Ikawa, M.
Osaka J. Math.
22 (1985), 657-689

# TRAPPING OBSTACLES WITH A SEQUENCE OF POLES OF THE SCATTERING MATRIX CONVERGING TO THE REAL AXIS

MITSURU IKAWA

(Received July 4, 1984)

**1. Introduction.** We consider the scattering of the acoustic equation by bounded obstacles. Let $\mathcal{O}$ be a bounded open set in $\boldsymbol{R}^3$ with sufficiently smooth boundary. We set $\Omega=\boldsymbol{R}^3-\bar{\mathcal{O}}$. Suppose that $\Omega$ is connected. Consider the following problem

$$\begin{cases} \square\, u = \dfrac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{3} \dfrac{\partial^2 u}{\partial x_j^2} = 0 & \text{in } (-\infty, \infty)\times\Omega \\ u(t, x) = 0 & \text{on } (-\infty, \infty)\times\Gamma\,. \end{cases}$$

Denote by $\mathcal{S}(z)$ the scattering matrix for this problem. About the definition and the fundamental properties of the scattering matrix, see Lax and Phillips [8], especially Theorems 5.1 and 5.6 of Chapter V.

On relationships between geometric properties of $\mathcal{O}$ and the location of poles of $\mathcal{S}(z)$ Lax and Phillips gave a conjecture [8, page 158] (see also Ralston [16, 17]), that is, for a nontrapping obstacle the scattering matrix $\mathcal{S}(z)$ is free for poles in $\{z;\ \text{Im}\ z\leqslant\alpha\}$ for some constant $\alpha>0$, and for a trapping obstacle $\mathcal{S}(z)$ has a sequence of poles $\{z_j\}_{j=1}^{\infty}$ such that $\text{Im} z_j\to 0$ as $j\to\infty$. Concerning this conjecture Morawetz, Ralston and Strauss [14] and Melrose [11] proved that the part for nontrapping obstacles is correct. On the other hand, Bardos, Guillot and Ralston [1], Petkov [15] and Ikawa [4, 5, 6] made considerations on some simple cases of trapping obstacles. Among them the result of Ikawa [4, 5] shows that the part of the conjecture for trapping obstacles is not correct in general, namely for two strictly convex objects $\mathcal{S}(z)$ is free for poles in $\{z;\ \text{Im} z\leqslant\alpha\}$ $(\alpha>0)$. Yet it seems very sure that the conjecture remains to be correct for a great part of trapping obstacles. In spite of the conjecture we have not known even an example of obstacle $\mathcal{O}$ for which is proved the existence of a sequence of poles of the scattering matrix converging to the real axis.[1)]

The purpose of this paper is to show an example of $\mathcal{O}$ whose scattering

1) Ralston [16] gives examples of the scattering by the inhomogeneity of medium such that the scattering matrix has a sequence of poles converging to the real axis.

matrix has such a sequence of poles.

**Theorem 1.** *Let $\mathcal{O}_j$, $j=1, 2$, be convex open sets in $\boldsymbol{R}^3$ with sufficiently smooth boundary $\Gamma_j$, and let $a_j \in \Gamma_j$, $j=1, 2$, be the point such that $|a_1-a_2|=\text{dis}(\mathcal{O}_1, \mathcal{O}_2)$. Suppose that the principal curvatures $\kappa_{jl}(x)$, $l=1, 2$ of $\Gamma_j$ at $x\in\Gamma_j$ satisfy*

$$(1.1) \qquad C|x-a_j|^e \geqslant \kappa_{jl}(x) \geqslant C^{-1}|x-a_j|^e \quad \text{for all } x\in\Gamma_j$$

*for some*

$$(1.2) \qquad \infty > e \geq 2$$

*and $C>0$. Then the scattering matrix for $\mathcal{O}=\mathcal{O}_1\cup\mathcal{O}_2$ has a sequence of poles $\{z_j\}_{j=1}^{\infty}$ such that*

$$\operatorname{Im} z_j \to 0 \qquad \text{as } j\to\infty\,.$$

In the proof of this theorem we start from a trace formula proved by Bardos, Guillot and Ralston [1]:

$$\begin{aligned} &\operatorname{tr}_{L^2(R^3)}\int \rho(t)\,(\cos t\sqrt{-\Delta}\oplus 0-\cos t\sqrt{-\Delta_0})\,dt \\ &= \frac{1}{2}\sum_{\text{poles}} \hat{\rho}(\lambda_j) \qquad\qquad \text{for } \rho\in C_0^\infty(2R, \infty)\end{aligned}$$

(explanation of the notation will be given in §2). The main differences of the treatment of this formula in this article from in [1] are (i) we substitute in the place of $\rho(t)$ a sequence of functions $\rho_q(t)$, $q=1, 2, \cdots$ such that $\min\{t;\, t\in \operatorname{supp}\rho_q\}\to\infty$ as $q\to\infty$, (ii) all the eigenvalues of the Poincaré mapping of the periodic ray are 1, which is a consequence of the assumption (1.1) subject to (1.2).

It should be remarked that the result in [4] can be extended to a case of two convex objects such that the Poincaré mapping of the periodic ray has not 1 as an eigenvalue. Namely, in this case all the poles of $\mathcal{S}(z)$ have the imaginary part$\geq\alpha$ for some $\alpha>0$. Therefore in order to find an example of an obstacle composed of two convex objects with a sequence of poles converging to the real axis we have to consider obstacles whose Poincaré mapping has 1 as an eigenvalue. Of course these differences give rise to an essential difficulty in the proof, especially in the estimate of the left hand side of the trace formula for large $q$. To overcome this difficulty we represent the kernel of $\cos t\sqrt{-\Delta}$ by a superposition of asymptotic solutions constructed following the process in [2, 4], and apply Varčenko's theorem [19, 7] to an estimation of integrals of asymptotic solutions.

## 2. On the trace formula and a reduction of the problem

We denote by $\Delta$ the selfadjoint realization in $L^2(\Omega)$ of the Laplacian in $\Omega$

with the Dirichlet boundary condition and by $\Delta_0$ the selfadjoint realization in $L^2(\boldsymbol{R}^3)$ of the Laplacian in $\boldsymbol{R}^3$. Bardos, Guillot and Ralston shows in [1] that the following trace formula

$$\begin{aligned}(2.1)\qquad & \operatorname{tr}_{L^2(\boldsymbol{R}^3)}\int_{\boldsymbol{R}}\rho(t)\,(\cos t\sqrt{-\Delta}\oplus 0-\cos t\sqrt{-\Delta_0}\,)\,dt \\ & =\frac{1}{2}\sum_{\text{poles}}\hat{\rho}(\lambda_j)\end{aligned}$$

holds for all $\rho\in C_0^\infty(2R,\infty)$[2], where $R$=diameter of $\mathcal{O}$,

$$\hat{\rho}(\lambda)=\int e^{i\lambda t}\rho(t)\,dt$$

and $\cos t\sqrt{-\Delta}\oplus 0$ is an operator in $L^2(\boldsymbol{R}^3)$ defined for $f=f_1+f_2$, $f_1\in L^2(\Omega)$, $f_2\in L^2(\mathcal{O})$ by

$$((\cos t\sqrt{-\Delta}\oplus 0)f)\,(x)=\begin{cases}(\cos t\sqrt{-\Delta}\,f_1)\,(x) & \text{for } x\in\Omega\\ 0 & \text{for } x\in\mathcal{O}\end{cases}.$$

Remark that an estimate of the right hand side of (2.1)

$$(2.2)\qquad \sum_{\text{poles}}|\hat{\rho}(\lambda_j)|\leqslant C(T)||\rho||_{H^4(\boldsymbol{R})}\,,\qquad \forall\rho\in C_0^\infty(2R,\,T)$$

is shown in §3 of [1], where $C(T)$ is a constant depending on $T$.

Let $\rho_0(t)\in C_0^\infty(-1,\,1)$ and define $\rho_q(t)$, $q=1, 2, \cdots$ by

$$(2.3)\qquad \rho_q(t)=\rho_0((q+1)^l(t-2dq))\,,$$

where $d=\operatorname{dis}(\mathcal{O}_1,\,\mathcal{O}_2)$ and $l$ is a positive integer determined later.

**Lemma 2.1.** *Suppose that all the poles $\{\lambda_j\}_{j=1}^\infty$ of $\mathcal{S}(z)$ verify*

$$(2.4)\qquad \operatorname{Im}\lambda_j\geqslant\alpha$$

*for some constant $\alpha>0$. Then we have*

$$(2.5)\qquad \sum_{j=1}^\infty|\hat{\rho}_q(\lambda_j)|\leqslant C(q+1)^{4l}e^{-2d\alpha q}\qquad \textit{for all } q$$

*where $C$ is a constant independent of $q$ and $l$.*

Proof. Set

$$\rho_{p,q}(t)=\rho_0((p+1)^l(t-2dq))\,.$$

Fix $q_0$ in such a way $2dq_0-1\geqslant 2R$. Then we have $\rho_{p,q_0}(t)\in C_0^\infty(2R,\,T)$ $(T=2dq_0+1)$ for all $p$. Applying (2.2) for $\rho_{p,q_0}$ we have

---

[1] Melrose [12] shows that (2.1) holds for all $\rho\in C_0^\infty(\boldsymbol{R}^+)$.

$$\sum_{j=1}^{\infty} |\hat{\rho}_{p,q_0}(\lambda_j)| \leqslant C(T)\|\rho_{p,q_0}\|_{H^4(\boldsymbol{R})}$$
$$\leqslant C(T)C(p+1)^{4l}\,.$$

Since $\hat{\rho}_{p,q}(\lambda)=e^{i2d(q-q_0)\lambda}\hat{\rho}_{p,q_0}(\lambda)$ we have, under the assumption (2.4), for all $\lambda_j$

$$|\hat{\rho}_{p,q}(\lambda_j)| \leqslant e^{-2d(q-q_0)\mathrm{Im}\lambda_j}|\hat{\rho}_{p,q_0}(\lambda_j)|$$
$$\leqslant e^{-2d\alpha(q-q_0)}|\hat{\rho}_{p,q_0}(\lambda_j)|\,.$$

Then

$$\sum_{j=1}^{\infty} |\hat{\rho}_{p,q}(\lambda_j)| \leqslant e^{-2d\alpha(q-q_0)} \sum_{j=1}^{\infty} |\hat{\rho}_{p,q_0}(\lambda_j)|$$
$$\leqslant e^{-2d(q-q_0)\alpha}\, C(T)C(p+1)^{4l}$$
$$\leqslant C(T)Ce^{2dq_0\alpha}(p+1)^{4l}e^{-2dq\alpha}\,.$$

Note that $\rho_q(t)=\rho_{q,q}(t)$. Then we have (2.5) by setting $p=q$ in the above estimate. Q.E.D.

Concerning the left hand side of (2.1) we have the following

**Proposition 2.2.** *Suppose that $\mathcal{O}$ satisfies the condition in Theorem* 1. *Choose $\rho_0(t)\in C_0^\infty(-1, 1)$ so that*

$$\rho_0(t)\geqslant 0, \quad \int_{-\infty}^{\infty}\rho_0(t)dt = 1 \tag{2.6}$$

*and*

$$\hat{\rho}_0(-k) = \hat{\rho}_0(k)\geqslant 0 \qquad \textit{for all } k\in\boldsymbol{R}\,. \tag{2.7}$$

*Then we have*

$$|\mathrm{tr}_{L^2(\boldsymbol{R}^3)}\int_{-\infty}^{\infty}\rho_q(t)\,(\cos t\sqrt{-\Delta}\oplus 0-\cos t\sqrt{-\Delta_0})dt| \geqslant cq^{(1-2/e_0)(l+1)-2}-C_l q^{(1-5/2e_0)l} \tag{2.8}$$

*for all $q\geqslant q_0$ if $l\geqslant l_0$, where $e_0=e+2$ and $l_0$ is a some fixed positive integer, $c$ is a positive constant independent of $l$.*

The remaining sections of this paper will be devoted to the proof of this proposition. Theorem 1 can be proved immediately by Lemma 2.1 and Proposition 2.2. Indeed, choose $\rho_0$ so that (2.6) and (2.7) are verified. Suppose that there is no sequence of poles which converges to the real axis. Then there exists $\alpha>0$ such that

$$\mathrm{Im}\,\lambda_j\geqslant\alpha \qquad \text{for all } j\,.$$

Then we have (2.5) for all large $q$. By using (2.5) and (2.8) we have from (2.1)

$$cq^{(1-2/e_0)(l+1)-2}-C_l q^{(1-5/2e_0)l}\leqslant C(q+1)^{4l}e^{-2d\alpha q}$$

for large $q$ if $l \geqslant l_0$. Letting $q$ tend to $\infty$ the above inequality shows a contradiction. Thus Theorem 1 is proved.

We would like to remark that if we use the result of Melrose [12] Theorem 1 can be made better in the following form.

**Theorem 2.** *Suppose that $\mathcal{O}$ satisfies the condition in Theorem 1. There exists a positive constant $\gamma$ such that for any $\varepsilon > 0$ a region*

$$\{z;\ \mathrm{Im}\, z \leqslant \varepsilon(|\mathrm{Re}\, z|+1)^{-\gamma}\}$$

*contains an infinite number of poles of $\mathcal{S}(z)$.*

Recall that Melrose [12] shows that

$$N(K) \leqslant C(1+K)^p \tag{2.9}$$

for some $p>0$ where $N(K)$=the number of $\lambda_j$ such that $|\lambda_j| \leqslant K$. By using (2.9) we have the following lemma, and Theorem 2 is derived immediately from Proposition 2.2 and Lemma 2.3.

