Hiramine, Y.
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ON TRANSLATION PLANES OF ORDER q^2 WHICH ADMIT AN AUTOTOPISM GROUP HAVING AN ORBIT OF LENGTH q^2-q

Dedicated to Professor Hirosi Nagao on his 60th birthday

YUTAKA HIRAMINE

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1. Introduction

Let π be a translation plane of order q^2 satisfying the following conditions:

(1.1) π has $GF(q)$ in its kernel.

(1.2) π admits a linear autotopism group of order q. (Here a linear autotopism group is a subgroup of the linear translation complement of π which fixes at least two points on the line l_∞ at infinity.)

Several classes of translation planes with these properties have been constructed in [1], [2] and [7]. Any of these planes can be coordinatized by a quasifield having a central kernel of order q and satisfy the following condition:

(1.2)′ π admits a linear autotopism group having an orbit of length q^2-q on the line at infinity.

The purpose of this paper is to investigate the translation planes with the properties (1.1) and (1.2), especially with (1.1) and (1.2)′ in the latter half of the paper.

In §2 we consider the quasifields corresponding to these planes. Let K be a field. Let $h(x)$ be a mapping from K into K and $r(y)$ and $s(y)$ mappings from $K^\#=K-\{0\}$ into K. Set $f(x, y)=-y^{-1}(x^2-r(y)x-s(y))$ and $g(x, y)=-x+r(y)$. Assume that $r(y)$, $s(y)$ and $h(x)$ satisfy the following conditions.

(1.3) $f(x, y_1)\neq f(x, y_2)$ whenever $x\in K$ and $y_1, y_2\in K^\#$, $y_1\neq y_2$.

(1.4) $K=f(x, K^\#)\cup h(x)$ (disjoint union) for any $x\in K$.

(1.5) $h(0)=h(1)=0$.

Let $\tilde{\Phi}_K$ be the set of such triples (r, s, h) and put $\Phi_K=\{(r, s, h)\mid(r, s, h)\in\tilde{\Phi}_K,$ $h(x)=0$ for any $x\in K\}$. An element $(r, s, 0)$ of Φ_K is often written (r, s) for brevity's sake.

A quasifield $Q_{(r,s,h)}((r, s, h)\in\tilde{\Phi}_K)$, which is a two dimensional left vector space over K with a basis $\{1, \lambda\}$, is defined by a multiplication

$$(1.6)\quad (z+t\lambda)\circ(x+y\lambda)=\begin{cases} zx+tf(x,y)+(zy+tg(x,y))\lambda & \text{if } y\neq 0\\ zx+th(x)+tx\lambda & \text{if } y=0, \end{cases}$$

for any x, y, z and t in K.

Such quasifields have been investigated by S.D. Cohen and M.J. Ganley [1] particularly in the case that K is a prime field and also by the author [3] in the case that $h(x)=0$ for any $x\in K$.

The quasifields Q_g, $g=(r, s, h)$ are characterized by the following theorem when $|K|<\infty$.

Theorem 2.4. *Let π be a translation plane of order q^2 having $GF(q)$ in its kernel. Then π is coordinatized by a quasifield Q_g for some $g=(r, s, h)\in\tilde{\Phi}_{GF(q)}$ if and only if π admits a linear autotopism group of order q.*

In the known examples as stated above, any of these planes can be coordinatized by a quasifield Q_g with $g\in\Phi_K$ (See [1]). In §3 we prove a theorem which is a generalization of Theorem 5.4 of [1].

Theorem 3.11. *Let π be a translation plane of order q^2 having $GF(q)$ in its kernel. If π admits a linear autotopism group having an orbit of length q^2-q on the line at infinity, then π is coordinatized by a quasifield Q_g for some $g=(r, s)\in\Phi_K$, where $r(y)=ay^n$ and $s(y)=by^{2n}$ for suitable $a, b\in K$ and an integer n, $0\leq n<q-1$.*

In §5 we determine the linear translation complement of the planes stated in Theorem 3.11 when $n\notin\{e(q-1)\mid e=0, 1/2, 1/3, 2/3, 1/4, 3/4\}$

2. The quasifields Q_g, $g\in\tilde{\Phi}_K$

Let K be a field. Let r and s be mappings from $K^\sharp$ into K and let h be a mapping from K into itself. Set $f(x, y)=-y^{-1}(x^2-r(y)x-s(y))$ and $g(x, y)=-x+r(y)$. Assume $(r, s, h)\in\tilde{\Phi}_K$. Then $f(x, y)$ and $h(x)$ satisfy the conditions (1.3)–(1.5). In this section first we show that Q_g, $g=(r, s, h)$ defined by (1.6) is actually a quasifield and secondly characterize the quasifield Q_g in terms of collineation groups.

Proposition 2.1. *Q_g is a quasifield for any $g\in\tilde{\Phi}_K$, which has K in its kernel.*

Proof. The proof is similar to that of Theorem 1 of [3]. Set $Q=Q_g$. By a definition of Q,

$$\begin{gathered} a(\xi+\eta)=a\xi+a\eta,\ (a\xi)\eta=a(\xi\eta),\ (\xi+\eta)\zeta=\xi\zeta+\eta\zeta,\\ \xi 1=1\xi=\xi \text{ and } \xi 0=0, \text{ for any } \xi, \eta, \zeta\in Q \text{ and } a\in K. \end{gathered}\tag{2.1}$$

Let $a, b, c, d\in K$ such that $(a, b)\neq(0, 0)$ and $(c, d)\neq(0, 0)$. Set $S=\{x+$

$y\lambda\in Q^{\sharp}=Q-\{0\}\,|\,(a+b\lambda)\,(x+y\lambda)=c+d\lambda,\ x,\ y\in K\}$. We show that

$$|S|=1\,. \tag{2.2}$$

Set $S_1=\{x+y\lambda\,|\,x,\ y\in K,\ y\neq 0,\ ax+bf(x,\ y)=c$ and $ay+bg(x,\ y)=d\}$ and $S_2=\{x\,|\,x\in K,\ ax+bh(x)=c,\ bx=d\}$. Then, by the multiplication (1.6), $S=S_1\cup S_2$.

Suppose $b=0$. Then, as $a\neq 0$, clearly $S=S_1\cup S_2=\{a^{-1}c+a^{-1}d\lambda\}$. Suppose $b\neq 0$. Then, by a similar argument as in Lemma 2.2 of [3], $S_2=\{x+y\lambda\,|\,f(b^{-1}d,\ y)=(bc-ad)/b^2,\ x=(ay+br(y)-d)/b\}$ and $S_2=\{b^{-1}d\}$ if $h(b^{-1}d)=(bc-ad)/b^2$ and $S_2=\phi$ if $h(b^{-1}d)\neq(bc-ad)/b^2$. By the hypotheses (1.3) and (1.4) we have $|S_1\cup S_2|=1$. Thus (2.2) holds.

Let $a,\ b,\ c,\ d\in K$ such that $(a,\ b)\neq(0,\ 0)$ and $(c,\ d)\neq(0,\ 0)$ and set $T=\{x+y\lambda\in Q^{\sharp}\,|\,x,\ y\in K,\ (x+y\lambda)\,(a+b\lambda)=c+d\lambda\}$. We show that

$$|T|=1\,. \tag{2.3}$$

Suppose $b\neq 0$. Then $T=\{x+y\lambda\in Q^{\sharp}\,|\,x,y\in K,\ xa+yf(a,\ b)=c,\ xb+yg(a,\ b)=d\}$ by (1.6). By (1.5), $h(0)=0$ and so (1.3) and (1.4) yield $f(0,\ y)=y^{-1}s(y)\neq 0$ for any $y\in K^{\sharp}$. In particular $ag(a,\ b)-bf(a,\ b)=-s(b)\neq 0$. This, together with $(c,\ d)\neq(0,\ 0)$ implies that $|T|=1$. Suppose $b=0$. Then $a\neq 0$ and hence $T=\{a^{-1}(c-a^{-1}dh(a))+a^{-1}d\lambda\}$.

We now prove the proposition. By (2.1)–(2.3), Q is a weak quasifield having K in its kernel. Since $\dim_K Q<\infty$, Q is a quasifield having K in its kernel by Theorem 7.3 of [4].

Lemma 2.2. *Let $Q=Q_g$, $g=(r,\ s,\ h)\in\tilde{\Phi}_K$ and let $x\in K$. Then the following three conditions are equivalent.*

(i) $h(x)=0$.

(ii) *$x\xi=\xi x$ for some $\xi\in Q-K$.*

(iii) *$x\xi=\xi x$ for any $\xi\in Q$.*

Proof. If $h(x)=0$, then $(z+t\lambda)x=zx+tx\lambda=x(z+t\lambda)$. Hence (i) implies (iii). Clearly (iii) implies (ii). Suppose $x\xi=\xi x$ for some $\xi\in Q-K$ and set $\xi=a+b\lambda$, where $a,\ b\in K$ and $b\neq 0$. Then $xa+xb\lambda=ax+gh(x)+bx\lambda$. Hence $h(x)=0$ and so (ii) implies (i).

