. Now σ_1 fixes exactly two components $V(\infty)$ and $V(0)$ in Π. Furthermore σ_1 fixes any G-block on Π. Since G is transitive on Π, there exists σ_2 such that σ_2 is conjugate to σ_1 and fixes exactly two components $V(M(1))$, $V(M(-1))$ in Π and all G-blocks on Π. Therefore $\sigma_1\sigma_2\tau_0\in G_{V(\infty),V(0)}$. But $V(M(1))^{\sigma_1\sigma_2\tau_0}=V(N(-1))$. This is contrary to Lemma 4.9.

Next assume that $\{V(M(1)),\ V(N(a))\}$ is a G-block. Since $\{V(M(1)),\ V(N(a))\}^{\varphi_\alpha}=\{V(M(\alpha)),\ V(N(a\alpha))\}$ is a G-block for any $\alpha\in S$, G is 2-transitive on the set of G-blocks. Therefore there exists $\varphi\in G$ such that φ interchanges $\{V(\infty),\ V(0)\}$ and $\{V(M(1)),\ V(N(a))\}$. Let $\varphi=\begin{pmatrix} A & C \\ B & D \end{pmatrix}$. Suppose that $\tilde{\varphi}=(V(\infty),\ V(M(1)))(V(0),\ V(N(a)))\cdots$ on Π. Then

$$\varphi=\begin{pmatrix} A & AN(a) \\ -M(1)^{-1}A & -M(1)^{-1}AM(1) \end{pmatrix}.$$

Let $b\in S\cap GF(q)$ with $b\neq 1$. Then

$$\begin{aligned} &V(M(b))^{\varphi} \\ &= V((A-M(b)M(1)^{-1}A)^{-1}(AN(a)-M(b)M(1)^{-1}AM(1))) \\ &= V((A-bA)^{-1}(AN(a)-bAM(1))) \\ &= V((1-b)^{-1}(N(a)-M(b)))\,. \end{aligned}$$

Hence $(1-b)^{-1}(N(a)-M(b))=(1-b)^{-1}\begin{bmatrix} a-b \\ a-b \\ -a \end{bmatrix}\in\Sigma$. Since $a\neq 0$, $a-b=a$ and so $b=0$, a contradiction. Suppose that $\tilde{\varphi}=(V(\infty),\ V(N(a)))(V(0),\ V(M(1)))\cdots$ on Π. Set $\tau=\tau_0\begin{pmatrix} I & 0 \\ 0 & aI \end{pmatrix}$. Since $\tilde{\tau}=(V(\infty),\ V(0))(V(M(1)),\ V(N(a)))\cdots$, $\widetilde{\varphi\tau}=(V(\infty),\ V(M(1)))(V(0),\ V(N(a)))\cdots$. This is the above case, a contradiction. Suppose that $\tilde{\varphi}=(V(\infty),\ V(M(1)),\ V(0),\ V(N(a)))\cdots$ on Π. Then $\varphi=\begin{pmatrix} A & AN(a) \\ -N(a)^{-1}A & -N(a)^{-1}AM(1) \end{pmatrix}$. Let $b\in S\cap GF(q)$ with $b\neq a$. Then $V(N(b))^{\varphi}=V((1-ba^{-1})^{-1}(N(a)-M(ba^{-1})))$. Thus $b=0$ by the similar argument above, a contradiction. Suppose that $\tilde{\varphi}=(V(\infty),\ V(N(a)),\ V(0),\ V(M(1)))\cdots$ on Π. Then $\widetilde{\varphi^3}=(V(\infty),\ V(M(1)),\ V(0),\ V(N(a)))\cdots$. This is a contradiction.

Theorem 4.11. *If $q\neq 3$, then* $|G|=3(q-1)(q^3-1)$.

Proof. By Lemma 4.4, Lemma 4.6, Lemma 4.9 and Theorem 4.10, $|G|=|V(\infty)^G||G_{V(\infty)}|=2|G_{V(\infty)}|=2|G_{V(\infty),V(M(1))}||V(M(1))^{G_{V(\infty)}}|=3(q-1)(q^3-1)$.

Theorem 4.12. *If $q=3$, then* $G\cong SL(2, 13)$.

Proof. Since G is transitive on Π [5], $|G|=28|G_{V(\infty)}|$. By Lemma 4.4, Lemma 4.6 and Lemma 4.9, $|G_{V(\infty)}|=|G_{V(\infty)\ V(M(1))}||V(M(1))^{G_{V(\infty)}}|=6\cdot 13$. Therefore $|G|=2^3\cdot 3\cdot 7\cdot 13$. On the other hand since $|SL(2, 13)|=|G|$ and $G\geq SL(2, 13)$ by [5], $G\cong SL(2, 13)$.

Theorem 4.13. *π is not a generalized André plane.*

Proof. Assume that π is a generalized André plane. Then there exist $\Sigma_1\subseteq\left\{\begin{bmatrix}\alpha\\0\\0\end{bmatrix}\middle|\alpha\in GF(q^3)^*\right\}$, $\Sigma_2\subseteq\left\{\begin{bmatrix}0\\\alpha\\0\end{bmatrix}\middle|\alpha\in GF(q^3)^*\right\}$ and $\Sigma_3\subseteq\left\{\begin{bmatrix}0\\0\\\alpha\end{bmatrix}\middle|\alpha\in GF(q^3)^*\right\}$ such that $\Sigma_1\ni I$ and $\Sigma_A=\Sigma_1\cup\Sigma_2\cup\Sigma_3\cup\{0\}$ is the spread set defining π. If $\alpha\neq\beta\in GF(q^3)^*$ and $n(\alpha)=n(\beta)$, then $\begin{bmatrix}\alpha\\0\\0\end{bmatrix}, \begin{bmatrix}\beta\\0\\0\end{bmatrix}\in\Sigma_1$, $\begin{bmatrix}0\\\alpha\\0\end{bmatrix}, \begin{bmatrix}0\\\beta\\0\end{bmatrix}\in\Sigma_2$ or $\begin{bmatrix}0\\0\\\alpha\end{bmatrix}, \begin{bmatrix}0\\0\\\beta\end{bmatrix}\in\Sigma_3$. Let $\pi(\Sigma_A)$ is the translation plane π which is constructed by Σ_A. Now since the order of π is q^3, π is an André plane by Corollary 12.5 of [4]. Let $G(\Sigma_A)$ is the linear translation complement of $\pi(\Sigma_A)$. Set $\tau=\begin{pmatrix}W & 0\\ 0 & W\end{pmatrix}$ where $W=\begin{bmatrix}t\\0\\0\end{bmatrix}$. Then $\langle\tau\rangle\leq G(\Sigma_A)_{V(0),V(\infty),V(I)}$ and $\langle\tau\rangle$ is transitive on $V(0)-\{(0, 0), V(0)\cap l_\infty\}$. This is contrary to Theorem 12.1 of [4].

Let $q\neq 3$. Since the translation complement of any proper semifield plane have an orbit of length 1 on l_∞, π differs from any semifield plane.

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