**Lemma 2.3.** *Suppose that $\{z;\ \mathrm{Im}\, z \leqslant \varepsilon_0(|\mathrm{Re}\, z|+1)^{-\gamma}\}$ $(\varepsilon_0>0)$ has no poles. Then it holds that*

$$\sum_{j=1}^{\infty} |\hat{\rho}_q(\lambda_j)| \leqslant C_{\varepsilon_0, l} \qquad \textit{for all } q$$

*if* $0<\gamma<l^{-1}$.

Proof. Let $0<\gamma<l^{-1}$. Choose $\alpha>0$ so that $1-\alpha\gamma>0$, $\alpha>l$. We classify the poles into three groups:

Group I $=\{\lambda_j;\ \mathrm{Im}\,\lambda_j \geqslant \varepsilon\}$ ,
Group II $=\{\lambda_j;\ \varepsilon > \mathrm{Im}\,\lambda_j \geqslant \varepsilon_0(|\mathrm{Re}\,\lambda_j|+1)^{-\gamma},\ |\mathrm{Re}\,\lambda_j| \leqslant q^\alpha\}$ ,
Group III$=\{\lambda_j\text{:}\ \varepsilon > \mathrm{Im}\,\lambda_j \geqslant \varepsilon_0(|\mathrm{Re}\,\lambda_j|+1)^{-\gamma},\ |\mathrm{Re}\,\lambda_j| \geqslant q^\alpha\}$ .

By the same argument as Lemma 2.1 we have

$$\sum_{\lambda_j \in \text{Group I}} |\hat{\rho}_q(\lambda_j)| \leqslant C_l (q+1)^{4l} e^{-2dq\varepsilon}\,.$$

From (2.9) the number of the poles of Group II is less than $C(1+q^\alpha)^p$. Then

$$\begin{aligned}\sum_{\lambda_j \in \text{Group II}} |\hat{\rho}_q(\lambda_j)| &\leqslant C_l e^{-2dq\varepsilon_0 (q^\alpha)^{-\gamma}} (1+q^\alpha)^p \\ &\leqslant C_l (1+q^\alpha)^p e^{-2d\varepsilon_0 q^{1-\alpha\gamma}}\,.\end{aligned}$$

Since an estimate $|\hat{\rho}_q(z)| \leqslant C_N \left(\dfrac{|z|}{q^l}\right)^{-N}$ holds for any $N$ we have

$$\sum_{n<\mathrm{Re}\lambda_j \leqslant (n+1)} |\hat{\rho}_q(\lambda_j)| \leqslant C_N (n+1)^p (nq^{-l})^{-N}\,,$$

and

$$\sum_{\lambda_j \in \text{Group III}} |\hat{\rho}_q(\lambda_j)| \leqslant \sum_{n=[q^{\alpha}]}^{\infty} C_N q^{lN}(n+1)^q n^{-N}$$
$$\leqslant C_N q^{lN}(q^{\alpha})^{-N+p+2}.$$

Then summing up these estimates, if we choose $N$ so large that $(-\alpha+l)N+p+2\leqslant 0$, it holds that

$$\sum_{\text{poles}} |\hat{\rho}_q(\lambda_j)| \leqslant C(1+q^{\alpha})^p e^{-2d\varepsilon_0 q^{(1-\alpha\gamma)}} + C_N \leqslant C'_N.$$

Q.E.D.

## 3. Program of the proof of Proposition 2.2

Denote the kernel distribution of $\cos t\sqrt{-\Delta_0}$ and $\cos t\sqrt{-\Delta}$ by $E_0(t; x, y)$ and $E(t, x, y)$ respectively. Then the kernel distribution $e(t; x, y)$ of $\cos t\sqrt{-\Delta} \oplus 0 - \cos t\sqrt{-\Delta_0}$ is written as

$$e(t; x, y) = \tilde{E}(t: x, y) - E_0(t: x, y)$$

where

$$\tilde{E}(t; x, y) = \begin{cases} E(t; x, y) & \text{for } x, y \in \Omega \\ 0 & \text{in } \boldsymbol{R}^3 \times \boldsymbol{R}^3 - \Omega \times \Omega. \end{cases}$$

Set

$$c_q(x, y) = \int_{-\infty}^{\infty} \rho_q(t) e(t; x, y) dt.$$

In order to show Proposition 2.2 it suffices to prove the following facts:

$$\text{supp } c_q \subset \bar{\Omega} \times \bar{\Omega}, \tag{3.1}$$

$$c_q(x, y) \in C_0^{\infty}(\bar{\Omega} \times \bar{\Omega}) \tag{3.2}$$

and

$$\left| \int_{\boldsymbol{R}^3} c_q(x, x) dx - c_0 q^{(1-2/e_0)(l+1)-2} \right| \leqslant C_l q^{(1-5/2e_0)l} \quad \text{for all } q \tag{3.3}$$

where $c_0$ is a positive constant determined by $\mathcal{O}$ and $\rho_0$.

Since $E_0(t; x, y)$ is well known the essential part of the proof is the consideration of $E(t; x, y)$. To take out properties of $E$, first we construct an approximation of $E$ as a superposition of asymptotic solutions, secondly we pick out the principal behavior of $E$ as $t \to \infty$. The construction of asymptotic solutions is done by a method essentially same as in [2] and [4]. But the assumption that all the principal curvatures of the boundary vanish at $a_1$ and $a_2$ gives rise to another behavior of asymptotic solutions than those in [2, 4]. Then in order to pick up this behavior of asymptotic solutions we have to make other considerations than in the previous papers.

Fix $\delta_2$, $\delta_3$ so that Corollary of Lemma 3.3 of [2] holds. Let $S_j(\delta_l)$, $j=1, 2$, $l=2, 3$ be the ones introduced in §3 of [2]. Denote by $\omega(\delta_l)$ a domain surrounded by $S_j(\delta_l)$, $j=1, 2$ and $\{y;\ \mathrm{dis}(y, L)=\delta_l\}$. Let

$$\psi(x)\in C_0^\infty(\Omega) \quad \text{such that} \quad \mathrm{supp}\,\psi\subset\omega(\delta_2)\,. \tag{3.4}$$

Then for $f\in C^\infty(\Omega)$ we have by Fourier's inversion formula

$$\psi(x)f(x) = w(x)\int_{S^2} d\omega\int_0^\infty k^2 dk\int_\Omega dy\ e^{ik\langle x-y,\omega\rangle}\psi(y)f(y)\,, \tag{3.5}$$

where $w(x)$ is a function in $C_0^\infty(\omega(\delta_3))$ verifying

$$w(x) = 1 \qquad \text{on} \quad \mathrm{supp}\,\psi\,. \tag{3.6}$$

Let $u(t, x; k, \omega)$ be the solution of an initial-boundary value problem

$$\begin{cases} \Box\, u = 0 & \text{in}\ \ (0, \infty)\times\Omega \\ u(t, x) = 0 & \text{on}\ \ (0, \infty)\times\Gamma \\ u(0, x) = w(x)e^{ik\langle x,\omega\rangle} & \\ \dfrac{\partial u}{\partial t}(0, x) = 0\,. & \end{cases} \tag{3.7}$$

Then

$$a(t, x) = \int_{S^2} d\omega\int_0^\infty k^2 dk\int_\Omega dy\ u(t, x; k, \omega)e^{-ik\langle y,\omega\rangle}\psi(y)f(y)$$

satisfies

$$\begin{cases} \Box\, a = 0 & \text{in}\ (0, \infty)\times\Omega \\ a(t, x) = 0 & \text{on}\ (0, \infty)\times\Gamma \\ a(0, x) = \psi(x)f(x) & \\ \dfrac{\partial a}{\partial t}(0, x) = 0\,. & \end{cases}$$

This means that $a(t, \cdot)=(\cos t\sqrt{-\Delta}\,\psi)f$. Therefore the kernel distribution $E(t; x, y)\psi(y)$ of $\cos t\sqrt{-\Delta}\,\psi$ is given by

$$E(t; x, y)\psi(y) = \int_{S^2} d\omega\int_0^\infty k^2 dk\ u(t, x; k, \omega)e^{-ik\langle y,\omega\rangle}\psi(y)\,, \tag{3.8}$$

here we interpret the integral as an oscillatory integral (cf. Kumano-go [8, §6 of Chapter 1]).

As an approximation of $u(t, x; k, \omega)$ we construct an asymptotic solution of (3.7) in a way that we can make clear the reflexion of geometric properties of $\mathcal{O}$ to the behavior of $u$. For the Cauchy problem with an oscillatory data

$$\begin{cases} \square h = 0 & \text{in } (0, \infty)\times \boldsymbol{R}^3 \\ h(0, x) = w(x)e^{ik\langle x, \omega\rangle} & \text{in } \boldsymbol{R}^3 \\ \dfrac{\partial h}{\partial t}(0, x) = 0 & \text{in } \boldsymbol{R}^3 \end{cases}$$

admits an asymptotic solution

$$\begin{aligned} h^{(N)}(t, x; k, \omega) &= e^{ik(\langle x, \omega\rangle - t)} \sum_{j=0}^{N} g_j(t, x; \omega)\,(ik)^{-j} \\ &\quad + e^{ik(\langle x, \omega\rangle + t)} \sum_{j=0}^{N} \tilde{g}_j(t, x; \omega)\,(ik)^{-j} \\ &= h_+^{(N)}(t, x; k, \omega) + h_-^{(N)}(t, x; k, \omega)\,. \end{aligned}$$

Set

$$\begin{aligned} m^{(N)}(t, x; k, \omega) &= h^{(N)}(t, x; k, \omega)\,|_{(0,\infty)\times\Gamma} \\ &= h_+^{(N)}|_{(0,\infty)\times\Gamma} + h_-^{(N)}|_{(0,\infty)\times\Gamma} = m_+^{(N)} + m_-^{(N)}\,. \end{aligned}$$

Note that from the location of the support of $h_\pm^{(N)}$, the support of $m_+^{(N)}$ is contained in one of $(0, \infty)\times\Gamma_1$ and $(0, \infty)\times\Gamma_2$. For example when $\omega_3<0$

$$\operatorname{supp} m_+^{(N)} \subset (0, \infty)\times\Gamma_1,\ \ \operatorname{supp} m_-^{(N)} \subset (0, \infty)\times\Gamma_2\,.$$

Since all the rays starting from supp $\psi$ and hitting $S(\delta_3)$ do not tangent to $\Gamma$ in $S(\delta_3)$ and the Gaussian curvature does not vanish in the outside of $S(\delta_3)$, the method of construction of asymptotic solution in [2] can be applied without any modification. We see from Corollary of Lemma 3.3 of [2] that it suffices to consider $z^{(N)}$ constructed in §8 of [2] when we consider the behavior in $\omega(\delta_3)$ of asymptotic solutions with oscillatory boundary data $m_\pm^{(N)}$. Let us denote by $z_\pm^{(N)} = w_\pm^{(N)} + y_\pm^{(N)}$ the asymptotic solution $z^{(N)}$ constructed by the process of Proposition 8.1 of [2] for boundary data $m_\pm^{(N)}$. Now consider $z_+^{(N)}$. For the simplicity of description we omit the suffix $+$. Recall that $w^{(N)}$ is of the form

$$\text{(3.9)}\qquad \begin{cases} w^{(N)} = \sum_{q=0}^{\infty} u_q^{(N)}\,, \\ u_q^{(N)}(t, x; k, \omega) = e^{ik(\varphi_q(x,\omega)-t)} \sum_{j=0}^{N} v_{q,j}(t, x; \omega)\,(ik)^{-j} \end{cases}$$

and that $y^{(N)}$ satisfies

$$\text{(3.10)}\qquad \operatorname{supp} y^{(N)} \cap ((0, \infty)\times\omega(\delta_2)) = \phi\,.$$

The fact that the principal curvatures of $\Gamma_1$ and $\Gamma_2$ vanish at $a_1$ and $a_2$ brings other behaviors of $\varphi_q$ and $v_{q,j}$ than those of [2, 4]. In this case $\{\nabla\varphi_q\}_{q=0}^{\infty}$ is not bounded in $C^\infty(\omega(\delta_3))$ and the sequences $v_{q,j}$, $q=0.\ 1,\ 2\ \cdots$ do not decrease exponentially. Concerning their estimate we have

**Lemma 3.1.** *There exist positive integers $l(j, m)$ depending on $j$ and $m$ such that*

$$\sum_{|\beta|\leqslant j} |\partial_\omega^\beta \nabla\varphi_q(\cdot\,;\omega)|_m(\omega(\delta_1)) \leqslant C_{j,m} q^{l(j,m)}, \tag{3.11}$$

$$\sum_{|\beta|\leqslant h} |\partial_\omega^\beta v_{q,j}(\cdot\,;\omega)|_m(\boldsymbol{R}\times\omega(\delta_1)) \leqslant C_{j+h,m} q^{l(j+h,m)} \tag{3.12}$$

*hold.*

There estimates are proved by induction of $j$, $h$, $m$ by using Lemmas 5.2, 5.3 and their remarks of [2].