In the rest of the paper we assume that K is a finite field. Let $g=(r,\ s,\ h)\in\tilde{\Phi}_K$. Set $M(x,\ 0)=\begin{pmatrix} x & 0\\ h(x) & x\end{pmatrix}$ and $M(x,y)=\begin{pmatrix} x & y\\ f(x,y) & g(x,y)\end{pmatrix}$, where $x,y\in K$, $y\neq 0$, $f(x,\ y)=-y^{-1}(x^2-r(y)x-s(y))$ and $g(x,\ y)=-x+r(y)$. We define Σ_g to be the set of all matrices $M(x,\ y)$, $x,\ y\in K$. Then the following holds.

Lemma 2.3. (i) *Σ_g is a spread set corresponding to Q_g for $g=(r,\ s,\ h)\in\tilde{\Phi}_K$.*

(ii) *Let* $M(x, y)\in\Sigma_g$. *If* $y\neq 0$, *then* $r(y)=tr(M(x, y))$ *and* $s(y)=-det(M(x, y))$. (*Here "tr" or "det" denotes the trace or the determinant of the matrix, respectively.*)

Proof. The spread set corresponding to $Q=Q_g$ is the set of all K-linear mappings M_m ($m\in Q$) such that $xM_m=xm$ for $x\in Q$. Set $m=a+b\lambda$ and $\lambda m=f'(a, b)+g'(a, b)\lambda$, where a, b, $f'(a, b)$ and $g'(a, b)$ are elements of K. Then, as $1m=a+b\lambda$, $\Sigma=\{\begin{pmatrix} a & b \\ f'(a, b) & g'(a, b)\end{pmatrix} \mid a, b\in K\}$ is the spread set corresponding to Q.

If $b=0$, $\lambda m=\lambda a=h(a)+a\lambda$ by (1.6). Hence $f'(a, 0)=h(a)$ and $g'(a, 0)=a$. If $b\neq 0$, then $\lambda m=f(a, b)+g(a, b)\lambda$ by (1.6). Hence $f'(a, b)=f(a, b)$ and $g'(a, b)=g(a, b)$. Thus $\Sigma=\Sigma_g$ and (i) holds. By a direct computation we have (ii).

Let $g\in\tilde{\Phi}_K$ and π_g a translation plane coordinatized by Q_g. Set $\pi=\pi_g$. Then π can be regarded as a 4-dimensional left vector space over K. Set $L(a, b)=\{(v, vM(a, b)) \mid v\in K\times K\}$ and $L(\infty)=\{(0, 0, v) \mid v\in K\times K\}$. Then $\mathcal{L}_g=\{L(a, b) \mid a, b\in K\}\cup\{L(\infty)\}$ is a spread of π, the set of lines of π passing through the origin. $\mathcal{L}_g$ is often identified with the set of points on l_∞. The linear translation complement of π is denoted by $LC(\pi)$. Since π is 4-dimensional over K, any element of $LC(\pi)$ is represented by a 4×4 matrix over K. Let $\sigma=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ be a nonsingular 4×4 matrix over K, where A, B, C, $D\in M_2(K)=\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \mid a, b, c, d\in K\}$. Then the following criterion is well known: σ is an element of $LC(\pi)$ if and only if the following conditions are satisfied.

(2.4.1) If C is nonsingular, then $C^{-1}D\in\Sigma_g$. (In this case $L(\infty)\sigma=L(u, v)$, where $C^{-1}D=M(u, v)$.)

(2.4.2) If C is singular, then C is a zero matrix. (In this case $L(\infty)\sigma=L(\infty)$.)

(2.4.3) If $A+M(x, y)C$ is nonsingular, then $(A+M(x, y)C)^{-1}(B+M(x, y)D)\in\Sigma_g$. (In this case $L(x, y)\sigma=L(u, v)$, where $(A+M(x, y)C)^{-1}(B+M(x, y)D)=M(u, v)$.)

(2.4.4) If $A+M(x, y)C$ is singular, then $A+M(x, y)C$ is a zero matrix. (In this case $L(x, y)\sigma=L(\infty)$.)

Let $q=p^n$ be a power of a prime p and set $K=GF(q)$. The translation plane coordinatized by the quasifields Q_g ($g=(r, s, h)\in\tilde{\Phi}_K$) are characterized as follows.

Theorem 2.4. *Let* π *be a translation plane of order* q^2 *having* K $(=GF(q))$ *in its kernel. Then* π *is coordinatized by a quasifield* Q_g *for some* $g=(r, s, h)\in\tilde{\Phi}_K$ *if and only if* π *admits a linear autotopism group of order* q.

Proof. Let $g=(r, s, h)\in\tilde{\Phi}_K$ and let π be a translation plane coordinatized by Q_g. Let $\Sigma=\{M(x, y)\,|\,x, y\in K\}$ or $\mathcal{L}=\{L(x, y)\,|\,x, y\in K\}\cup\{L(\infty)\}$ be the spread set or the spread of π as defined above. Let $t\in K$ and set $\sigma=\begin{pmatrix}X & O\\ O & X\end{pmatrix}$, where $X=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$. Since $X^{-1}M(x, 0)X=\begin{pmatrix}1 & 0\\ -t & 1\end{pmatrix}\begin{pmatrix}x & 0\\ h(x) & x\end{pmatrix}\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}=\begin{pmatrix}x & 0\\ h(x) & x\end{pmatrix}$, $L(x, 0)\sigma=L(x, 0)$ for any $x\in K$. Similarly $X^{-1}M(x, y)X=M(x+yt, y)$ for $y\neq 0$. Hence $L(x, y)\sigma=L(x+yt, y)$ for any $x, y\in K, y\neq 0$ and clearly $L(\infty)\sigma=L(\infty)$. By (2.4.1) and (2.4.2), σ is a collineation of π which fixes a subset $\{L(x, 0)\,|\,x\in K\}\cup\{L(\infty)\}$ of l_∞ pointwise. Therefore the set P of such collineations forms a subgroup of the linear translation complement of π. Clearly $|P|=q$ and P fixes $L(\infty)$ and $L(0, 0)$. Thus π admits a linear autotopism group of order q.

Conversely, assume $q\,\big|\,|N|$, where $N=LC(\pi)_{A,B}$ and $A, B\in l_\infty$, $A\neq B$. Let P be a Sylow p-subgroup of N. Then $|P|\geq q$. Since $|l_\infty-\{A, B\}|=q^2-1\not\equiv 0 \pmod p$, P fixes another point C on l_∞, $C\neq A, B$. We coordinatize π with $A=\{(0, 0, v)\,|\,v\in K\times K\}$, $B=\{(v, 0, 0)\,|\,v\in K\times K\}$ and $C=\{(v, v)\,|\,v\in K\times K\}$ (cf. Lemma 2.1 of [6]). Then $P\leq G$, where $G=\{\begin{pmatrix}X & O\\ O & X\end{pmatrix}|\,X\in GL(2, q)\}$. Hence $|P|=q$ and P is a Sylow p-subgroup of G. As A, B and C are G-invariant, we may assume $P=\{\begin{pmatrix}Y & O\\ O & Y\end{pmatrix}|\,Y=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}, t\in K\}$ by a Sylow's theorem. Let $\Sigma=\{N(x, y)=\begin{pmatrix}x & y\\ a(x, y) & b(x, y)\end{pmatrix}|\,x, y\in K\}$ be a spread set such that $L'(x, y)=\{(v, vN(x, y))\,|\,v\in K\times K\}$ is a line through the origin O for any $x, y\in K$.

Let $\begin{pmatrix}Y & O\\ O & Y\end{pmatrix}\in P$, where $Y=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$. Then $Y^{-1}\begin{pmatrix}x & 0\\ z & u\end{pmatrix}Y=\begin{pmatrix}x & 0\\ z' & u\end{pmatrix}\in\Sigma$, where $z=a(x, 0)$, $u=b(x, 0)$ and $z'=(u-x)t+z$. By a property of the spread set, $z=z'$ and therefore $x=u=b(x, 0)$. Setting $h(x)=a(x, 0)$, $\begin{pmatrix}x & 0\\ h(x) & x\end{pmatrix}\in\Sigma$ for any $x\in K$. Since Σ contains a zero matrix and a unit matrix, $h(0)=h(1)=0$. Similarly $Y^{-1}\begin{pmatrix}0 & y\\ z & u\end{pmatrix}Y=\begin{pmatrix}yt & y\\ z' & u'\end{pmatrix}\in\Sigma$ for any $t\in K$ and $y\in K^\sharp$, where $z=a(0, y)$, $u=b(0, y)$, $z'=-t^2y+tu+z$ and $u'=-ty+u$. Hence $N(x, y)$ and $N(x', y)$ are P-conjugate, so that $tr(N(x, y))=tr(N(x', y))$ and $\det(N(x, y))=det(N(x', y))$ for any $x, x'\in K$ and $y\in K^\sharp$. Set $r(y)=tr(N(x, y))$ and $s(y)=-det(N(x, y))$, $y\neq 0$. Then $x+b(x, y)=r(y)$ and $ya(x, y)-xb(x, y)=s(y)$. Therefore $a(x, y)=-y^{-1}(x^2-r(y)x-s(y))$ and $b(x, y)=-x+r(y)$. Set $f(x, y)=a(x, y)$ and $g(x, y)=b(x, y)$ for $x\in K$ and $y\in K^\sharp$. Let $x\in K$ and $y, z\in K^\sharp$, $y\neq z$. Then $det(N(x, y)-N(x, z))\neq 0$ and $det(N(x, 0)-N(x, y))\neq 0$ as Σ is a spread set. Hence $f(x, y)\neq f(x, z)$ and $h(x)\neq f(x, y)$. Thus the triple (r, s, h) satisfies the conditions (1.3)–(1.5) and so $(r, s, h)\in\tilde{\Phi}_K$ and π is a coordinatized by $Q_{(r,s,h)}$.