Taking account of the location of the support of $z^{(N)}$ the estimates (3.11) and (3.12) give

$$\operatorname{supp} z^{(N)} \subset (0, \infty)\times\Omega$$

$$\sum_{|\beta|\leqslant h} |\partial_\omega^\beta (z_\pm^{(N)}(\cdot, \cdot\,; k, \omega)|_m(t, \Omega) \leqslant C_{N,j,m} k^{m+h}\left(\frac{t}{2d}\right)^{l(N+2+h,m)}, \tag{3.13}$$

$$\sum_{|\beta|\leqslant h} |\partial_\omega^\beta (\Box z_\pm^{(N)}(\cdot, \cdot\,; k, \omega)|_m(t, \Omega) \leqslant C_{N,j,m} k^{-N+m}\left(\frac{t}{2d}\right)^{l(N+2+h,m)}, \tag{3.14}$$

$$z_\pm^{(N)} = h_\pm^{(N)} \qquad \text{on } (0, \infty)\times\Gamma\,. \tag{3.15}$$

Set $u^{(N)} = -(z_+^{(N)} + z_-^{(N)}) + h^{(N)}$. Then

$$u^{(N)}(0, x; k, \omega) = w(x)e^{ik\langle x,\omega\rangle},$$
$$\frac{\partial u^{(N)}}{\partial t}(0, x; k, \omega) = 0$$

and $\Box u^{(N)}$ has an estimate of the type (3.14). Concerning the difference between the actual solution $u$ of (3.7) and $u^{(N)}$ we have from the above remarks

$$\sum_{|\beta|\leqslant h} |\partial_\omega^\beta (u-u^{(N)})\,(\cdot, \cdot\,; k, \omega)|_m(t, \Omega) \leqslant C_{N,h,m} k^{-N+m+2}\left(\frac{t}{2d}\right)^{l(N+2+h,m)+1}. \tag{3.16}$$

We see immediately from Lemma 3.1 and (3.16) that

$$\int \rho(t)E(t; x, y)\psi(y)dt \in C^\infty(\bar\Omega\times\bar\Omega) \qquad \text{for any } \rho\in C_0^\infty(\boldsymbol{R})\,.$$

Since $\operatorname{supp} E_0(t; \cdot, \cdot) \subset \{(x, y);\ |x-y| = |t|\}$

$$\int_{\boldsymbol{R}^3} c_q(x, x)\psi(x)dx = \iint_\Omega E(t, x, x)\psi(x)\rho_q(t)dt\,dx \tag{3.17}$$

for large $q$. From (3.8), (3.17)

$$\int_{\boldsymbol{R}^3} c_q(x, x)\psi(x)dx$$

$$= \int_\Omega dx \int dt \int_{S^2} d\omega \int_0^\infty k^2 dk\ \rho_q(t) u(x, t; k, \omega) e^{-ik\langle x,\omega\rangle} \psi(x)$$
$$= \int \cdots \int_{k>1} \rho_q(t) w_+^{(N)}(t, x; k, \omega) e^{-ik\langle x,\omega\rangle} \psi(x) dx dt d\omega k^2 dk$$
$$+ \int \cdots \int_{k>1} \rho_q(t) w_-^{(N)}(t, x; k, \omega) e^{-ik\langle x,\omega\rangle} \psi(x) dx dt d\omega k^2 dk$$
$$+ \int \cdots \int_{k>1} \rho_q(t)\ (y_+^{(N)} + y_-^{(N)})\ (t, x; k, \omega) e^{-ik\langle x,\omega\rangle} \psi(x) dx dt d\omega k^2 dk$$
$$+ \int \cdots \int_{k>1} \rho_q(t)\ (u(t, x; k, \omega) - u^{(N)}(t, x; k, \omega)) e^{-ik\langle x,\omega\rangle} \psi(x) dx dt d\omega k^2 dk$$
$$+ \int_\Omega dx \int dt \int_{S^2} d\omega \int_0^1 k^2 dk\ \rho_q(t) u(t, x; k, \omega) e^{-ik\langle x,\omega\rangle} \psi(x)$$
$$= I_+ + I_- + II + III + IV .$$

Since we have for $0 \leqslant k \leqslant 1$

$$|u(t, x; k, \omega)| \leqslant C \qquad \text{in } [0, \infty) \times \Omega ,$$

it holds that

$$|IV| \leqslant C \int \psi(x) dx \int \rho_q(t) dt \leqslant C \int \psi(x) dx\ q^{-l} .$$

From (3.4) and (3.10) the integrand of $II$ vanishes identially. Thus $II=0$. Next consider $III$. Set

$$\int \cdots \int dx dt d\omega \int_1^\infty k^2 dk \{\quad\} = \int \cdots \int dx dt d\omega \int_1^q k^2 dk \{\quad\} + \int \cdots \int dx dt d\omega \int_q^\infty k^2 dk \{\quad\}$$
$$= III_1 + III_2 .$$
$$|III_1| \leqslant C \int \rho_q(t) dt \int_\Omega \psi(x) dx \int_1^q k^2 dk \leqslant C\, q^{-l} q^3 \leqslant C\, q^{-l+3} .$$

Since supp $\rho_q \subset [2dq - q^{-l}, 2dq + q^{-l}]$, by using (3.16)

$$|III_2| \leqslant C \int \psi(x) dx \int \rho_q(t) dt\ q^{l(N+2,0)} \int_q^\infty k^{-N+2} dk \leqslant C\, q^{-l+l(N+2,0)-N+3} .$$

Thus we have

**Lemma 3.2.** *If we choose $l > l(N+2, 0) - N + 3$ it holds that*

$$|\int_{R^3} c_q(x, x) \psi(x) dx - (I_+ + I_-)| \leqslant C_{N,l} , \tag{3.18}$$

*for all $q$.*

Now we set about the estimation of $I_+$. Set

$$I_{r,j}(t, k) = \int_{S^2} d\omega \int_\Omega dx\ e^{ik\Phi_r(x,\omega)} v_{r,j}(t, x; \omega) \psi(x) , \tag{3.19}$$

$$\Phi_r(x, \omega) = \varphi_r(x, \omega) - \langle x, \omega \rangle . \tag{3.20}$$

Then we have

$$I_+ = \int_1^\infty k^2 dk \sum_{j=0}^{N} (ik)^{-l} \int e^{-ikt} \rho_q(t) I_{r,j}(t, k) dt . \tag{3.21}$$

Note that except $r=2q-1,\ 2q,\ 2q+1$ $\operatorname{supp} \rho_q \cap \operatorname{supp} v_{r,j}(t, x; \omega) = \phi$. Since for $r=2q\pm 1$

$$|\partial_{x_3} \Phi_r(x, \omega)| \geqslant 1 \qquad \text{for all } (x, \omega) \in \omega(\delta_3) \times S^2 ,$$

if $v_{r,j}(t, x; \omega) \neq 0$, we have

$$|I_{r,j}(t, k)| \leqslant C_M k^{-M} q^{l(M)} \qquad \text{for } r = 2q \pm 1$$

where $l(M)$ is an integer depending on $M$. Therefore we have

$$|I_+ - \int_1^\infty k^2 dk \sum_{j=0}^{N} (ik)^{-j} \int e^{-ikt} I_{2q,j}(t, k) \rho_q(t) dt| \leqslant C \quad \text{for all } q \tag{3.22}$$

if $l$ is large. Set

$$J_{q,j}(x_3, t; k) = \int_{S^2} d\omega \int_{R^2} dx' e^{ik\Phi_{2q}(x,\omega)} v_{2q,j}(t, x; \omega) \psi(x) . \tag{3.23}$$

**Proposition 3.3.**

$$\begin{aligned} |J_{q,j}(x_3, t; k) - k^{-1-2/e_0 i2dq} \{c^0_{q,j}(x_3, t) \\ + \sum_{h=1}^{[3e_0/2]} \sum_{m=1}^{mh} c^{h,m}_{q,j}(x_3, t) k^{-h/e_0} (\log k)^{m-1}\}| \leqslant C q^{l_1} k^{-4} \end{aligned} \tag{3.24}$$

*where $l_1$ is a constant, $c^{h,m}_{q,j}(x_3, t)$ are determined by $\Phi_{2q}$ and $v_{q,j}$ and they satisfy*

$$\sum_{l=0}^{2} |\partial_t^l c^{h,m}_{q,j}(x_3, t)| \leqslant C\, q^{l_1} \qquad \textit{for all } x_3 \in (0, d) \textit{ and } t > 0 ,$$

*especially*

$$c^0_{q,j}(x_3, t) = c\, v_{q,j}(t, 0, x_3; \omega_0)\, q^{-1-2/e_0}$$

*for some fixed non zero constant $c$ determined by the shape of $\Gamma_j$ near $a_j$ and $\omega_0 = (0, 0, 1)$.*

The above proposition will be proved in sections 4 and 5. Now admit this result. To evaluate $v_{2q,0}$ we use (5.9) of [4]. For $\omega_0$ we see from Lemma 4.1 of [2] that the principal curvatures at $a_1$ and $a_2$ of the wave front of $\varphi_r$ are zero for all $r$. Then we have $\Lambda_{2q-j}(X_{-j}(x, \nabla\varphi_{2q}))=1$ for all $j$ when $x'=0$. Therefore we have for $\omega_0$

$$v_{2q,0}(t, 0, 0, x_3; \omega_0) = w(0, 0, (2qd+x_3)-t) .$$

Note that from (3.6) $w(0, 0, (2dq+x_3)-t))=1$ holds for $(0, 0, x_3) \in \operatorname{supp} \psi$ and

$t\in\operatorname{supp}\rho_q\subset[2dq-q^{-l}, 2dq+q^{-l}]$. Therefore

$$\int dx_3\int_0^\infty k^2dk\int dt\; c^0_{q,0}(x_3, t)e^{-ikt}k^{-1-2/e_0}e^{ik2dq}\rho_q(t)\psi(0, x_3) \tag{3.25}$$
$$= c\int\psi(0, x_3)dx_3\int_1^\infty k^{1-2/e_0}\hat{\rho}_0(k/q^l)q^{-l}\,dk\;q^{-1-2/e_0}$$
$$= c_0\int_0^d\psi(0, x_3)dx_3\;q^{(1-2/e_0)(l+1)-2}+O(q^{-l}),$$

where $c_0=c\int_0^\infty k^{1-2/e_0}\hat{\rho}_0(k)dk\neq 0$ from (2.7). Next we shall show the following estimate for $h\geqslant 1$ and for all $j$, $m$

$$\left|\int_0^d dx_3\int_1^\infty k^2dk\int e^{-ikt}k^{-1-j-(2+h)/e_0}(\log k)^{m-1}c^{h,m}_{q,j}(x_3, t)\rho_q(t)dt\right| \tag{3.26}$$
$$\leqslant C_l\,q^{l_1}q^{(1-11/4e_0)l}.$$

Set

$$I=\int_1^\infty k^2dk\int e^{-ikt}k^{-1-j-(2+h)/e_0}(\log k)^{m-1}c^{h,m}_{q,j}(x_3, t)\rho_q(t)dt$$
$$=\int_1^{(q+1)^l}k^2dk\int\cdots dt+\int_{(q+1)^l}^\infty k^2dk\int\cdots dt$$
$$=I_1+I_2.$$

Substituting an estimate $|c^{h,m}_{q,j}|\leqslant C\,q^{l_1}$ we have

$$|I_1|\leqslant C\,q^{l_1}\int_1^{(q+1)^l}k^{1-j-(h+2)/e_0}(\log(q+1)^l)^{m-1}dk\cdot\int\rho_q(t)dt$$
$$\leqslant C\,q^{l_1}(l\log(1+q))^m q^{l(1-j-(2+h)/e_0)}.$$

About $I_2$, we make integration by parts in $t$ variable two times, and we have

$$I_2=\int_{(q+1)^l}^\infty k^2dk\int(ik)^{-2}e^{-ikt}k^{-1-j-(2+h)/e_0}$$
$$\cdot(\log k)^{m-1}\left(\frac{\partial}{\partial t}\right)^2(c^{h,m}_{q,j}(x_3, t)\rho_q(t))dt.$$

By using estimates of $c^{h,m}_{q,j}$ and the definition of $\rho_q(t)$ we have

$$\int\left|\left(\frac{\partial}{\partial t}\right)^2(c^{h,m}_{q,j}(x_3, t)\rho_q(t))\right|dt\leqslant C\,q^{l_1}(q+1)^l.$$

Thus it follows that

$$|I_2|\leqslant C\,q^{l_1}(q+1)^l\int_{(q+1)^l}^\infty k^{-1-j-(2+h)/e_0}(\log k)^{m-1}dk$$
$$\leqslant C_\varepsilon\,q^{l_1}(q+1)^{l\,(1-j-(2+h)/e_0+\varepsilon)}\;(\varepsilon>0).$$

Taking account of $h\geqslant 1$ we have

$$|I| \leqslant C_l\, q^{l_1} q^{(1-11/4e_0)l}\,.$$

Since $C_l$ is independent of $x_3$ the above estimate implies (3.26). Combining the above estimates we have

$$(3.27)\qquad |I_+ - c_0\int_0^d \psi(0,\, x_3)dx_3\, q^{(1-2/e_0)(l+1)-2}| \leqslant C_l\, q^{(1-5/2e_0)l}$$

if $l$ is sufficiently large. For $I_-$ we have the same estimate as $I_+$. Then form (3.27) and Lemma 3.2 it follows that

$$(3.28)\qquad |\int_{R^3} c_q(x,\, x)\psi(x)dx - 2c_0\int_0^d \psi(0,\, x_3)dx_3\, q^{(1-2/e_0)(l+1)-2}| \leqslant C_l\, q^{(1-5/2e_0)l}$$

for any $\psi \in C_0^\infty(\omega(\delta_3))$ when $l$ is large.

When $\psi \in C_0^\infty(\omega(\delta_3) \cup S(\delta_3))$ we have to modify the procedure of the construction of the kernel of $\cos t\sqrt{-\Delta}\,\psi$. Namely, when $\operatorname{supp}\psi \cap S(\delta_3) \neq \phi$ we cannot choose $w(x)$ in (3.6) as a function in $C_0^\infty(\Omega)$. Therefore the solution of (3.7) is not smooth function, and $z_\pm^{(N)}$ has discontinuities, which make the argument more complicated. But as we shall show in §6 the same estimate also holds in this case. Thus

**Lemma 3.4.** *For any* $\psi \in C_0^\infty(\omega(\delta_3) \cup S(\delta_3))$ *the estimate* (3.28) *holds if we choose* $l$ *sufficiently large.*

Next consider the case $\psi \in C_0^\infty(\bar{\Omega})$ and

$$(3.29)\qquad \operatorname{supp}\psi \cap \omega(\delta_3) = \phi\,.$$

Suppose that in addition to (3.29) any ray starting from supp $\psi$ does not tangent to $\Gamma$ at $S(\delta_3)$. Then the procedure of construction of an approximation of $c_q(x,\, y)\psi(y)$ is same as before. In the representation of $I_{r,j}(t,\, k)$ the amplitude function $\Phi_r(x,\, \omega)$ has no critical point, that is,

$$|\partial_x \Phi_r(x,\, \omega)| + |\partial_\omega \Phi_r(x,\, \omega)| \neq 0 \qquad \text{for all } (x,\, \omega) \in \operatorname{supp}\psi \times S^2\,.$$

Thus we have for any $M$

$$|I_+(t,\, k)| \leqslant C_M\, q^{l(M)}\, k^{-M}$$

where $l(M)$ is a constant depending on $M$. Therefore we have

$$|\int_{R^3} c_q(x,\, x)\psi(x)dx| \leqslant C \qquad \text{for all } q.$$

By employing the argument in §6 of [2] the additional condition may be removed easily. Then

**Lemma 3.5.** *Let* $\psi \in C_0^\infty(\bar{\Omega})$ *such that*

$$\operatorname{supp}\psi \cap \omega(\delta_3) = \phi\,.$$

*Then an estimate*

(3.30) $$|\int_{R^3} c_q(x, x)\psi(x)dx| \leqslant C$$

*holds where the constant C depends of $\mathcal{O}$ and $\psi$ but independent of q.*

Note that for $\psi$ of the form $\psi(x)=\psi_0(x-\zeta)$ for a fixed $\psi_0\in C_0^\infty(\boldsymbol{R}^3)$ and some $\zeta\in\boldsymbol{R}^3$ the constant $C$ in (3.30) is independent of $\psi$, namely $C$ is depends on $\mathcal{O}$ and $\psi_0$ only. Since

$$\text{supp}(E(t;\cdot,\cdot)-E_0(t;\cdot,\cdot))\subset\{(x, y);\ |x|,\ |y|\leqslant R+|t|\}$$

the estimate (2.8) is derived from Lemmas 3.3 and 3.4.

## 4. On the critical points of $\Phi_{2q}(x, \omega)$

Let $\varphi_0(x, \omega)=\langle x, \omega\rangle$ and let $\varphi_1, \varphi_2 \cdots, \varphi_{2q} \cdots$ be the sequence of phase functions in (3.9). For $x\in\omega(\delta_3)$ set $X_0(x, \omega)=x$ and, if $\{x+l\omega;\ l\geqslant 0\}\cap\Gamma\neq\phi$

$$\begin{aligned} l_0(x, \omega) &= \inf\{l;\ l\geqslant 0,\ x+l\omega\in\Gamma\}\ , \\ X_1(x, \omega) &= x+l_0\omega\ , \\ \Xi_1(x, \omega) &= \omega-2\langle n(X_1(x, \omega)), \omega\rangle n(X_1(x, \omega))\ . \end{aligned}$$

Following the process of §3 of [2] define successively $l_j(x, \omega)$, $X_j(x, \omega)$, $\Xi_j(x, \omega)$, $L_j(x, \omega)$, $\mathcal{L}_j(x, \omega)$ for $j=1, 2, \cdots$. For $x\in\omega(\delta_3)$ set

$$\begin{aligned} l_{-1}(x, \nabla\varphi_{2q}(x, \omega)) &= \inf\{l;\ l\geqslant 0,\ x-l\,\nabla\varphi_{2q}(x, \omega)\in\Gamma\}\ , \\ X_{-1}(x, \nabla\varphi_{2q}(x, \omega)) &= x-l_{-1}(x, \nabla\varphi_{2q}(x, \omega))\nabla\varphi_{2q}(x, \omega)\ . \end{aligned}$$

Define successively $X_{-j}(x, \nabla\varphi_{2q}(x, \omega))$ following §4 of [4]. For $x\in\boldsymbol{R}^3$ and $\omega\in S^2$ set

$$\mathcal{P}(x, \omega) = \{y;\ \langle y-x, \omega\rangle = 0\}\ .$$

Let us denote by $Y_{2q}(x, \omega)$ the point

$$\mathcal{P}(x, \omega)\cap\{X_{-2q}(x, \nabla\varphi_{2q}(x, \omega))-l\omega;\ l\geqslant 0\}\ .$$

Remark that, if we set $y=Y_{2q}(x, \omega)$, we have

$$X_{2q-j}(y, \omega) = X_{-1-j}(x, \nabla\varphi_{2q}(x, \omega))\ , \qquad j=1, 2, \cdots, 2q-1\ .$$

$Y_{2q}(x, \omega)\in\rho(x, \omega)$ means that

(4.1) $$\langle Y_{2q}(x, \omega), \omega\rangle = \langle x, \omega\rangle\ .$$

Now we have by using (4.1)

(4.2) $$\Phi_{2q}(x, \omega) = |X_1(y, \omega)-y|+|X_2(y, \omega)-X_1(y, \omega)|$$

$$+\cdots+|X_{2q}(y,\omega)-X_{2q-1}(y,\omega)|+|x-X_{2q}(y,\omega)|$$

where we put $y=Y_{2q}(x,\omega)$. Recall that the broken ray $\mathcal{X}(Y_{2q}(x,\omega),\omega)$ is a path starting from a point on a plane $\mathcal{P}(x,\omega)$ and reach at $x$ after $2q$ times reflexion on $\Gamma$ according to the geometric optics. The path of the geometric optics can be characterized as a path that has a minimal length among the ones which start from on $\mathcal{P}(x,\omega)$ and arrive at $x$ after passing $2q$ times points on $\Gamma$. Namely,

$$\Phi_{2q}(x,\omega)=\inf\{|x^{(1)}-x^{(0)}|+|x^{(2)}-x^{(1)}| \\ +\cdots+|x^{(2q)}-x^{(2q-1)}|+|x-x^{(2q)}|\} \tag{4.3}$$

where the infimum is taken on $x^{(0)}, x^{(1)}, \cdots, x^{(2q)}$ running over

$$x^{(0)}\in\mathcal{P}(x,\omega), \\ x^{(1)}, x^{(3)}, \cdots, x^{(2q-1)}\in\Gamma_2\ (\Gamma_1), \\ x^{(2)}, x^{(4)}, \cdots, x^{(2p)}\in\Gamma_1\ (\Gamma_2),$$

if $\omega_3>0$ (if $\omega_3<0$). Let us set

$$S^2_\pm=\{(\omega_1,\omega_2,\pm\sqrt{1-\omega_1^2-\omega_2^2});\ \omega_1^2+\omega_2^2<1\}.$$

**Lemma 4.1.** *Let $\omega\in S^2_+$ and $x\in\omega(\delta_3)$. Suppose that*

$$Y_{2q}(x,\omega)\ \textit{exists.} \tag{4.4}$$

*Then for $\omega=\omega(\omega_1,\omega_2)=(\omega_1,\omega_2,\sqrt{1-\omega_1^2-\omega_2^2})$*

$$\frac{\partial\Phi_{2q}(x,\omega)}{\partial\omega_j}=\langle y-x,\frac{\partial\omega}{\partial\omega_j}\rangle \\ =(y_j-x_j)-\omega_j(1-\omega_1^2-\omega_2^2)^{-1/2}(y_3-x_3) \tag{4.5}$$

*for $j=1, 2$, where $y=(y_1,y_2,y_3)=Y_{2q}(x,\omega)$.*

Proof. Let $\tilde\omega=\omega(\omega_1+\Delta\omega_1,\omega_2)$ and $\tilde y=Y_{2q}(x,\tilde\omega)$. Since $X_{-j}(x,\nabla\varphi_{2q}(x,\omega))$ is continuous in $x$ and $\omega$ we have

$$\tilde y\to y \quad \text{as } \Delta\omega_1\to 0. \tag{4.6}$$

Set

$$z=\mathcal{P}(x,\tilde\omega)\cap\{y+l\omega;\ l\in\boldsymbol{R}\} \\ \tilde z=\mathcal{P}(x,\omega)\cap\{\tilde y+l\tilde\omega;\ l\in\boldsymbol{R}\}.$$

Then from (4.3)

$$\Phi_{2q}(x,\tilde\omega)=\inf\{|x^{(1)}-x^{(0)}|+\cdots+|x-x^{(2q)}|\} \\ \leqslant|X_1(y,\omega)-z|+|X_2(y,\omega)-X_1(y,\omega)|$$

$$+\cdots+|X_{2q}(y, \omega)-X_{2q-1}(x, \omega)|+|x-X_{2q}(y, \omega)| .$$

Since we have $|X_1(y,\omega)-z|=|X_1(y,\omega)-y|+|y-z|$ if $z$ is on the prolongation of a segment $X_1(y, \omega)y$ it holds that

$$\Phi_{2q}(x, \tilde{\omega})\leqslant\Phi_{2q}(x, \omega)+|y-z| . \tag{4.7}$$

If $z$ is on the prolongation of $X_1(y, \omega)y$ $\tilde{z}$ must be on a segment $X_1(\tilde{y}, \tilde{\omega})\tilde{y}$, and we have

$$|X_1(\tilde{y}, \tilde{\omega})-\tilde{z}| = |X_1(\tilde{y}, \tilde{\omega})-\tilde{y}|-|\tilde{y}-\tilde{z}| .$$

Then similarly we have

$$\Phi_{2q}(x, \omega)\leqslant\Phi_{2q}(x, \tilde{\omega})-|\tilde{y}-\tilde{z}| . \tag{4.8}$$

Taking account of $\overline{X_1(y, \omega)y}\perp\mathcal{P}(x, \omega)$ and $\overline{X_1(\tilde{y}, \tilde{\omega})\tilde{y}}\perp\mathcal{P}(x, \tilde{\omega})$ we have

$$\begin{aligned}|y-z| &= \langle y-x, \tilde{\omega}-\omega\rangle+o(|\tilde{\omega}-\omega|)\\ |\tilde{y}-\tilde{z}| &= \langle \tilde{y}-x, \tilde{\omega}-\omega\rangle+o(|\tilde{\omega}-\omega|) .\end{aligned}$$

Thus from (4.7), (4.8) and (4.6) it follows that

$$\lim_{\Delta\omega_1\to 0}\frac{\Phi_{2q}(x, \tilde{\omega})-\Phi_{2q}(x, \omega)}{\Delta\omega_1}=\langle y-x, \frac{\partial\omega}{\partial\omega_1}\rangle . \qquad \text{Q.E.D.}$$

In the rest of this section we shall use the notation as in §3 of [2].

**Lemma 4.2.** *Let* $y=Y_{2q}(x, \omega)$. *Suppose that* $|x'-y'|\leqslant|x'|/2$ *and*

$$y'\cdot\omega'\geqslant 0 . \tag{4.9}$$

*Then it holds that*

$$|x'-y'|\geqslant cq|x'|^{e} .$$

Proof. First note that from the assumption on the principal curvatures we have

$$|n(x)'|\geqslant c|x'|^{e+1} \qquad \text{for } x\in S(\delta_3) .$$

Since $d|x'(s)|^2/ds\geqslant d|x'(s)|^2/ds|_{s=0}=y'\cdot\omega'\geqslant 0$ for all $s>0$ we have

$$\frac{d}{ds}|x'(s)|^2\geqslant c|y'|^{e+1} \qquad \text{for } s\geqslant s_1 .$$

Therefore

$$|x'|^2-|y'|^2\geqslant 2dqc|y'|^{e+1} ,$$

from which it follows that

$$|x'|-|y'|\geqslant 2dqc|y'|^{e}.$$

By using $|y'|\geqslant|x'|/2$, which is a consequence of the assumption, the assertion of Lemma follows. Q.E.D.