Remark 2.5. Let π be a translation plane of order q^2 having $GF(q)$ in its kernel. By a similar argument as in the proof of the theorem above, the following holds: Let G be a linear autotopism group of π. Then a Sylow p-subgroup of G is of order at most q and fixes a Bear subplane pointwise.

As we have stated in §1, many classes of the planes are known which satisfy the assumption of Theorem 2.4 and any of these satisfies $h=0$.

(2.5.1) The Hall planes of order q^2 ([2]): $K=GF(q)$, $q=p^n$, p a prime, $r(y)=c$, $s(y)=d$, where c, $d\in K$ and x^2-cx-d is irreducible over K.

(2.5.2) The planes constructed by Narayana Rao and Satyanarayana ([7]): $K=GF(5^{2n+1})$, $r(y)=3y^{-1}$, $s(y)=-3y^{-2}$.

(2.3.5) The classes of planes constructed by S.D. Cohen and M.J. Ganley ([1]): (i) $K=GF(p^n)$, p an odd prime, $r(y)=0$, $s(y)=ky^{1-p^m}$, $k\notin K^2$, $0\leq m<n$. (If $m=0$, then (2.5.1) is obtained.)

(ii) $K=GF(q)$, $q\equiv-1 \pmod 6$, $r(y)=3y^{-1}$, $s(y)=-3y^{-2}$. (If $q=5^{2n+1}$, then (2.5.2) is obtained)

(iii) $K=GF(q)$, $q\equiv\pm 3 \pmod{10}$, $r(y)=5y^{-2}$, $s(y)=-5y^{-4}$.

Here we show that the case (ii) holds for $q=2^{2n+1}$.

Lemma 2.6. *Let $K=GF(q)$, $q\equiv-1 \pmod 3$ and set $r(y)=3y^{-1}$, $s(y)=-3y^{-2}$, $h(x)=0$ for $x\in K$, $y\in K^\sharp$. Then $(r, s)\in\Phi_K$.*

Proof. As $q\equiv-1 \pmod 3$, $f(a,y)=-y^{-1}(a^2-3y^{-1}a+3y^{-2})=-3((y^{-1}-a/3)^3+(a/3)^3)$. Hence $f(a, y)\neq 0$ for $a\in K$ and $y\in K^\sharp$ and $f(a, y)$ is a bijection from $K^\sharp$ onto $K^\sharp$. Thus $(r, s)\in\Phi_K$.

3. Some sufficient conditions for $h=0$

In this section we prove a theorem which gives a sufficient condition under which $h(x)=0$ for any x in the field. Throughout this section q is a power of a prime p.

Proposition 3.1. *Let π be a translation plane of order q^2 having $K=GF(q)$ in its kernel and let G be a linear autotopism group of π which fixes at least three points on l_∞. If $q\,\big|\,|G|$ and $q<|G^{l_\infty}|$, then π is coordinatized by a quasifield Q_g for some $g\in\Phi_K$. (Here G^{l_∞} denotes the restriction of G on l_∞.)*

Proof. Let A, B and C be distinct fixed points of G on l_∞. We coordinatize π with $A=\{(0, 0, v)\,|\,v\in K\times K\}$, $B=\{(v, 0, 0)\,|\,v\in K\times K\}$ and $C=\{(v, v)\,|\,v\in K\times K\}$. As we have seen in the proof of Theorem 2.4, we may assume that $P=\{\begin{pmatrix} Y & O\\ O & Y\end{pmatrix}\,|\, Y=\begin{pmatrix} 1 & 0\\ t & 1\end{pmatrix},\ t\in K\}\leq G\leq\{\begin{pmatrix} X & O\\ O & X\end{pmatrix}\,|\,X\in GL(2, q)\}$ and a spread set of π is $\Sigma_g=\{M(x, y)\,|\,x, y\in K\}$ for some $g=(r, s, h)\in\tilde{\Phi}_K$. We argue that $g\in\Phi_K$.

Suppose P is not a normal subgroup of G. Then, by a Dickson's theorem (cf. Chapter II of [5]), G contains $H=\{\begin{pmatrix} X & O \\ O & X\end{pmatrix} \mid X\in SL(2, q)\}$. Applying (2.4.3), $X^{-1}M(x, 0)X\in\Sigma_g$ for any $M(x, 0)\in\Sigma_g$ and $X\in SL(2, q)$. Since $\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix}^{-1}\begin{pmatrix} x & 0 \\ h(x) & x\end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix}=\begin{pmatrix} x & 0 \\ u^2h(x) & x\end{pmatrix}$, we have $u^2h(x)=h(x)$ for each $u\in K^{\sharp}$. Hence $h(x)=0$, $x\in K$, if $|K|>3$. If $|K|=2$, then $h(x)=0$ by the hypothesis (1.5). Assume $|K|=3$. Then r and s can be written in the form $r(y)=a+by$ and $s(y)=c+dy$ for some a, b, c and d in K. By the hypothesis (1.4), $h(x)=-(-(x^2-(a+b)x-(c+d))-((x^2-(a-b)x-(c-d))=bx+d$. Hence $b=d=0$ by the hypothesis (1.5).

Suppose P is a normal subgroup of G. Then $G\leq\{\begin{pmatrix} X & O \\ O & X\end{pmatrix} \mid X=\begin{pmatrix} a & 0 \\ b & c\end{pmatrix}$, a, $c\in K^{\sharp}$, $b\in K\}$. Let K_1 be the group of (O, l_∞)-homologies of π, where O denotes the origin of π. By assumption, $G\neq PK_1$. Let $\sigma\in G-PK_1$ and set $\sigma=\begin{pmatrix} D & O \\ O & D\end{pmatrix}$, where $D=\begin{pmatrix} a & 0 \\ b & c\end{pmatrix}$ and a, $c\in K^{\sharp}$, $b\in K$. By (2.4.3), $\sigma^{-1}M(x, 0)\sigma=\begin{pmatrix} x & 0 \\ ac^{-1}h(x) & x\end{pmatrix}\in\Sigma_g$. Hence $h(x)=ac^{-1}h(x)$ for any $x\in K$. If $ac^{-1}=1$, then $D=\begin{pmatrix} 1 & 0 \\ a^{-1}b & 1\end{pmatrix}\cdot a\in PK_1$, contrary to the choice of σ. Thus $ac^{-1}\neq 1$, so that $h(x)=0$ for all $x\in K$.

Lemma 3.2. *Let π be a translation plane of order q^2 having $K=GF(q)$ in its kernel and let G be a linear autotopism group of π. If $q \mid |G|$, then a Sylow p-subgroup of G is of order q and fixes a Baer subplane of π pointwise.*

Proof. The lemma follows immediately from Remark 2.5.

In the remainder of this section we assume the following.

HYPOTHESIS 3.3 (i) π is a translation plane coordinatized by a quasifield Q_g, where $g=(r, s, h)\in\tilde{\Phi}_K$ and $h\neq 0$.

(ii) π admits a linear autotopism group $G\leq(LC(\pi))_{L(0,0),L(\infty)}$ such that G contains a group of (O, l_∞)-homologies of order $q-1$ and has an orbit of length q^2-q on l_∞.

(iii) A Sylow p-subgroup P of G fixes $L(1, 0)\in\mathcal{L}_g$.

Lemma 3.4. *Let Γ be the set of fixed points of P on l_∞ and put $\Gamma'=\Gamma-\{L(0, 0), L(\infty)\}$. Then*

(i) $|\Gamma|=q+1$.

(ii) $|G|=q(q-1)^2$ *and P is a normal subgroup of G. Moreover $G^{\Gamma'}$ and $G^{l_\infty-\Gamma}$ are regular.*

Proof. By Lemma 3.2, $|P|=q$ and the fixed structure of P is a Baer subplane of π. Therefore (i) holds.