**Lemma 4.3.** *Suppose that*

$$|x'-y'|\leqslant|x'|/2 \text{ and } x'\cdot\Xi_{2q}(y,\omega)\leqslant 0. \tag{4.10}$$

*Then it holds that* $y'\cdot\omega\leqslant -qc|x'|^{e+1}$ *and*

$$|x'-y'|\geqslant cq|x'|^{e}.$$

Proof. Since $d|x'(s)|^2//ds$ is an increasing function and

$$0\geqslant d|x'(s)|^2/ds|_{s=s_{2q}+0}\geqslant d|x'(s)|^2/ds|_{s=s_{2q}-0}+2c(1-\delta)|x'|^{e-1}$$

we have

$$\frac{d}{ds}|x'(s)|^2\leqslant -2(1-\delta)c|x'|^{e+1}. \qquad \text{for all } s<s_{2q},$$

which implies

$$|y'|^2-|x'|^2=|x'(0)|^2-|x'(s_{2q})|^2\geqslant 2dqc(1-\delta)|x'|^{e+1}.$$

Thus we have

$$|y'|-|x'|\geqslant 2dqc(1-\delta)|x'|^{e}.$$

**Lemma 4.4.** *When*

$$|\omega'|\geqslant Cq(|x'|^{e+1}+|y'|^{e+1})$$

*holds for some constant C independent of q, we have*

$$|x'-y'|\geqslant dq|\omega'| \text{ and } x'\cdot\Xi_{2q}\geqslant 2dq|y'|^{e+1}.$$

Proof. Since $|x'(s)|^2$ is a convex function we have $|x'(s)|\leqslant\max(|x'|, |y'|)$ for all $s$. Denote the right hand side by $M$. From the law of reflexion

$$\Xi_j(y,\omega)-\Xi_{j-1}(y,\omega)=2(X_j(y,\omega), n(X_j(y,\omega))n(X_j(y,\omega)),$$

we have for $j=1$

$$|\Xi_1(y,\omega)'-\omega'|\leqslant 2|n(X_1(y,\omega))'|\leqslant 2cM^{e+1}.$$

Similarly we have for all $j\leqslant 2q$

$$|(\Xi_j(y,\omega)-\Xi_{j-1}(y,\omega))'|\leqslant 2CM^{e+1}.$$

Then by using the assumption we have

$$|(\Xi_j-\omega)'|\leqslant 2qCM^{e+1}\leqslant|\omega'|/2 \qquad \text{for all} \quad j\leqslant 2q\,.$$

$$\begin{aligned}|(x-y)'| &= |(\sum_{j=1}^{2q} l_j\Xi_{j-1})'| \\ &\geqslant |(\sum_{j=1}^{2q} l_j\omega)'|-|\sum_{j=1}^{2q} l_j(\Xi_j-\omega)'| \\ &\geqslant 2dq|\omega'|-dq|\omega'|\geqslant dq|\omega'|\,.\end{aligned}$$

Q.E.D.

**Lemma 4.5.** *Let* $x=(0, 0, x_3)$, $0<x_3<d$. *If* $q^2|\omega'|<1$ *it holds that*

$$|(x-Y_{2q}(x, \omega))'-2dq\,\omega'|\leqslant Cq^2|\omega'|^2 \tag{4.11}$$

*where* $C$ *is a constant independent of* $q$.

Proof. Let us set $y=Y_{2q}(x, \omega)$, $-\Xi_{2q}(y, \omega)=\tilde{\omega}$. Then we have

$$X_j(x, \tilde{\omega}) = X_{2q-j}(y, \omega),\ \Xi_j(x, \tilde{\omega}) = -\Xi_{2q-j}(y, \omega)\,.$$

First we show that

$$|X_j(x, \tilde{\omega})'|\leqslant Cj|\omega'|, \quad |\Xi_j(x, \tilde{\omega})'-\tilde{\omega}'|\leqslant Cj|\tilde{\omega}'|^2 \tag{4.12}$$

holds for all $j\leqslant 2q$. Suppose that $q^2|\tilde{\omega}'|<1$ and (4.12) holds for $j\leqslant h$. Then

$$\begin{aligned}|X_{h+1}(x, \tilde{\omega})'| &\leqslant |X_h(x, \tilde{\omega})'|+l_h|\Xi_h(x, \tilde{\omega})'| \\ &\leqslant Ch|\omega'|+C(2d+\delta_3)|\omega'|\leqslant C(h+1)|\tilde{\omega}'|\,, \\ |\Xi_{h+1}(x, \tilde{\omega})'-\tilde{\omega}'| &\leqslant C|X_{h+1}(x, \tilde{\omega})'|^{e+1} \\ &\leqslant C(h+1)^3|\tilde{\omega}'|^3\leqslant C(h+1)|\tilde{\omega}'|^2\,.\end{aligned}$$

Thus (4.12) holds for $j=h+1$. By induction (4.12) holds for all $j\leqslant 2q$. Since

$$\begin{aligned}X_{j+1}(x, \tilde{\omega})-X_j(x, \tilde{\omega}) &= l_j(x, \tilde{\omega})\Xi_j(x, \tilde{\omega})\,, \\ (X_{2q}(x, \tilde{\omega})-x)' &= \sum_{j=1}^{2q} l_j(x, \tilde{\omega})\Xi_j(x, \tilde{\omega})' \\ &= \sum_{j=1}^{2q} l_j(x, \tilde{\omega})\tilde{\omega}'+\sum_{j=1}^{2q} l_j(x, \tilde{\omega})\,(\Xi_j(x, \tilde{\omega})-\tilde{\omega})'\,.\end{aligned}$$

Note that $|l_j(x, \tilde{\omega})-d|\leqslant C|X_j(x, \tilde{\omega})'|^2\leqslant Cq^2|\tilde{\omega}'|^2$.
Then

$$|(X_{2q}(x, \tilde{\omega})-x)'-2dq\tilde{\omega}'|\leqslant 2dq|\tilde{\omega}'|^2+Cq^2|\tilde{\omega}'|^2\leqslant C'q^2|\tilde{\omega}'|^2\,. \tag{4.13}$$

Now from (4.12) and $\Xi_{2q}(x, \tilde{\omega})=\omega$

$$|(\omega-\tilde{\omega})'|\leqslant C2q|\tilde{\omega}'|^2\leqslant Cq^{-1}|\tilde{\omega}'|\,,$$

which implies $|(\omega-\tilde{\omega})'|\leqslant Cq^{-1}|\omega'|$ for large $q$. From (4.13) and the above estimate (4.11) follows immediately.

**Corollary.** *On the assumption of Lemma 4.5 we have*

$$\left|\frac{\partial\Phi_{2q}}{\partial\omega_j}(0, x_3, \omega)-2dq\omega_j\right|\leqslant Cq^2|\omega'|^2.$$

Proof. Since $x$ and $y$ are on $\mathscr{P}(x, \omega)$ $|x_3-y_3|\leqslant|(x-y)'||\omega'|$. From (4.11) $x_j-y_j=2dq\omega_j+0(q^2|\omega'|^2)$, and from (4.5)

$$\frac{\partial\Phi_{2q}}{\partial\omega_j}(0, x_3, \omega)-(y_j-x_j)=O(q^2|\omega'|^2).$$

Combining these relations we have the assertion.

**Lemma 4.6.** *Suppose that* $q^2|x'|<1$, $|\omega'|<|x'|^3$. *Then*

$$|(X_j(x, \omega)-x)'|\leqslant C|x'|^2,$$
$$|\Xi_j(x, \omega)'|\leqslant Cj|x'|^3$$

*hold for all* $j\leqslant 2q$, *where* $C$ *is a constant independent of* $q$.

Proof. From (4.11) we have

$$|\Xi_1(x, \omega)'|\leqslant|\omega'|+C|x'|^3\leqslant(C+C_1)|x'|^3,$$
$$|X_1(x, \omega)'|\leqslant|x'|+2(d+\delta_3)|\omega'|\leqslant|x'|(1+Cq^{-4})\leqslant|x'|(1+q^{-2}).$$

Suppose that

$$|X_j(x, \omega)'|\leqslant|x'|(1+jq^{-2}),\ |\Xi_j(x, \omega)'|\leqslant C_2j|x'|^3 \tag{4.14}$$

holds for $j\leqslant h$. Then by the same reasoning as the above

$$\begin{aligned}|X_{h+1}(x, \omega)'|&\leqslant|X_h(x, \omega)'|+2(d+\delta_3)C_2h|x'|^3\\&\leqslant|x'|(1+hq^{-2}+2(d+\delta_3)C_2q^{-4})\\&\leqslant|x'|(1+(h+1)q^{-2})\end{aligned}$$

if $2(d+\delta_3)C_2q^{-2}<1$, and

$$\begin{aligned}|\Xi_{h+1}(x, \omega)'|&\leqslant|\Xi_h(x, \omega)'|+C|X_{h+1}(x, \omega)'|^3\\&\leqslant C_2h|x'|^3+C|x'|^3(1+(h+1)q^{-2})^3\\&\leqslant C_2(h+1)|x'|^3\end{aligned}$$

if $C2^3<C_2$. Thus (4.14) holds for all $j\leqslant 2q$. Therefore

$$\begin{aligned}|(X_{2q}(x, \omega)-x)'|&\leqslant\sum_{j=1}^{2q}l_j|\Xi_j(x, \omega)'|\\&\leqslant 2d|x'|^3C_2\sum_{j=1}^{2q}j\leqslant C|x'|^2.\end{aligned}$$

Q.E.D.

**Lemma 4.7.** *Let $x$ and $y=Y_{2q}(x, \omega)\in\omega(\delta_3)$. Then we have*

$$|\mathrm{grad}_{x',\omega}\Phi_{2q}(x', x_3; \omega)|\geqslant c\min(|x'|^{e+1}, q^{-1}|\omega'|). \tag{4.15}$$

Proof. When $|\omega'|\geqslant Cq(|x'|^{e+1}+|y'|^{e+1})$ Lemma 4.4 shows

$$|\partial_\omega\Phi_{2q}(x, \omega)|\geqslant(1-C|\omega'|)|(x-y)'|\geqslant(1-C|\omega'|)2dq|\omega'|.$$

Thus (4.15) holds. Now let

$$|\omega'|\leqslant Cq(|x'|^{e+1}+|y'|^{e+1})\leqslant 1. \tag{4.16}$$

If $|(x-y)'|\geqslant\frac{1}{2}|x'|$, (4.15) follows immediately from (4.5). Then hereafter we suppose $|x'-y'|\leqslant 1/2|x'|$. Note that from the above inequality $|y'|\leqslant 3/2|x'|$. When $|x(s)'|^2$ is monotonically increasing or decreasing Lemma 4.2 or 4.3 can be applied and we have $|\partial_\omega\Phi_{2q}(x, \omega)|\geqslant(1-C|\omega'|)|x'|^e$, which implies (4.15). If $|x(s)'|^2$ is not monotone, set

$$|X_j(y, \omega)'|^2=\min|x(s)'|^2.$$

Suppose that $|X_j'|\geqslant 1/2|x'|$. Under the condition (4.16) applying Lemma 4.3 to a broken ray $y\to X_j$, we have

$$y\cdot\omega'\leqslant -Cj|X_j|^{e+1}\leqslant -Cj|x'|^{e+1}.$$

Similarly applying Lemma 4.4 to a broken ray $X_j\to x$ we have

$$\Xi_{2q}(y, \omega)\cdot x'\geqslant c(2q-j)|x'|^{e+1}.$$

Therefore

$$\begin{aligned}(x', (\omega-\nabla\varphi_{2q}(x, \omega))') &= (x, \omega'-\Xi_{2q}(y, \omega)')\\ &= (x-y, \omega')+(y, \omega')-(x', \nabla\varphi_{2q}(y, \omega)),\end{aligned}$$

from which it follows that

$$\begin{aligned}|x'||\omega'-\nabla\varphi_{2q}(x, \omega)'| &\geqslant -\frac{|x'|}{2}cq(|x'|^{e+1}+|y'|^{e+1})+2q|x'|^{e+1}\\ &\geqslant cq|x'|^{e+1}.\end{aligned}$$

Then $|\partial_{x'}\Phi_{2q}(x, \omega)|=|\omega'-(\nabla\varphi_{2q}(x, \omega))'|\geqslant cq|x'|^e$, which implies (4.15).

Consider the case $|X_j'|\leqslant\frac{1}{2}|x'|$. Since

$$d|x(s)'|^2/ds|_{s=s_j+0}\geqslant 0,\ d|x(s)'|^2/ds|_{s=s_j-0}\leqslant 0$$

we have $y\cdot\omega'<0$. Suppose that $j\leqslant q$.

$$2\Xi_{2q-1}(y, \omega)'\cdot x'=d|x(s)'|^2/ds\Big|_{s=s_{2q}}$$

$$\geqslant \frac{1}{q}(|x'|^2-|X_j'|^2) \geqslant \frac{1}{2q}|x'|^2,$$

$$\begin{aligned}\frac{1}{2q}|x'|^2 &\leqslant \Xi_{2q-1}(y,\omega)'\cdot X_{2q}(y,\omega)'-y'\cdot\omega' \\ &= (\Xi_{2q}(y,\omega)'-\omega')\cdot x'+(x-y)'\cdot\omega' \\ &\leqslant |\Xi_{2q}(y,\omega)'-\omega'|\,|x'|-\frac{|x'|}{2}Cq|x'|^{e+1}.\end{aligned}$$

Then we have

$$|\Xi_{2q}(y,\omega)'-\omega'| \geqslant \frac{Cq}{2}|x'|^{e+1}, \text{ or } |\Xi_{2q}(y,\omega)'-\omega'| \geqslant \frac{|x'|}{q}.$$

This shows (4.15). Q.E.D.

**Corollary.** *For any fixed $0<x_3<d$, $\Phi_{2q}(x', x_3; \omega)$ as a function of $x'$ and $\omega$, the critical points of $\Phi_{2q}$ are $(x', \omega)$ such that $x'=0$, $\omega=(0, 0, \pm 1)$.*

**Lemma 4.8.** *For $\omega=(0, 0, \pm 1)$ it holds that for $q^2|x'|<1$*

$$Cq|x'|^{e+2} \geqslant \Phi_{2q}(x,\omega)-2dq \geqslant cq|x'|^{e+2} \tag{4.17}$$

$$\left|\frac{\partial \Phi_{2q}}{\partial \omega}(x,\omega)\right| \leqslant Cq|x'|^2. \tag{4.18}$$

Proof. Let $\omega'=0$ and $q^2|x'|<1$. For a broken ray $\mathcal{X}(y,\omega)$, $y=Y_{2q}(x,\omega)$, since $y'\cdot\omega'=0$ $|x(s)'|^2$ is increasing. Therefore $|x'|\geqslant|y'|$, which implies $q^2|y'|<1$. Apply Lemma 4.6 to $\omega$ and $y$ and we have

$$|(X_j(y,\omega)-y)'| \leqslant C|y'|^2 \leqslant C|x'|^2.$$

Setting $j=2q$ we have $|x'-y'|\leqslant C|x'|^2$ which shows (4.18). By using the above estimate we have

$$|X_j(y,\omega)'-x'| \leqslant C|x'|^2.$$

Therefore we have

$$C|x'|^{e+2} \geqslant |X_{j+1}(y,\omega)-X_j(y,\omega)|-d \geqslant c|x'|^{e+2}.$$

Summing up this inequality from $j=0$ to $2q-1$ and we have (4.17).

## 5. Proof of Proposition 3.3

From Corollary of Lemma 4.7 it suffices to consider the integration (3.23) near $x'=0$, $\omega=(0, 0, \pm 1)$. Since $x_3$ and $t$ are fixed we shall omit in the rest of this section to write them in the expression of calculus. First we apply the stationary phase method to the integration in $\omega$ variables. Let us set

$$\omega(\omega') = (\omega_1, \omega_2, \sqrt{1-\omega_1^2-\omega_2^2}),\ \ \omega' = (\omega_1, \omega_2),$$

$$\frac{\partial \Phi_{2q}}{\partial \omega_j}(x', x_3, \omega(\omega')) = f_{q,j}(x', \omega'),\ j = 1, 2 .$$

From Corollary of Lemma 4.5 we have

$$f_{q,j}(0, 0) = 0,\ j = 1, 2 , \tag{5.1}$$

$$\frac{\partial f_{q,j}}{\partial \omega_h}(0, 0) = 2qd\delta_{jh},\ j, h = 1, 2 . \tag{5.2}$$

Concerning Lemma 3.1 we can easily verify from Lemmas 5.2 and 5.3 of [2] that $l(2, 0)=2$, i.e.

$$\left|\frac{\partial f_{q,j}}{\partial \omega_h}(x', \omega')\right|_1 \leqslant Cq^2 . \tag{5.3}$$

Then the implicit function theorem assures the existence of solution of the equations

$$f_{q,j}(x', \omega') = 0 , \qquad j = 1, 2 \quad \text{for} \quad |x'| \leqslant q^{-2} . \tag{5.4}$$

Let us denote this solution by $\omega_q'(x')$. Then from (3.11) we have

$$|\partial_{x'}^{\alpha}\omega_q'(x')| \leqslant C_\alpha q^{l(\alpha)} \qquad \text{for } |x'| \leqslant q^{-2} \tag{5.5}$$

where $l(\alpha)$ denotes an integer depending on $\alpha$. In the rest of this section we shall use notation $l(\alpha)$ for various integer depending on $\alpha$. For the phase function we have

$$\Phi_{2q}(x, \omega) = \Phi_{2q}(x, \omega(\omega_q'(x'))) + \frac{1}{2}\sum_{|\alpha|=2}\frac{1}{\alpha!}(\omega' - \omega_q'(x'))^{\alpha} F_{q,\alpha}(x', \omega') ,$$

where

$$F_{q,(j,h)}(x', \omega') = \int_0^1 \frac{\partial f_{q,j}}{\partial \omega_h}(x', \theta\omega_q'(x') + (1-\theta)\omega')d\theta .$$

Then from (5.2) and (5.3) it holds that

$$\mathcal{F}_q(x', \omega') = [F_{q,(j,h)}(x', \omega')]_{j,h=1,2} \geqslant dqI .$$

By making a change of variables

$$\zeta = \mathcal{F}_q(x', \omega')^{1/2}(\omega' - \omega_q'(x'))$$

we have

$$\Phi_{2q}(x, \omega(\omega')) = \Phi_{2q}(x, \omega(\omega_q'(x'))) + \frac{1}{2}\zeta^*\zeta \tag{5.6}$$

and an estimate

$$|\partial_{x'}^{\alpha}\zeta| \leqslant C_\alpha q^{l(\alpha)} . \tag{5.7}$$

Let $\chi$ be a $C^\infty$ function verifying

$$\chi(\omega') = \begin{cases} 1 & |\omega'| \leqslant 1 \\ 0 & |\omega'| \geqslant 2 . \end{cases}$$

**Lemma 5.1.** *Let* $|x'| \leqslant q^{-2}$ *and* $g(x', \omega') \in C^\infty(\boldsymbol{R}^2 \times \boldsymbol{R}^2)$. *An oscillatory integral*

$$H_q(k, x') = \int_{\boldsymbol{R}^2} e^{ik\Phi_{2q}(x,\omega(\omega'))} g(x', \omega') \chi(\omega'/\delta) d\omega' \quad (\delta > 0)$$

*has an expansion*

$$H_q(k, x') = e^{ik\Psi_q(x')} \{ \sum_{j=0}^{6} k^{-1-j/2} h_{q,j}(x') + k^{-4} h_q(x', k) \}$$

*where*

$$\Psi_q(x') = \Phi_{2q}(x, \omega(\omega_q'(x'))) , \tag{5.8}$$

$$|\partial_{x'}^\alpha h_{q,j}(x')| \leqslant C_\alpha q^{l(\alpha)} |g|_{|\alpha|+2j} , \tag{5.9}$$

$$|\partial_{x'}^\alpha h_q(x'; k)| \leqslant C_\alpha q^{l(\alpha)} |g|_{|\alpha|+12} \qquad \textit{for all } k . \tag{5.10}$$

*Especially for* $j=0$

$$h_{q,0}(x') = \frac{1}{2\pi} (\det \mathcal{F}_q(x', \omega_q'(x')))^{-1/2} g(x', \omega_q'(x')) .$$

Proof. By (5.6) we can write

$$H_q(k, x') = e^{ik\Psi_q(x')} \int_{\boldsymbol{R}^2} e^{ik\zeta^*\zeta} g(x', \omega') \frac{D\omega'}{D\zeta} d\zeta .$$

By using (5.7) we have the assertion by a standard argument.

Then the proof of (3.24) is reduced to obtain an expansion of an oscillatory integral

$$H_{q,j}(k) = \int e^{ik\Psi_q(x')} h_{q,j}(x') dx' . \tag{5.11}$$

To this end we apply Varčenko's theorem [18, 7]. First consider properties of $\Psi_q(x')$.

Let $x_3 = -\gamma(x')$ be a representation of $\Gamma_1$ near $a_1$ and $x_3 = d + \tilde{\gamma}(x')$ be a representation of $\Gamma_2$ near $a_2$.

**Lemma 5.2.** *It holds that*

$$|\Psi_q(x') - 2q(d + \gamma(x') + \tilde{\gamma}(x'))| \leqslant C_q(\gamma(x') + \tilde{\gamma}(x'))|x'|^2 , \tag{5.12}$$

*where* $C_q$ *has an estimate* $C_q \leqslant Cq^a$ *for some* $a > 0$.

Proof. Let $x(s)$ be a representation of $\mathcal{X}(x, \omega(\omega'_q(x')))$. Setting $|X'_j| = \min|x(s)'|$ we have $\Xi_j X'_j \geqslant 0$, $\Xi_{j-1}\cdot X'_j \leqslant 0$. Note that we have $x = X_{2q}(x, \omega(\omega'_q(x')))$ from the definition of $\omega'_q(x')$. Since $\Xi_j - 2(\Xi_j, n(X_j))n(X_j) = 0$ it holds that

$$|\Xi_j| \leqslant C|y'|^{e+1} = C|x'|^{e+1} \leqslant C|x'|^3 .$$

Applying Lemma 4.6 to broken rays $X_j$ to $X_{2q}$ and $y=x$ to $X_j$ we have, if $q^2|x'| \leqslant 1$,

$$\begin{aligned} &|X_h(x, \omega)' - x'| \leqslant C|x'|^2 , \\ &|\Xi_h(x, \omega)'| \geqslant Cq|x'|^3 \end{aligned} \tag{5.13}$$

for all $h$. Evidently we have

$$(X_h)_3 = \begin{cases} -\gamma(X'_h) & \text{if } X_h \in \Gamma_1 \\ d + \tilde{\gamma}(X'_h) & \text{if } X_h \in \Gamma_2 . \end{cases}$$

Thus we have

$$\begin{aligned} ((X_{h+1})_3 - (X_h)_3)^2 &= \{(d+\gamma(x')+\gamma(x')) + ((X_{h+1})_3 - (-\gamma(x')) \\ &\qquad - ((X_h)_3 - (d+\tilde{\gamma}(x'))\}^2 \\ &= (d+\gamma(x')+\gamma(x'))^2(1+(O(\operatorname{grad}(\gamma+\tilde{\gamma})\,(x')|x'|^2)^2) . \end{aligned}$$

On the other hand

$$|X'_{h+1} - X'_h| \leqslant Cq|x'|^{e+1} .$$

Then taking account of (1.1) we have

$$\Big|\sum_{h=0}^{2q-1} |X_{h+1} - X_h| - 2q(d+\gamma(x')+\gamma(x'))\Big| \leqslant C_q|x'|^{2(e+1+2)} .$$

For $x'$ such that $q^2|x'| > 1$ (5.12) holds for $C_q = q^a$ if we choose $a$ sufficiently large. Q.E.D.

Let $\chi_1$ and $\chi_2$ be functions in $C^\infty(\boldsymbol{R}^2)$ such that

$$\chi_1 + \chi_2 = 1 \qquad \text{on } \boldsymbol{R}^2 ,$$

$$\operatorname{supp} \chi_1 \subset \{x';\ |x'| \leqslant 2\},\ \chi_1 = 1 \qquad \text{for } |x'| \leqslant 1 .$$

Set

$$H^{(p)}_{q,j}(k) = \int e^{ik\Psi_q(x')} \chi_p(q^3 x') h_{q,j}(x') dx',\ p = 1, 2 .$$

From (5.12) it follows

$$|\nabla_{x'} \Psi_q(x')| \geqslant \frac{q}{2} |\operatorname{grad}(\gamma(x')+\gamma(x'))| \geqslant \frac{cq}{2} |x'|^{e+1} .$$

Therefore on the support of $\chi_2$ we have $|\nabla_{x'}\Psi_q(x')|\geqslant cq^{-a}$ for some $a>0$. Then using (5.9) we have

$$|H^{(2)}_{q,j}(k)|\leqslant C_N q^{l(N)}k^{-N}\,. \tag{5.13}$$

When we apply Varčenko's theorem to $H^{(1)}_{q,j}$ we have to pay attention to parameter $q$, in other words, we have to obtain an expansion in $k$ of $H^{(1)}_{q,j}$ which is uniform in $q\to\infty$. To this end first we consider the Newtonian polyhedra of $\Psi_q$. Here we use freely the notation in [7]. (5.12) implies

$$\Psi_{q\Gamma}=q(\gamma+\tilde{\gamma})_\Gamma \qquad \text{for large } q\,.$$

Let $Y$ and $\pi$ be an analytic manifold and a projection constructed following Chapter II of [7] for $(\gamma+\tilde{\gamma})$, that is,

$$\pi\colon\, Y\to U$$
$$(\gamma+\tilde{\gamma})\circ\pi(y)=+y_1^{l_1}y_2^{l_2}$$
$$J(y)=y_1^{m_1}y_2^{m_2}J_\pi(y),\ J_\pi(0)\neq 0\,.$$

Then it follows that

$$\Psi_q\circ\pi(y)=+qy_1^{l_1}y_2^{l_2}(1+\psi_q(y))\,,$$

where $\psi_q(0)=0$ and $|\partial_y\psi_q(y)|\leqslant C_\alpha q^{l(\alpha)}$. Then we can find a change of variables $\pi_q\colon y=y_q(z)$ for $|z|\leqslant Cq^a$ such that

$$\Psi_q\circ\pi\circ\pi_q(z)=+qz_1^{l_1}z_2^{l_2}\,,$$
$$y_q(0)=0,\ \frac{\partial y_q}{\partial z}(0)=I\,,$$
$$|\partial_z y_q|\leqslant C_\alpha q^{l(\alpha)}\,.$$

Then $H^{(1)}_{q,j}$ may be represented in finite sum of integrals of the form

$$\int e^{ikqz_1^{l_1}z_2^{l_2}}(h_{q,j}\circ\pi\circ\pi_q)\,(z)J_q(z)dz\,.$$

Then Proposition 3.3 follows from Theorem 3.23 of [7] beside the representation of $c^0_{q,j}$. Note that $\Psi_q$ verifies the condition 3) of Theorem 3.23 of [7] because $\partial^\alpha_{x'}\Psi_q(0)=0$ for $|\alpha|\leqslant 4$, and $\Psi_q(x')>\Psi_q(0)$ for $x'\neq 0$. In Lemma 5.1 when we set $g=v_{q,j}$ we have

$$h_{q,0}=\frac{1}{2\pi}q^{-1}v_{q,j}(x,\,\omega(\omega_q'(x')))\,.$$

Then we have from Theorem 3.23 of [7] the desired relation.