Let $C\in\Gamma'$ and assume $(G_C)^{l_\infty}\neq P^{l_\infty}$. Then, as we have seen in the proof of Proposition 3.1, $h(x)=0$ for any $x\in K$. Hence $(G_C)^{l_\infty}=P^{l_\infty}$ and so $G_C=PK_1$. Here K_1 is the group of (O, l_∞)-homologies of order $q-1$. Since $q^2-q\,\big|\,|G/K_1|$, $q(q-1)^2\,\big|\,|G|$. Hence $q-1\leq|G\colon G_C|$. Moreover $|G\colon G_C|\leq q-1$, for otherwise $|G\colon G_C|=q^2-q$, contrary to Lemma 3.2. From this $|G\colon G_C|=q-1$. In particular $|G|=q(q-1)^2$ and Γ' and $l_\infty-\Gamma$ are G-orbits. Hence $PK_1=G_{\Gamma'}$, the pointwise stabilizer of Γ' in G. Therefore P is a normal subgroup of G.

REMARK: By Hypothesis 3.3, $P=\{\begin{pmatrix} Y & O \\ O & Y\end{pmatrix}\mid Y=\begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix},\ t\in K\}$ and as P is normal in G, $G\leq\{\begin{pmatrix} A & O \\ O & B\end{pmatrix}\mid A=\begin{pmatrix} a & 0 \\ b & c\end{pmatrix},\ B=\begin{pmatrix} u & 0 \\ v & w\end{pmatrix},\ a, c, u, w\in K^\sharp,\ b, v\in K\}$.

Lemma 3.5. *Set* $W=\{\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in G\mid A=\begin{pmatrix} 1 & 0 \\ b & c\end{pmatrix},\ B=\begin{pmatrix} u & 0 \\ v & w\end{pmatrix},\ c, u, w\in K^\sharp,$ $b, v\in K\}$. *Then the following hold.*

(i) $G=K_1W$, *where K_1 is the group of (O, l_∞)-homologies of π.*

(ii) $K^\sharp=\{u\in K\mid\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in W,\ B=\begin{pmatrix} u & 0 \\ * & *\end{pmatrix}\}$.

(iii) $K^\sharp=\{w\in K\mid\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in W,\ B=\begin{pmatrix} * & 0 \\ * & w\end{pmatrix}\}$.

Let $\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in W$, *where* $A=\begin{pmatrix} 1 & 0 \\ b & c\end{pmatrix}$ *and* $B=\begin{pmatrix} u & 0 \\ v & w\end{pmatrix}$. *Then*

(iv) $w=uc$.

(v) $h(ux)=c^{-1}uh(x)+c^{-1}(v-bu)x$ *for any* $x\in K$.

(vi) $r(cuy)=ur(y)+(v-bu)y$ *and* $s(cuy)=u^2s(y)$ *for any* $y\in K^\sharp$.

Proof. Let g be a mapping from G into $K^\sharp$ such that $g((x_{ij}))=x_{11}$, where $(x_{ij})\in G$. As $|K_1|=q-1$, it follows from the remark above that g is a homomorphism and $g(G)=K^\sharp$. Hence $|G/W|=q-1$. Clearly $W\cap K_1=1$. Thus (i) holds.

Since $L(x, 0)\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in\Gamma'$ for $x\in K^\sharp$ and $\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in W$, $A^{-1}M(x, 0)B\in\Sigma$. Set $A=\begin{pmatrix} 1 & 0 \\ b & c\end{pmatrix}$ and $B=\begin{pmatrix} u & 0 \\ v & w\end{pmatrix}$. Then $A^{-1}\begin{pmatrix} x & 0 \\ h(x) & x\end{pmatrix}B=\begin{pmatrix} ux & 0 \\ t & c^{-1}wx\end{pmatrix}\in\Sigma$, where $t=c^{-1}(v-bu)x+c^{-1}uh(x)$. Hence $w=cu$ and $h(ux)=c^{-1}uh(x)+c^{-1}(v-bu)x$. Thus we have (iv) and (v). In particular $L(1, 0)G=L(1, 0)W=\{L(u, 0)\mid\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in G,\ B=\begin{pmatrix} u & 0 \\ * & *\end{pmatrix}\}$. By Lemma 3.4 (ii), we have (ii).

Similarly $L(0, y)\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in l_\infty-\Gamma$ for $y\in K^\sharp$ and $\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in G$. Hence A^{-1} $L(0, y)B\in l_\infty-\Gamma$. Set $A=\begin{pmatrix} 1 & 0 \\ b & c\end{pmatrix}$ and $B=\begin{pmatrix} u & 0 \\ v & uc\end{pmatrix}$. Then $A^{-1}\begin{pmatrix} 0 & y \\ y^{-1}s(y) & r(y)\end{pmatrix}$

$B=\begin{pmatrix} vy & ucy \\ t_1 & t_2 \end{pmatrix}$, where $t_1=c^{-1}(-bvy+uy^{-1}s(y)+vr(y))$ and $t_2=-buy+ur(y)$. Hence $L(0,y)\begin{pmatrix} A & O \\ O & B \end{pmatrix}=L(vy, ucy)$. Therefore $r(ucy)=vy+t_2=ur(y)+(v-bu)y$ and $s(ucy)=t_1ucy-t_2vy=u^2s(y)$ by Lemma 2.3. Thus we have (vi). Moreover, by Lemma 3.4 (ii), $l_\infty-\Gamma=L(0, 1)G=L(0, 1)W=\{L(v, uc)\mid\begin{pmatrix} A & O \\ O & B \end{pmatrix}\in W,\ A=\begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix},\ B=\begin{pmatrix} u & 0 \\ v & uc \end{pmatrix}\}$. Thus we have (iii).

Lemma 3.6. *Let $a, b\in K^\sharp$ and $c\in K$ with $K^\sharp=\langle a\rangle$. Let $g(x)$ be a mapping from $K^\sharp$ into K satisfying $g(ax)=bg(x)+cx$ for any $x\in K^\sharp$. Then we have*

(i) *If $c\neq 0$, then $a\neq b$ and $g(x)=kx^m+(c/a-b)x$, where $k\in K$ and m is an integer such that $a^m=b$, $0\leq m<q-1$, $m\neq 1$.*

(ii) *If $c=0$, then $g(x)=kx^m$, where $k\in K$ and m is an integer such that $a^m=b$, $0\leq m<q-1$.*

Proof. Any mapping from $K^\sharp$ into K is uniquely represented in the form $\sum_{n=0}^{q-2}p_nx^n$ for suitable p_n's in K, $0\leq n<q-1$. Set $g(x)=\sum_{n=0}^{q-2}p_nx^n$. Then $\sum_{n=0}^{q-2}p_na^nx^n=\sum_{n=0}^{q-2}p_nbx^n+cx$ for all $x\in K^\sharp$. Hence $p_n(a^n-b)=0$ if $n\neq 1$ and $p_1(a-b)=c$. Since $K^\sharp=\langle a\rangle$, there is a unique integer m satisfying $0\leq m<q-1$ and $b=a^m$.

Suppose $c\neq 0$. Then $a\neq b$, $p_1=c/(a-b)$ and $p_n=0$ for each $n\neq 1$, m. Thus (i) holds. Suppose $c=0$. Then $p_n=0$ for each $n\neq m$. Thus (ii) holds.

Lemma 3.7. (i) *$h(x)=ax^m-ax$, $r(y)=iy^d-ay$ and $s(y)=jy^{2d}$ for some $a, j\in K^\sharp$ and $i\in K$ and integers m, d, where $0\leq m$, $d<q-1$, $m\neq 1$, $d\neq 1$.*

(ii) *Let $\begin{pmatrix} A & O \\ O & B \end{pmatrix}\in W$, where $A=\begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix}$ and $\begin{pmatrix} u & 0 \\ v & uc \end{pmatrix}$. Then $c=u^{1-m}$ and $(2-m, q-1)=1$. Moreover $iu^{(2-m)d}=iu$, $u^{2(2-m)d}=u^2$ and $j\neq 0$.*

Proof. By Lemmas 3.5 and 3.6, $h(x)=ax^m+a'x$, $r(y)=iy^d+ly$ and $s(y)=jy^{e'}$ for suitable $a, a', i, j, l\in K$ and integers m, d, e', where $0\leq m, d, e'<q-1$, $m\neq 1$, $d\neq 1$. Moreover, $(cu)^d=u$ and $(cu)^{e'}=u^2$ if $K^\sharp=\langle cu\rangle$. Hence $(cu)^{2d-e'}=1$ and so $2d\equiv e'$ (mod $q-1$). From this $s(y)=jy^{2d}$. By the hypotheses (1.4) and (1.5), $h(1)=a+a'=0$, $s(y)\neq 0$. Therefore $h(x)=ax^m-ax$ and $j\neq 0$. We have $a\neq 0$, for otherwise $h(x)=0$, $x\in K$, a contradiction.

Using Lemma 3.5 (v) (vi),

$$au^mx^m-aux = c^{-1}u(ax^m-ax)+c^{-1}(v-bu)x\,, \tag{3.1}$$

$$i(cu)^dy^d+lcuy = iuy^d+luy+(v-bu)y \tag{3.2}$$

and

$$j(cu)^{2d}y^{2d} = u^2jy^{2d} \tag{3.3}$$

for any $x, y\in K^\sharp$.

From (3.1), $c=u^{1-m}$ and $v-bu=ua(1-c)$. (3.4)

Since $cu=u^{2-m}$, $(2-m, q-1)=1$ by Lemma 3.5 (iii), (iv). Substituting (3.4) into (3.2) and (3.3) we have $i(u^{(2-m)d}-u)=0$, $u^{2-m}(u^{m-1}-1)(a+l)=0$ and $j(u^{2(2-m)d}-u^2)=0$. As $m\neq 1$ and $j\notin 0$, $a+l=0$ and $u^{2(2-m)d}-u^2=0$. Thus the lemma holds.

Lemma 3.8. *Set $g(x)=-x^{m-2}+ix^{m-1}+jx^m$. Then $g(x)$ is an injection from $K^\sharp$ into K and $a\notin g(K^\sharp)$.*

Proof. By the hypotheses (1.3) and (1.4), a mapping $f_1(y)=-y^{-1}(v^2-r(y)v-s(y))-h(v)$ is an injection from $K^\sharp$ into itself for a fixed $v\in K$. Let $v=1$. It follows from Lemma 3.7 (i) that $f_1(y)=-y^{-1}(1-iy^d-jy^{2d})-a$. Set $f_2(t)=f_1(t^{2-m})$. Then, by Lemma 3.7 (ii), $f_2(t)$ is injective and moreover $it^{(2-m)d}=it$, $t^{2(2-m)d}=t^2$. Hence $f_2(t)=-t^{m-2}+it^{m-1}+jt^m-a$ and $f_2(t)$ is an injection from $K^\sharp$ into itself. Thus the lemma holds.

Lemma 3.9. (i) $m\neq 0$.
(ii) *$i\neq 0$ and $j=-i^2/4$ if $2\nmid q$ and $i=0$ if $2\mid q$.*

Proof. Assume $m=0$. Then $g(x)=-(1/x)^2+i(1/x)+j$ and $2\mid q$ by Lemma 3.7 (ii). If $q>2$, then $g(x)=g(x/(ix-1))$ and $x\neq x/(ix-1)$ for $x\in K-\{0, 1/i\}$. Hence $q=2$ and therefore $h(x)=0$ for $x\in K=\{0, 1\}$, a contradiction. Thus $m\neq 0$.

If the quadratic polynomial jx^2+ix-1 is irreducible over K, then $0\notin g(K^\sharp)$. This implies $a=0$ by Lemma 3.8, contrary to Lemma 3.7 (i). Hence $jx^2+ix-1=j(x-b)(x-b')$ for some $b, b'\in K^\sharp$. Then $g(b)=g(b')$. By Lemma 3.8, $b=b'$. Therefore we have (ii).

Lemma 3.10. *There exists no $(r, s, h)\notin \Phi_K$ satisfying Hypothesis 3.3.*

Proof. By Lemmas 3.8 and 3.9, $g(x)=-(1/4)x^{m-2}(ix-2)^2$ or $g(x)=x^{m-2}(bx+1)^2$, $b^2=j$ according as $2\nmid q$ or $2\mid q$, respectively. Set $K^\sharp=\langle c\rangle$.

Assume $2\nmid q$ and set $x_1=2c^2(c^{m-2}-1)/i(c^m-1)$ and $x_2=2(c^{m-2}-1)/i(c^m-1)$. Then $g(x_1)=g(x_2)$. Moreover $x_1, x_2\in K^\sharp$ and $x_1\neq x_2$ as $(m-2, q-1)=1$ and $m\neq 0$. Hence $g(x)$ is not injective, contrary to Lemma 3.8.

Assume $2\mid q$ and set $x_1=c^2(c^{m-2}+1)/b(c^m+1)$ and $x_2=(c^{m-2}+1)/b(c^m+1)$. Then $g(x_1)=g(x_2)$ and $x_1, x_2\in K^\sharp$, $x_1\neq x_2$ as $(m-2, q-1)=1$ and $m\neq 0$. This is also a contradiction.

We now prove the following theorem.

Theorem 3.11. *Let π be a translation plane of order q^2 having $K=GF(q)$ in its kernel. If π admits a linear autotopism group G having an orbit of length q^2-q on l_∞, then π is coordinatized by a quasifield Q_g for some $g=(r, s)\in\Phi_K$,* *where* $r(y)=ay^n$, $s(y)=by^{2n}$ *for suitable* a, $b\in K$ *and an integer* n, $0\leq n<q-1$.

Proof. Let A and B be distinct fixed points of G on l_∞ and C a fixed point of a Sylow p-subgroup P of G such that $C\in l_\infty-\{A, B\}$. Set $A=\{(0, 0, v)\mid v\in K\times K\}$, $B=\{(v, 0, 0)\mid v\in K\times K\}$ and $C=\{(v, v)\mid v\in K\times K\}$. Then π is coordinatized by Q_g for some $g=(r, s, h)\in\tilde{\Phi}_K$ by Theorem 2.4. Assume $h(x)\neq 0$ for an $x\in K$. Then π satisfies Hypothesis 3.3. Applying Lemma 3.10, we obtain a contradiction. Therefore $h(x)=0$ for any $x\in K$. The theorem follows from Theorem 2 of [3].

4. Some properties of $(r, s)\in\Phi_K$

Let $K=GF(q)$, where q is a power of a prime p. Let $g=(r, s)\in\Phi_K$. As we have defined in §2, $\Sigma_g=\{\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix}\mid x\in K\}\cup\{\begin{pmatrix} x & y \\ f(x,y) & g(x,y)\end{pmatrix}\mid x, y\in K, y\neq 0\}$, where $f(x, y)=-y^{-1}(x^2-r(y)x-s(y))$ and $g(x, y)=-x+r(y)$. Set $\Sigma=\Sigma_g$. In this section we list several lemmas which will be required in the sequel.

Lemma 4.1. *Let* $P, M\in M_2(K)$, $\det(P)\neq 0$ *and set* $P^{-1}MP=\begin{pmatrix} * & y \\ * & *\end{pmatrix}$. *Assume* $y\neq 0$. *Then* $P^{-1}MP\in\Sigma$ *if and only if* $r(y)=tr(P^{-1}MP)$ *and* $s(y)=-\det(P^{-1}MP)$.

Proof. See Lemma 3.1 of [3].

Lemma 4.2. *Let* P, $Q\in M_2(K)$, $|K|>3$. *If* $P+xQ\in\Sigma$ *for any* $x\in K$, *then either* (i) Q *is a zero matrix and* $P\in\Sigma$ *or* (ii) P *and* Q *are scalar matrices.*

Proof. See Lemma 3.3 of [3].

Lemma 4.3. (i) $s(y)\neq 0$ *for* $y\in K^\sharp$.
(ii) *If* $2\mid q$, *then* $r(y)\neq 0$ *for* $y\in K^\sharp$.

Proof. Since $(r, s)\in\Phi_K$, $f(x, y)=-y^{-1}(x^2-r(y)x-s(y))\neq 0$ for all $x\in K$ and $y\in K^\sharp$. Hence (i) holds.

Assume $2\mid q$ and $r(y)=0$ for some $y\in K^\sharp$. Then $f(x, y)=-y^{-1}(x-w)^2$, where w is a unique element of K such that $w^2=s(y)$. Thus $f(w, y)=0$, a contradiction.

Let π_g and $\mathcal{L}_g$ be as defined in §2. Set $\Delta_g=\{L(x, 0)\mid x\in K\}\cup\{L(\infty)\}$ and $\Omega_g=\mathcal{L}_g-\Delta_g$.

Lemma 4.4. *Assume that either* r *or* s *is not a constant function. Let* $\sigma\in LC(\pi_g)$. *Then* σ *fixes* $L(\infty)$ *if and only if* σ *is of the form* $\begin{pmatrix} A & iA \\ O & jA\end{pmatrix}$, $A=\begin{pmatrix} a & 0 \\ c & d\end{pmatrix}$, i, $c\in K$, j, a, $d\in K^\sharp$, *where* $r(a^{-1}djy)=jr(y)+2i$ *and* $s(a^{-1}djy)=j^2s(y)-ijr(y)-i^2$ *for all* $y\in K^\sharp$. *Under the condition,* $L(x, y)\sigma=L(u, v)$, *where* $u=i+jx+a^{-1}cjy$

and $v=a^{-1}djy$.

Proof. Using (1.3) and Lemma 4.3, we can easily verify that $\Phi_{GF(2)}=\{(1, 1)\}$ and $\Phi_{GF(3)}=\{(0, 2), (1, 1), (2, 1)\}$. Therefore r and s are constant functions for any $(r, s)\in\Phi_K$ when $|K|\leq 3$. Hence $|K|>3$.