## 6. Representation of the kernel of cos $t\sqrt{-\Delta}$ near $a_1$ and $a_2$

Let $\psi(x)$ be a $C^\infty$ function with support contained in a small neighborhood of $a_1$. We consider the behavior of

$$\int_\Omega \left(\int \rho_q(t)E(t, x, x)\psi(x)dt\right)dx \qquad \text{as} \quad q\to\infty .$$

In this section we denote by $s$ a point of $\Gamma_1$ and by $n(s)$ the unit outer normal of $\Gamma_1$ at $s$. Correspond $(r, s)$ to $x$ near $a_1$ by $x=s+rn(s)$. First we state a result on the propagation of the solutions for oscillatory boundary data.

**Lemma 6.1.** *Let $m$ be an oscillatory boundary data on $\boldsymbol{R}\times\Gamma_1$ of the form*

$$m(t, s; p, p') = e^{i(p\zeta(s)-p't)}h(t, s; p)$$

*satisfying* supp $h\subset(0, 1)\times S_1(\delta_3)$ *and*

$$|\partial_{t,s}^{\alpha}h| \leqslant C_\alpha p^{(1/2-\varepsilon_0)|\alpha|} \qquad (\varepsilon_0>0) . \tag{6.1}$$

*If $|p\nabla_s\zeta|/|p'|\geqslant 4\,\delta_3/d$, the solution of*

$$\begin{cases} \Box u = 0 & \text{in } \boldsymbol{R}\times\Omega \\ u = m & \text{on } \boldsymbol{R}\times\Gamma_1 \\ u = 0 & \text{on } \boldsymbol{R}\times\Gamma_2 \\ \text{supp } u\subset\{t\geqslant 0\} \end{cases} \tag{6.2}$$

*verifies an estimate for any $N$*

$$|\partial_{t,x}^{\alpha}u(t, x; p)| \leqslant C_{\alpha,N}q^{l(\alpha)}p^{-N} \qquad \text{on } [2d, 2dq]\times\omega(\delta_3) . \tag{6.3}$$

Except the case that $|p\nabla_s\zeta|/|p'|$ is near 1 an asymptotic solution of (6.2) can be constructed by a usual manner and checked the propagation of solutions. For exceptional case we make use of the result of Melrose-Sjöstrand [13] on the propagation of singularities. We omit the proof.

As in §3 denoting by $u(t, x; k, \omega)$ the solution of

$$\begin{cases} \Box u = 0 & \text{in } (0, \infty)\times\Omega \\ u = 0 & \text{on } (0, \infty)\times\Gamma \\ u(0, x) = e^{ik\langle x,\omega\rangle}w(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial t}(0, x) = 0 & \text{in } \Omega \end{cases} \tag{6.4}$$

where $w(x)=1$ on supp $\psi$, we have

$$E(t; x, y)\psi(y) = \int_0^\infty k^2dk\int_{|\omega|=1} d\omega\, u(t, x; k, \omega)e^{-ik\langle y,\omega\rangle}\psi(y) .$$

In consideration of the behavior of $u(t, x; k, \omega)$ the difference of the case (6.4) from $u$ of §3 is that the initial data $w(x)e^{ik\langle x,\omega\rangle}$ does not verify the compatibility condition of an initial-boundary value problem at $\{t=0\}\times\Gamma$. Therefore the solution of (6.4) is not regular, and this fact gives rise to difficulties.

Let $\chi_1, \chi_2 \in C^\infty(\boldsymbol{R})$ such that

$$\chi_1 = \begin{cases} 1 & |r| \leqslant 1 \\ 0 & |r| \geqslant 2 \end{cases}$$

and $\chi_1(r)^2 + \chi_2(r)^2 = 1$ on $\boldsymbol{R}$. For $\varepsilon > 0$ we have in $\Omega$

$$\begin{aligned} & w(x)e^{ik\langle x,\omega\rangle} \\ &= Y(r)e^{ik\langle x,\omega\rangle}\chi_1(k^{1/2-\varepsilon}r)^2 w(x) + Y(r)e^{ik\langle x,\omega\rangle}\chi_2(k^{1/2-\varepsilon}r)^2 w(x) \\ &= f_1 + f_2 \end{aligned}$$

where $Y(r)=1$ for $r \geq 0$ and $=0$ for $r<0$. For $u_2(t, x; k, \omega) = \cos t\sqrt{-\Delta} f_2$ we can use the method in §3~5 without large modification and we have

**Lemma 6.2.** *It holds that*

$$\begin{aligned} \Big| \int_\Omega dx \int_0^\infty k^2 dk \int_{|\omega|=1} d\omega\, \chi_2(k^{1/2-\varepsilon)}r) e^{-ik\langle x,\omega\rangle} \Big( \int \rho_q(t) u_2(t, x; k, \omega) dt \Big) \\ - c_0 q^{(1-2/e_0)(l+1)-2} \int_0^d \psi(0, x_3) dx_3 \Big| \leqslant C_l q^{(1-5/2e_0)l}. \end{aligned}$$

Hereafter we consider the behavior of $u_1(x, t; k, \omega) = \cos t\sqrt{-\Delta}\, f_1$. The asymptotic solution $u_0$ for Cauchy problem

$$\begin{cases} \Box u = 0 & \text{in } \boldsymbol{R}\times\boldsymbol{R}^3 \\ u(0, x) = e^{ik\langle x,\omega\rangle}\chi_1(k^{1/2-\varepsilon}r)\psi(x) & \text{in } \boldsymbol{R}^3 \\ \dfrac{\partial u}{\partial t}(0, x) = 0 & \text{in } \boldsymbol{R}^3 \end{cases}$$

is obtained in a form

$$\begin{aligned} u_0(t, x; k, \omega) &= e^{ik(\langle x,\omega\rangle - t)} \sum_{j=1}^N v_j^+(t, x; k)\,(ik)^{-j} \\ &\quad + e^{ik(\langle x,\omega\rangle + t)} \sum_{j=1}^N v_j^-(t, x, k)\,(ik)^{-j} \\ &= u_0^+ + u_0^-. \end{aligned}$$

Then $m^\pm = u_0^\pm|_{(0,\infty)\times\Gamma}$ is of the form

$$m^\pm(t, s; k) = e^{ik(\langle s,\omega\rangle \pm t)} h^\pm(t, s; k) \tag{6.5}$$

$$|\partial_{t,s}^\alpha h^\pm(t, s; k)| \leqslant C_\alpha k^{(1/2-\varepsilon)|\alpha|}. \tag{6.6}$$

Extend $m^\pm$ to a function on $\boldsymbol{R}\times\Gamma$ by setting $m^\pm=0$ for $t<0$. Denote by $u^\pm$ the solution of

$$\begin{cases} \Box u = 0 & \text{in } \boldsymbol{R}\times\Omega \\ u = m^\pm & \text{on } \boldsymbol{R}\times\Gamma \\ \operatorname{supp} u \subset \{t \geq 0\}\times\Omega. \end{cases} \tag{6.7}$$

Then we have $u_1=-u^+-u^-$ on $\omega(\delta_3)$ for $t\geqslant 2R$. Then it suffices to consider $u^\pm$.

Since $m^\pm|_{(0,\infty)\times\Gamma_2}\in C_0^\infty$ we can apply the method in §3~5 for $m^\pm$ on $\Gamma_2$. Therefore we consider only the solution for $m^\pm$ on $\Gamma_1$. First consider the case $|\omega'|\geqslant 1/2$. Since there is no difference for $m^+$ and $m^-$ we consider the solution for $m^+$ and omit $+$ for brevity. By Fourier's inversion formula

$$m(t, s; k, \omega) = w(t)\iint e^{ik'(t-t')}m^+(t', s; k, \omega)dt'dk' \tag{6.8}$$
$$= \int w(t)e^{ik't}e^{ik\langle s,\omega\rangle}\hat{h}(k'-k, s; k, \omega)dk'$$

for $w(t)\in C_0^\infty(\boldsymbol{R})$ such that $w(t)=1$ on supp $m^+$. Let us denote by $b(t, x; k, \omega, k')$ the solution of (6.7) whose $m^+$ is replaced by $w(t)e^{ik't}e^{ik\langle s,\omega\rangle}\hat{h}(k'-k, s; k, \omega)$. For $|k'|\leqslant 2|k|$ we have

$$|k\nabla_s\langle s, \omega\rangle|/|k'|\geqslant 4\delta_3/d\,,$$

from which

$$|b(t, x; k, \omega, k')|\leqslant C_N q^{l(\alpha)}k^{-N} \qquad \text{in } [2d, 2dq]\times\omega(\delta_3) \tag{6.9}$$

follows by an application of Lemma 6.1. For $|k'|\geqslant 2|k|$

$$|\hat{h}(k'-k, s; k, \omega)|\leqslant C|k'-k|^{-1}\leqslant C'|k'|^{-1}\,.$$

As an approximation of $b$ we have an asymptotic solution of the form

$$b' = \sum_{q=0}^{\infty}\sum_{j=0}^{N} e^{ik'(\varphi_q(x;\omega,k/k')-t)}v_{q,j}(t, x; k, \omega, k')\,(ik')^{-j}$$

where

$$|v_{q,j}|\leqslant C(k^{1/2-\varepsilon})^{2j}|k'-k|^{-1}\,. \tag{6.10}$$

Since $|\partial\varphi_q/\partial x_3|\geqslant 1-C\delta_3$ on $\omega(\delta_3)$ and $|\omega_3|\leqslant\sqrt{3}/2$, we have

$$\Big|\int_\Omega\Big(\int\rho_q(t)b(t, v; k, \omega, k')dt\Big)\chi_1(k^{1/2-\varepsilon}r)e^{ik\langle x,\omega\rangle}dx\Big|$$
$$\leqslant C(\hat{\rho}(k'/q^{-l})+q^{-l}C|\zeta(k'/q^{-l})|)|k'-k|^{-3}k^{1/2-\varepsilon}$$

by using (6.10) and the fact $b'=0$ on $\Gamma_1$, where $\zeta$ is a rapidly decreasing function. Thus we have

$$\Big|\int_\Omega dx\int_{|\omega'|>1/2}d\omega\int_0^\infty k^2dk\int_{|k'|>2k}dk'\Big(\int\rho_q(t)bdt\Big)\chi_1(k^{1/2-\varepsilon}r)e^{ik\langle x,\omega\rangle}\Big|$$
$$\leqslant C\iint_{|k'|>2k}k'^{-3}k^{2+1/2+\varepsilon}(\hat{\rho}(k'/q^{-l})+q^{-l}\zeta(k'/q^{-l}))dk'dk$$
$$\leqslant Cq^{l(1/2-\varepsilon)}\,.$$

Combining this estimate and (6.9) we have

**Lemma 6.3.** *It holds that*

$$|\int_{\Omega}dx\int k^2dk\int_{|\omega'|>1/2}d\omega\int dk'(\int\rho_q(t)bdt)\chi_1(k^{1/2-\varepsilon}r)e^{ik\langle x,\omega\rangle}|$$
$$\leqslant C_l(q^l)^{1/2-\varepsilon}.$$

Next we consider the case of $|\omega'|\leqslant 1/2$. In this case in addition to (6.6) another estimate

$$|\partial^{\alpha}_{s,t}h^+(t, s; k)|\leqslant C_{\alpha} \qquad \text{for } (t, s)\in[0, t_0k^{-(1/2-\varepsilon)}]\times S_1(\delta_3) \tag{6.11}$$

holds if we choose $t_0>0$ small. Let us set

$$m^{\pm} = Y(t)\chi_1(Tk^{1/2-\varepsilon}t)^2m^{\pm}+Y(t)\chi_2(Tk^{1/2-\varepsilon}t)^2m^{\pm}$$
$$= m^{\pm}_1+m^{\pm}_2 .$$

Denote by $b^{\pm}_p$, $p=1, 2$, the solution of (6.7) replaced $m^{\pm}$ by $m^{\pm}_p$. Concerning $b^{\pm}_2$ we can apply the method in §3∼5 for construction of asymptotic solution and acheive the parallel argument.