Let $\sigma\in(LC(\pi_g))_{L(\infty)}$. Since σ fixes $L(\infty)$, σ can be written in the form $\begin{pmatrix} A & B \\ O & C\end{pmatrix}$ for some $A, C\in GL(2, q)$ and $B\in M_2(K)$. Then, by (2.4.2) and (2.4.3), $\begin{pmatrix} A & B \\ O & C\end{pmatrix}\in LC(\pi_g)$ if and only if $A^{-1}(B+MC)=A^{-1}B+A^{-1}MC\in\Sigma$ for all $M\in\Sigma$. Applying Lemma 4.2, $A^{-1}B=i$ and $A^{-1}C=j$ for some $i, j\in K$, $j\neq 0$. Hence, for any $x, y\in K$, there exist $u, v\in K$ such that

$$A^{-1}(i+jM(x, y))A = M(u, v)\,. \tag{4.1}$$

In particular $L(x, 0)\sigma=L(i+jx, 0)$.

Assume $y\neq 0$. Then $v\neq 0$ by (4.1). Set $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $a, b, c, d\in K$. By (4.1), $\begin{pmatrix} i+jx & jy \\ jf & i+jg\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} u & v \\ f' & g'\end{pmatrix}$, where $f=f(x, y)$, $g=g(x, y)$, $f'=f(u, v)$ and $g'=g(u, v)$. From this, we have

$$b(i+jx)+d(jy) = av+bg'\,, \tag{4.2}$$

$$b(jf)+d(i+jg) = cv+dg' \tag{4.3}$$

and
$$a(i+jx)+cjy = au+bf'\,. \tag{4.4}$$

(4.2) and (4.3) yield $(ad-bc)v=F(x, y)$, where $F(x, y)=(b^2jy^{-1})x^2+(2bdj-b^2jy^{-1}r(y))x+(d^2jy-bdjr(y)-b^2jy^{-1}s(y))$. (4.5)

By Lemma 4.1, $r(v)=2i+jr(y)$ (4.6)

and $s(v)=-i^2-ijr(y)+j^2s(y)$. (4.7)

Set $\Psi_y=\{v\in K^{\#}\,|\,r(v)=2i+jr(y)\}$ for $y\in K^{\#}$. Assume $b\neq 0$. Then, by (4.5), $|\Psi_y|\geq(q+1)/2$ when $p>2$ and $|\Psi_y|\geq q/2$ when $p=2$. Hence $\Psi_y\cap\Psi_z\neq\phi$, $y, z\in K^{\#}$. This implies $\Psi_y=K^{\#}$ for $y\in K^{\#}$. Thus $r(y)$ is a constant function. Similarly, using (4.7), it can be shown that $s(x)$ is also a constant function. But this contradicts the assumption. Therefore $b=0$. In particular $ad\neq 0$ and so by (4.2) and (4.4),

$$u = i+jx+a^{-1}cjy,\ v = a^{-1}djy \tag{4.8}$$

and $L(x, y)\sigma=L(u, v)$ when $y\neq 0$.

Conversely, assume $\sigma=\begin{pmatrix} A & iA \\ O & jA\end{pmatrix}$, $A=\begin{pmatrix} a & 0 \\ c & d\end{pmatrix}$, $i, c\in K$, $j, a, d\in K^{\#}$ and assume (4.6)–(4.8). Then $A^{-1}(i+jM(x, y))A=a^{-1}d^{-1}\begin{pmatrix} d & 0 \\ -c & a\end{pmatrix}\begin{pmatrix} i+jx & jy \\ * & *\end{pmatrix}\begin{pmatrix} a & 0 \\ c & d\end{pmatrix}=\begin{pmatrix} u & u \\ * & *\end{pmatrix}$.

By (4.6)–(4.8), $tr(A^{-1}(i+jM(x,y))A)=2i+jr(y)=r(v)$ and $\det(A^{-1}(i+jM(x,y))A)=i^2+ijr(y)-j^2s(y)=-s(v)$ as $tr(M(x,y))=r(y)$ and $\det(M(x,y))=-s(y)$. Applying Lemma 4.1, $A^{-1}(i+jM(x,y))A=M(u,v)\in\Sigma$. Thus $\sigma\in(LC(\pi_g))_{L(\infty)}$.

5. The linear translation complement of π_g, $\pi_g\in\Pi_K$

In this section we continue the notations of the previous section. Let Π_K denote the set of the planes π_g $(g\in\tilde{\Phi}_K)$ with a linear autotopism group acting transitively on Ω_g. Let $\pi_g\in\Pi_K$, $g=(r,s,h)\in\tilde{\Phi}_K$. Then, by Theorem 3.11, $h=0$, $r(y)=ky^n$ and $s(y)=ly^{2n}$ for some $k,l\in K$ and an integer n, $0\leq n<q-1$. Set $\pi=\pi_g$, $\Sigma=\Sigma_g$, $\Delta=\Delta_g$ and $\Omega=\Omega_g$.

Lemma 5.1. *Let $r_0(y)=iy^m$ and $s_0(y)=jy^{2m}$, where $i,j\in K$ and m is an integer, $0\leq m<q-1$.*

(i) *Assume $2\not| q$. Then $(r_0,s_0)\in\Phi_K$ if and only if*

 (a) *$i^2+4j\notin K^2$ and*

 (b) *$i^2(t^m-t)^2+4j(1-t)(t^{2m}-t)\notin K^2$ for all $t\in K-\{0,1\}$.*

(ii) *If $(r_0,s_0)\in\Phi_K$, then $i^2+4j\neq 0$.*

Proof. Assume $2\not| q$. By Lemma 3.2 (ii) of [3], $(r_0,s_0)\in\Phi_K$ if and only if (1) $(i^2+4j)y^{2m}\notin K^2$ and (2) $i^2(xy^m-yx^m)^2+4(x-y)\,j(xy^{2m}-yx^{2m})\notin K^2$ for any $x,y\in K^\sharp$, $x\neq y$. Set $t=y/x$. Then $t\neq 0$, 1 and $(i^2(t^m-t)^2+4j(1-t)(t^{2m}-t))x^{2m+2}\notin K^2$. Thus (i) holds and (ii) follows immediately from (i) and Lemma 4.3 (ii).

Lemma 5.2. *Set $G=LC(\pi)$ and $H=G_{L(\infty),L(0,0)}$. If either r or s is not a constant function, then $G_{L(\infty)}=H$.*

Proof. Let $\sigma\in G_{L(\infty)}$. By Lemma 4.4, $\sigma=\begin{pmatrix}A & iA\\ O & jA\end{pmatrix}$ and $A=\begin{pmatrix}a & 0\\ c & d\end{pmatrix}$ for suitable $i,c\in K$ and $j,a,d\in K^\sharp$ and we have

$$k(a^{-1}djy)^n=jky^n+2i \tag{5.1}$$

$$l(a^{-1}djy)^{2n}=j^2ly^{2n}-ijky^n-i^2 \text{ for any } y\in K^\sharp. \tag{5.2}$$

As $0<n<q-1$, the equation (5.1) yields

$$kj(a^{-n}d^nj^{n-1}-1)=0,\ \ 2i=0 \tag{5.3}$$

By (5.2), $(l(a^{-1}dj)^{2n}-j^2l)y^{2n}+ijky^n+i^2=0$ for any $y\in K^\sharp$. There exists n', $0\leq n'<q-1$, such that $y^{2n}=y^{n'}$ for $y\in K^\sharp$. Hence

$$(l(a^{-1}dj)^{2n}-j^2l)y^{n'}+ijky^n+i^2=0 \text{ for any } y\in K^\sharp. \tag{5.4}$$

Assume $i\neq 0$. Then $p=2$ by (5.3). In particular $|K^\sharp|$ is odd and therefore $n'\equiv 2n\not\equiv 0 \pmod{q-1}$. Hence $i^2=0$ by (5.4), a contradiction. Thus $i=0$ and so $\sigma\in H$.

Lemma 5.3. *Set* $N=\{a\begin{pmatrix} A & O \\ O & c^n A\end{pmatrix} \mid A=\begin{pmatrix} 1 & 0 \\ b & c^{1-n}\end{pmatrix},\ a,\ c\in K^\#,\ b\in K\}$ *and* $\tau=\begin{pmatrix} J & O \\ O & -J\end{pmatrix}$, $J=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. *Then* $H=N$ *if* $k\neq 0$ *and* $H=\langle\tau\rangle N$ *if* $k=0$.

Proof. If r and s are constant functions, π is a Hall plane. Hence we may assume that either r or s is not a constant function. Let $\sigma\in H$. Applying Lemmas 4.4 and 5.2, $\sigma=\begin{pmatrix} A & O \\ O & jA\end{pmatrix}$, $A=\begin{pmatrix} a & 0 \\ c & d\end{pmatrix}$ for some a, d, $j\in K^\#$ and $c\in K$, where $r(a^{-1}djy)=jr(y)$ and $s(a^{-1}djy)=j^2s(y)$ for all $y\in K^\#$. Hence $\sigma\in H$ if and only if $k(a^{-1}djy)^n=jky^n$ and $l(a^{-1}djy)^{2n}=j^2ly^{2n}$ for all $y\in K^\#$. These are equivalent to

$$k((a^{-1}dj)^n-j) = 0 \tag{5.5}$$

and $$l((a^{-1}dj)^{2n}-j^2) = 0, \text{ respectively.} \tag{5.6}$$

It follows from (5.6) and Lemma 4.3 (i) that $l\neq 0$ and therefore $j=\pm(a^{-1}dj)^n$.

Assume $j=(a^{-1}dj)^n$ and set $a^{-1}dj=c_1$. Then $j=c_1^n$ and $d=ac_1^{1-n}$. Hence $\sigma=a\begin{pmatrix} A & O \\ O & c_1^n A\end{pmatrix}$, $A=\begin{pmatrix} 1 & 0 \\ b_1 & c_1^{1-n}\end{pmatrix}$, where $b_1=c/a$. Therefore $\sigma\in N$.

Assume $j\neq(a^{-1}dj)^n$. Then $k=0$ by (5.5) and so it is not difficult to verify that $\tau\in H$. Set $\sigma'=\tau\sigma$. Then $\sigma'=\begin{pmatrix} A_1 & O \\ O & j_1A_1\end{pmatrix}$, $A_1=\begin{pmatrix} a_1 & 0 \\ c_1 & d_1\end{pmatrix}$, where $j_1=-j$, $a_1=a$, $c_1=c$ and $d_1=-d$. Hence $j_1=(a_1^{-1}d_1j_1)^n$ as $j=-(a^{-1}dj)^n$. By a similar argument as in the previous paragraph, we have $\sigma'\in N$. Therefore $\sigma\in\langle\tau\rangle N$. Thus we have the lemma.

Lemma 5.4. *Let* K_1 *be a subgroup of the multiplicative group* $K^\#$ *of index* $(n,\ q-1)$ *and set* $K_2=\langle -1,\ K_1\rangle$. *Assume either* r *or* s *is not a constant function. Then, for each* $w\in K^\#$,

$$L(w,\ 0)H = \{L(x,\ 0)\mid x\in K_1w\} \text{ if } k\neq 0 \text{ and}$$
$$L(w,\ 0)H = \{L(x,\ 0)\mid x\in K_2w\} \text{ if } k=0\,.$$

Proof. Applying Lemmas 4.4, 5.2, and (5.3), $L(w,\ 0)H=\{L(wc^n,\ 0)\mid c\in K^\#\}$ if $k\neq 0$ and $L(w,\ 0)H=\{L(\pm wc^n,\ 0)\mid c\in K^\#\}$ if $k=0$. Since $\{c^n\mid c\in K^\#\}$ is a subgroup of $K^\#$ of index $(n,\ q-1)$, the lemma holds.

Lemma 5.5. *Assume there exists an element* $\sigma\in G$ *which exchanges* $L(\infty)$ *for* $L(0,\ 0)$. *Then one of the following occurs.*

(i) *Both* r *and* s *are constant functions.*

(ii) $2n(n-1)\equiv 0 \pmod{q-1}$, $k\neq 0$, $\sigma=\begin{pmatrix} O & A \\ wA & O\end{pmatrix}$, $A=\begin{pmatrix} a & 0 \\ c & d\end{pmatrix}$ *and* $(a^{-1}d)^n=-(wl)^{n-1}$ *for some* w, a, $d\in K^\#$ *and* $c\in K$.

(iii) $4n(n-1)\equiv 0 \pmod{q-1}$, $k=0$, $\sigma=\begin{pmatrix} O & A \\ wA & O\end{pmatrix}$, $A=\begin{pmatrix} a & 0 \\ c & d\end{pmatrix}$ *and* $(a^{-1}d)^{2n}$

$=(wl)^{2n-2}$ *for some* w, a, $d\in K^\#$ *and* $c\in K$.

Proof. As we have seen in the proof of Lemma 4.4, if $|K|\leq 3$, then (i) holds. Therefore we may assume $|K|>3$. Since σ exchanges $L(\infty)$ for $L(0, 0)$, there are A, $B\in GL(2, K)$ such that $\sigma=\begin{pmatrix} O & A \\ B & O \end{pmatrix}$. Moreover, by (2.4.3), $B^{-1}M^{-1}A\in\Sigma$ for any $M\in\Sigma$, $M\neq\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Applying Lemma 4.2, $B^{-1}A=w^{-1}$ for some $w\in K^\#$. Hence $B=wA$ so $\sigma=\begin{pmatrix} O & A \\ wA & O \end{pmatrix}$ and $w^{-1}A^{-1}M^{-1}A\in\Sigma$ for any $M\in\Sigma$, $M\neq\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Set $M=M(x, 1)$ and $M(u, v)=w^{-1}A^{-1}M^{-1}A$, $v\neq 0$. Then $M=\begin{pmatrix} x & 1 \\ f & g \end{pmatrix}$, where $f=-(x^2-kx-l)$ and $g=-x+k$. Hence $M(u, v)=w^{-1}(\det(A))^{-1}(\det(M))^{-1}$ $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} g & -1 \\ -f & x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=(1/t)\begin{pmatrix} dg+bf & -d-bx \\ * & * \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $t=-wl(ad-bc)$. Hence we have $v=(1/t)\,(-b^2x^2+(kb^2-2bd)x+kbd-d^2+b^2l)$. By Lemma 4.1, $r(v)=tr(M(u, v))=tr(w^{-1}M^{-1})=w^{-1}(tr(M))/\det(M)=k/(-wl)$.

Assume $b\neq 0$. Set $v=\rho(x)$ and $\Psi=\{\rho(x)\,|\,x\in K\}$. Then $|\Psi|\geq(q+1)/2$ when $p>2$ and $|\Psi|\geq q/2$ when $p=2$. Hence $\{-k/wl\}=\{r(y)\,|\,y=\rho(x),\ x\in K\}$ $=\{kv^n\,|\,v\in\Psi\}$. Since $0\leq n<q-1$, either $n=0$ or $k=0$. Assume $n\neq 0$. Then $k=0$. But $s(v)=-det(M(u, v))=-det(w^{-1}M^{-1})=-w^{-2}/det(M)=w^{-2}l^{-1}$. Hence $\{w^{-2}l^{-1}\}=\{s(y)\,|\,y=\rho(x),\ x\in K\}=\{lv^{2n}\,|\,v\in\Psi\}$. From this, $2n=q-1$ and (i) holds.

Assume $b=0$ and deny (i). Then, for $x\in K$ and $y\in K^\#$, $w^{-1}A^{-1}M(x,y)^{-1}A=$ $=(-1/adws(y))\begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}\begin{pmatrix} g & -y \\ -f & x \end{pmatrix}\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}=(-1/adlwy^{2n})\begin{pmatrix} dg & -dy \\ * & * \end{pmatrix}\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}=\begin{pmatrix} * & v \\ * & * \end{pmatrix}$, where $v=(d/alw)y^{1-2n}$. By Lemma 4.1, $r(v)=w^{-1}tr(M(x, y)^{-1})=r(y)/(-ws(y))$ $=(-k/lw)y^{-n}$ and $s(v)=-w^{-2}\det(M(x, y)^{-1})=(1/w^2s(y))=(1/lw^2)y^{-2n}$. Hence $k((d/alw)y^{1-2n})^n=(-k/lw)y^{-n}$ and $l((d/alw)y^{1-2n})^{2n}=(1/lw^2)y^{-2n}$ for all $y\in K^\#$. Thus, for any $y\in K^\#$, we have

$$k((1/lw)y^{2n(n-1)}+(d/alw)^n)=0 \quad\text{and}$$
$$(1/lw^2)y^{4n(n-1)}-l(d/alw)^{2n}=0\,.$$

If $k\neq 0$, then $2n(n-1)\equiv 0 \pmod{q-1}$ and $1/lw+(d/alw)^n=0$. If $k=0$, then $4n(n-1)\equiv 0 \pmod{q-1}$ and $1/lw^2-l(d/alw)^{2n}=0$. Therefore (ii) or (iii) holds. Thus we have the lemma.

Lemma 5.6. *Assume either r or s is not a constant function. Then* $L(\infty)$ $G_{L(0,0)}\cap\Delta=\{L(\infty)\}$.

Proof. Suppose false and let σ be an element of G such that $L(0, 0)\sigma$ $=L(0, 0)$ and $L(\infty)\sigma=L(h, 0)$ for some $h\in K^\#$. Then $\sigma=\begin{pmatrix} A & O \\ B & hB \end{pmatrix}$ for some

$A, B\in GL(2, K)$ and so $\sigma^{-1}=\begin{pmatrix} A^{-1} & O \\ -h^{-1}A^{-1} & h^{-1}B^{-1}\end{pmatrix}$. Hence, by (2.4.3), $(A^{-1}-Mh^{-1}A^{-1})^{-1}Mh^{-1}B^{-1}\in\Sigma$ for any $M\in\Sigma$ such that $\det(A^{-1}-Mh^{-1}A^{-1})\neq 0$. Set $M=M(x, 0)$, $x\neq h$. Then $(xh^{-1}/(1-xh^{-1}))AB^{-1}\in\Sigma$ for any $x\in K$, $x\neq h$. This implies $tAB^{-1}\in\Sigma$ for any $t\in K$, $t\neq -1$. By (2.4.1), $-AB^{-1}=(-h^{-1}A^{-1})^{-1}h^{-1}B^{-1}\in\Sigma$. Thus $tAB^{-1}\in\Sigma$ for any $t\in K$. Applying Lemma 4.2, $AB^{-1}=g^{-1}$ for some $g\in K^{\#}$ so $\sigma=\begin{pmatrix} A & O \\ gA & ghA\end{pmatrix}$ and $A^{-1}(E+gM)^{-1}ghMA\in\Sigma$ for any $M\neq -g^{-1}E$.