**Lemma 6.4.** *We have an estimate*

$$|\int_{\Omega}dx\int_{|\omega'|<1/2}d\omega\int k^2dk(\int\rho_q(t)b^{\pm}_2dt)\chi_1(k^{1/2-\varepsilon}r)e^{ik\langle x,\omega\rangle}|$$
$$\leqslant C_lq^{(1/2-\varepsilon)l}.$$

Note that $m^{\pm}_1$ is of the form

$$m^{\pm}_1 = e^{ik(\langle s,\omega\rangle\mp t)}h^{\pm}_1(t, s; k, \omega),$$

$$|\partial^{\alpha}_t\partial^{\beta}_s h_1(t, s; k, \omega)|\leqslant C_{\alpha,\beta}k^{(1/2-\varepsilon)\alpha} \qquad \textit{for } t>0 . \tag{6.12}$$

We consider only for $m^+_1$, and hereafter we omit the suffix + and 1 for brevity. In a same way as (6.8) we have

$$m(t, s; k, \omega) = \int w(t)e^{-ik't}e^{ik\langle s,\omega\rangle}\hat{h}(k'-k, s; k, \omega)dk'$$

where

$$\hat{h}(k'-k, s; k, \omega) = \int e^{-i(k't'-kt')}h(t', s; k, \omega)dt' . \tag{6.13}$$

Denote by $b(t, x; k, \omega, k')$ the solution of (6.7) replaced $m^{\pm}$ by $w(t)e^{ik't}e^{ik\langle s,\omega\rangle}$ $\hat{h}(k'-k, s; k, \omega)$. Then

$$b^+_1(t, x; k, \omega) = \int b(t, x; k, \omega, k')dk' . \tag{6.14}$$

Taking account of (6.12) we have for all $k'$

$$|\partial^{\alpha}_s\hat{h}(k'-k, s; k, \omega)|\leqslant C_{\alpha}k^{-(1/2-\varepsilon)} , \tag{6.15}$$

and for $k' \neq k$ we have by integration by parts in (6.13)

$$|\partial_s^\alpha \hat{h}(k'-k, s; k, \omega)| \leqslant C_\alpha |k'-k|^{-1}. \tag{6.16}$$

For small $\gamma$ let $\varphi_1(x; \omega, \gamma)$ be a function verifying

$$\begin{cases} \varphi_1 = (1+\gamma)\langle s, \omega\rangle & \text{on } \Gamma_1 \\ \dfrac{\partial \varphi_1}{\partial n} > 0 & \text{on } \Gamma_1 \\ |\nabla \varphi_1| = 1\,. \end{cases}$$

Then for $\varphi_1$ we can define a sequence of phase functions $\varphi_j(x; \omega, \gamma)$, $j=2, 3, \cdots$ following the process in §3. Set

$$\Phi_{2q}(x; \omega, \gamma) = \varphi_{2q}(x; \omega, \gamma) - \langle x, \omega\rangle\,.$$

As a modification of considerations in §4 we have

**Lemma 6.5.** *Let $\gamma_0$ and $r_0$ be small positive constants. Then there exists $\omega(x, \gamma)$ satisfying*

$$\nabla_{\omega'} \Phi_{2q}(x; \omega(x, \gamma), \gamma) = 0 \qquad \text{for } |x-a_1| \leqslant r_0$$

*and this critical point is non-degenerate. If we set*

$$\psi_q(x, \gamma) = \Phi_{2q}(x; \omega(x, \gamma), \gamma)\,,$$

*the critical point with respect to $x'$ is only $x'=0$ and concerning the Newtonian polyhedra of $\psi_q$ we have the same assertions as in §4 for all $|\gamma| \leqslant \gamma_0$.*

For $k' \in [(1-\gamma_0)k, (1+\gamma_0)k]$, with the aid of the above lemma we estimate an oscillatory integral following the process of §5. Applying Varčenko's theorem we have

$$\begin{aligned} J(t, r; k, k') &= \int ds \int_{|\omega'|<1/2} d\omega\, b(t, x; k, \omega, k') e^{ik\langle x, \omega\rangle} \chi_1(k^{1/2-\varepsilon} r) \\ &= \chi_1(k^{1/2-\varepsilon} r) e^{ik'(t-(2dq+\gamma r))} \{c_0(r; k, k') k'^{-1-2/e_0} + O(k'^{-1-5/3e_0})\} \end{aligned} \tag{6.17}$$

where

$$|c_0(r; k, k')| \leqslant C k^{-1/2+\varepsilon}$$

holds because of (6.15). Then

$$\begin{aligned} &\left|\int dr \int k^2 dk \int \rho_q(t) dt \int_{|k'-k| \leqslant k^{1/2+\varepsilon}} J(t, r; k, k') dk'\right| \\ &\leqslant C \iint_{|k'-k| \leqslant k^{1/2+\varepsilon}} (k'/q^{-l}) k'^{-1-2/e} k^2 k^{-(1/2-\varepsilon)} k^{-(1/2-\varepsilon)} dk dk' \\ &\leqslant C_l q^{(1/2-2/e_0+\varepsilon)l}\,. \end{aligned} \tag{6.18}$$

For $k'\in[k+k^{1/2+\varepsilon}, (1+\gamma_0)k]$ use (6.16) and make an integration by parts with respect to $r$ in the left hand side of (6.18). Then since $c_0(0; k, k')=0$ we have

$$(6.19) \qquad |\int k^2dk\int\rho_q(t)dt\int_0^{r_0}dr\int_{k+k^{1/2+\varepsilon}}^{(1+\gamma_0)k}J(t, r; k, k')dk'|$$
$$\leqslant C\int dk\int_{k+k^{1/2+\varepsilon}}^{(1+\gamma_0)k}\hat{\rho}\Big(\frac{k'}{q^l}\Big)\frac{1}{|k'-k|^3}k'^{-1-2/\varepsilon_0}k^{1/2-\varepsilon}k^2dk'$$
$$\leqslant C_l q^{(1/2+\varepsilon-2/\varepsilon_0)l}.$$

We have the same estimate for $k'\in[(1-\gamma_0)k, k-k^{1/2+\varepsilon}]$. Thus it remains us to consider for $|\omega'|<1/2$ and $|k'-k|\geqslant\gamma_0 k$. For $k'\geqslant(1+\gamma_0)k$ set

$$\tilde{J}(k, k')=\int dt\int dr\,\rho_q(t)J(t, r; k, k')$$

and we have from (6.16)

$$\tilde{J}(k, k')\leqslant\zeta(k'/q^l)|k'-k|^{-3}k^{1/2-\varepsilon}$$

where $\zeta\in\mathcal{S}(\boldsymbol{R})$. Thus

$$(6.20) \qquad |\int k^2dk\int_{(1+\gamma_0)k}^{\infty}\tilde{J}(k, k')dk'|\leqslant C_l q^{(1/2-\varepsilon)l}.$$

Suppose $|k'|\leqslant(1-\gamma_0)k$. When $|k\omega'|\leqslant k^{\varepsilon}$, $|k'|\leqslant k^{\varepsilon}$ we have immediately

$$|\partial_{t,x}^{\alpha}b(x, t; k, \omega, k')|\leqslant C_{\alpha}k^{(|\alpha|+2)\varepsilon}$$

from the energy estimate of solution of (6.7). Thus we have

$$|\int J(t, r; k, k')dr|\leqslant Ck^{-3+3\varepsilon},$$

from which it follows that

$$|\int k^2dk\int_{|\omega'|<k^{-1+\varepsilon}}d\omega\int_{|k'|<k^{\varepsilon}}dk'\int dt\,\rho_q(t)J(t, r; k, k')|\leqslant C$$

for all $q$. Let us suppose $|k\omega'|\geqslant k^{\varepsilon}$, $|k'|\geqslant k^{\varepsilon}$. If $|k\omega'|/|k'|\geqslant 4\delta_3/d$ an application of Lemma 6.1 gives

$$|J(t, r; k, k')|\leqslant C_N k^{-\varepsilon N}.$$

Thus

$$(6.21) \qquad |\int k^2dk\int_{(1-\gamma_0)k>|k'|>k^{\varepsilon}}dk'\int_{|k\omega'|/|k'|>d_0}d\omega\int dr\int\rho_q(t)J(t, r; k, k')dt|\leqslant C.$$

Let $|k\omega'|/|k'|\leqslant d_0=4\delta_3/d$, $(1-\gamma_0)k\geqslant|k'|\geqslant k^{\varepsilon}$. Then we have

$$|J(t, r; k, k')|\leqslant Ck^{-3}k^{1/2-\varepsilon}.$$

Therefore

$$(6.22)\qquad |\int k^2dk\int_{k^\varepsilon}^{(1-\gamma_0)k}dk'\int_{|k\omega'|/|k'|\leqslant d_0}d\omega\int dt\,dr\,\rho_q(t)J(t,r;k,k')|$$
$$\leqslant C\int k^2dk\int_{k^\varepsilon}^{(1-\gamma_0)k}dk'\zeta\left(\frac{k'}{q^l}\right)k^{-3}k^{1/2-\varepsilon}\left(\frac{k'}{k}\right)^2$$
$$\leqslant C\int_{-\infty}^{\infty}\zeta(k'q^{-l})k'^2\left(\int_{(1-\gamma_0)k'}^{\infty}k^{-5/2-\varepsilon}dk\right)dk'$$
$$\leqslant C\int_{-\infty}^{\infty}\zeta(k'q^{-l})k'^{2-3/2-\varepsilon}dk'\leqslant Cq^{(1/2-\varepsilon)l}\,.$$

Then the estimates (6.18)$\sim$(6.22) imply the following

**Lemma 6.6.** *We have*

$$|\int_\Omega dx\int_{|\omega|<1/2}d\omega\int k^2dk\left(\int\rho_q(t)b_1^+dt\right)\chi_1(k^{1/2-\varepsilon}r)e^{ik\langle x,\omega\rangle}|\leqslant C_lq^{(1/2-\varepsilon)l}\,.$$

From Lemmas 6.2$\sim$6.6 we have

**Proposition 6.7.** *Let $\psi(x)$ be a $C^\infty$ function with support in a small neighborhood of $a_1$. Then an estimate*

$$|\int_\Omega\left(\int\rho_q(t)E(t,x,x)dt\right)\psi(x)dx-c_0q^{(1-2/e_0)(l-1)}\int_0^d\psi(0,x_3)dx_3|\leqslant C_lq^{(1-5/2e_0)l}$$

*holds.*

## References


[1] C. Bardos, J.C. Guillot and J. Ralston: *La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion,* Comm. Partial Differential Equations **7** (1982), 905–958.

[2] M. Ikawa: *Decay of solutions of the wave equation in the exterior of two convex obstacles,* Osaka J. Math. **19** (1982), 459–509.

[3] ————: *Mixed problems for the wave equation,* Proc. NATO Advanced Study Institute, Singularities in boundary value problems, Reidel Publ., (1981), 97–119.

[4] ————: *On the poles of the scattering matrix for two strictly convex obstacles,* J. Math. Kyoto Univ. **23** (1983), 127–194.

[5] ————: *On the poles of the scattering matrix for two strictly convex obstacles, Addendum,* J. Math. Kyoto Univ. **23** (1983), 795–802.

[6] ————: *On the distribution of the poles of the scattering matrix for two strictly convex obstacles,* Hokkaido Math. J. **12** (1983), 343–359.

[7] A. Kaneko: Newton polyhedora, singularities, oscillatory integrals, (in Japanese), Lecture note of Sofia Univ., No. 11, 1981.

[8] H. Kumano-go: Pseudo differential operators, MIT Press, 1983.

[9] P.D. Lax and R.S. Phillips: Scattering theory, Academis Press, New York, 1967.
[10] P.D. Lax and R.S. Phillips: *A logarithmic bound on the location of the poles of the scattering matrix*, Arch. Rational Mech. Anal. **40** (1971), 268–280.
[11] R.B. Melrose: *Singularities and energy decay in acoustical scattering,* Duke Math. J. **46** (1979), 43–59.
[12] ————: *Polynomial bound on the number of scattering poles*, J. Func. Anal. **53** (1983), 287–303.
[13] R.B. Melrose and J. Sjöstrand: *Singularities of boundary value problems*. I, Comm. Pure Appl. Math. **31** (1978), 593–617.
[14] C.S. Morewetz, J. Ralston and W.A. Strauss; *Decay of solutions of the wave equation outside nontrapping obstacles,* Comm. Pure Appl. Math. **30** (1977), 447–508.
[15] V.M. Petkov: *La distribution des poles de la matrice de diffusion,* Séminaire Goulaouic-Meyer-Schwartz, 1982–1983, Exposé N° VII.
[16] J. Ralston: *Trapped rays in spherical symmetric media and poles of the scattering matrix,* Comm. Pure Appl. Math. **24** (1971), 571–582.
[17] ————: *The first variation of the scattering matrix*; *An addendum*, J. Differential Equations **28** (1978), 155–162.
[18] ————: *Propagation of singularities and the scattering matrix,* Proc. NATO Advanced Study Institute, Singularities in boundary value problems, Reidel Publ., (1981), 161–181.
[19] A.N. Varcenko: *Newton polyhedra and estimation of oscillating integrals,* Funct. Anal. Appl. **10** (1976), 175–196.

Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560, Japan