Set $M(u, v)=A^{-1}(E+gM(x, y))^{-1}ghM(x, y)A$ and Assume $y\neq 0$. Then $v\neq 0$. Hence $tr(M(u, v))=ght_1/(1+gt_1)+ght_2/(1+gt_2)$ and $\det(M(u, v))=(ght_1/(1+gt_1))\,(ght_2/(1+gt_2))$, where $t_1+t_2=tr(M(x, y))$ and $t_1t_2=\det(M(x, y))$. By Lemma 2.3 (ii), $kv^n=(ghky^n-2g^2hly^{2n})/(1+gky^n-g^2ly^{2n})$ and $-lv^{2n}=-g^2h^2ly^{2n}/(1+gky^n-g^2ly^{2n})$. Therefore $k^2/(-l)=k^2v^{2n}/(-lv^{2n})=(k-2gly^n)^2/(-l)\,(1+gky^n-g^2ly^{2n})$, so we have $(k^2+4l)\,(gly^n-k)gy^n=0$ for $y\in K^{\#}$. By Lemma 5.1 (ii), $k^2+4l\neq 0$. Thus $n=0$ and $gl-k=0$. Then r and s are constant functions, contrary to the hypothesis.

Lemma 5.7. *Assume* $|K|>3$ *and let* $\sigma\in G$. *Then* $L(0, 0)\sigma=L(\infty)$ *if and only if* $L(\infty)\sigma=L(0, 0)$.

Proof. Assume $L(0, 0)\sigma=L(\infty)$. Then $\sigma=\begin{pmatrix} O & A \\ B & C\end{pmatrix}$ for some $A, B\in GL(2, K)$ and $C\in M_2(K)$. By (2.4.3), $(MB)^{-1}(A+MC)\in\Sigma$ for any $M\in\Sigma$, $M\neq\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$. Set $M=M(x, 0)$, $x\in K^{\#}$. Then $B^{-1}Ax^{-1}+B^{-1}C\in\Sigma$ for all $x\in K$, $x\neq 0$. On the other hand $B^{-1}C\in\Sigma$ by (2.4.1). Therefore $B^{-1}C+B^{-1}Ax\in\Sigma$ for all $x\in K$. Applying Lemma 4.2, $B^{-1}C=i$ and $B^{-1}A=j$ for some $i\in K$ and $j\in K^{\#}$. Hence $\sigma=\begin{pmatrix} O & jB \\ B & iB\end{pmatrix}$ and $B^{-1}\,(iE+jM^{-1})B\in\Sigma$ for any $M\in\Sigma$, $M\neq\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$.

Set $M(u, v)=B^{-1}(iE+jM(x, y)^{-1})B$ and assume $y\neq 0$. Clearly $v\neq 0$. Moreover $tr(M(u, v))=2i+jtr(M(x, y))/\det(M(x, y))=2i-kj/ly^n$ and $\det(M(u, v))=i^2+ijtr(M(x, y))/\det(M(x, y))+j^2/\det(M(x, y))=i^2-ijk/ly^n-j^2/ly^{2n}$. On the other hand $kv^n=2i-kj/ly^n$ and $-lv^{2n}=i^2-kij/ly^n-j^2/ly^{2n}$. Hence $k^2/(-l)=(kv^n)^2/(-lv^{2n})=(2ily^n-kj)^2/(-l)\,(j^2+ijky^n-i^2ly^{2n})$ so we have $i(k^2+4l)\,(ily^n-kj)y^n=0$ for any $y\in K^{\#}$. By Lemma 5.1 (ii), $k^2+4l\neq 0$ and by Lemma 4.3 (i), $l\neq 0$. Therefore $i=0$, whence $L(\infty)\sigma=L(0, 0)$.

Conversely, assume $L(\infty)\sigma=L(0, 0)$. Then $\sigma^{-1}\in G$ and $L(0, 0)\sigma^{-1}=L(\infty)$. By the result as above, we have $L(\infty)\sigma^{-1}=L(0, 0)$, which implies $L(0, 0)\sigma=L(\infty)$. Thus the lemma holds.

Lemma 5.8. *If* $|K|>3$, *then* G *is not transitive on* $\Delta\cup\Omega$.

Proof. Suppose false. If r and s are constant functions, then π is a Hall plane so the Lemma follows from Theorem 13.10 of [6]. Therefore we may assume that either r or s is not a constant function. Applying Lemma 5.2, $G_{L(\infty)}=H$ and $|G: H|=|\Delta\cup\Omega|=q^2+1$.

Let P be a Sylow p-subgroup of H. By Lemma 5.3, $P=\{\begin{pmatrix}A & O\\ O & A\end{pmatrix}|A=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}, t\in K\}$ and so P fixes Δ pointwise and is semiregular on Ω. By a Witt's theorem, $N_G(P)$ is transitive on Δ. In particular, $q+1$ divides $|N_G(P): N_H(P)|$. By Lemma 5.3, P is a normal subgroup of H and so $N_H(P)=H$. Hence $q+1$ divides $|G: H|=q^2+1$, while $(q+1, q^2+1)=(q+1, 2)\leq 2$. This is a contradiction. Thus the lemma holds.

Lemma 5.9. *Let $a, b, c, d, e\in K$ such that $(a, b, c, d, e)\neq(0, 0, 0, 0, 0)$. If $ay^{4n}+by^{3n}+cy^{2n}+dy^n+e=0$ for all $y\in K^\sharp$, then one of the following occurs.*

(i) *$n=0$ and $a+b+c+d+e=0$.*

(ii) *$n=(q-1)/2$ and $a+c+e=b+d=0$.*

(iii) *$n=(q-1)/3$ or $2(q-1)/3$ and $a+d=b+e=c=0$.*

(iv) *$n=(q-1)/4$ or $3(q-1)/4$ and $a+e=b=c=d=0$.*

Proof. There exist u, v and w with $0\leq u, v, w<q-1$ such that $y^{4n}=y^u$, $y^{3n}=y^v$ and $y^{2n}=y^w$ for any $y\in K^\sharp$. If u, v, w, n and 0 are all distinct, then the equation $ay^u+by^v+cy^w+dy^n+e=0$ has at most $q-2$ solutions for y, contrary to the assumption. Hence $in\equiv i'n \pmod{q-1}$ for some integers i, i', $0\leq i<i'\leq 4$. From this, $jn\equiv 0 \pmod{q-1}$ for some integer j, $1\leq j\leq 4$.

Assume $n\equiv 0 \pmod{q-1}$. Then $n=0$ as $0\leq n\leq q-2$ and so we have (i).

Assume $2n\equiv 0 \pmod{q-1}$. Then $n=(q-1)/2$ as $0\leq 2n\leq 2q-4$ and so the equation is equivalent to $(b+d)y^n+a+c+e=0$. Thus we have (ii) in this case.

Similarly we have (iii) or (iv) according as $3n\equiv 0 \pmod{q-1}$ or $2n\not\equiv 0$, $4n\equiv 0 \pmod{q-1}$, respectively.

Lemma 5.10. *Assume $|K|>5$. Let Ω' be the G-orbit on l_∞ which contains Ω. If Ω' contains $L(\infty)$, then $|\Omega'|\neq q^2-1$ or $n\in\{e(q-1)|e=0, 1/2, 1/3, 2/3, 1/4, 3/4\}$.*

Proof. By Lemmas 4.4 and 5.3, H is transitive on Ω. Assume $|\Omega'|=q^2-1$ and set $\{L(a, 0), L(b, 0)\}=\Delta\cup\Omega-\Omega'$, where $a, b\in K$, $a\neq b$. By Lemma 5.6, $a\neq 0$ and $b\neq 0$ and by Lemma 5.2, $G_{L(\infty)}=H$. Let P be a Sylow p-subgroup of H. Using a Witt's theorem, $N_G(P)$ is transitive on $\Omega'\cap\Delta$. Let R be the stabilizer of $L(a, 0)$ and $L(b, 0)$ in $N_G(P)$ and Ψ the R-orbit on l_∞ which contains $L(\infty)$. As R is a normal subgroup of $N_G(P)$ of index at most 2, $|\Psi|=(q-1)/2$ or $q-1$. Let $L(z, 0)\in\Psi$. There is an element $\sigma\in R$ such that $L(\infty)\sigma=L(z, 0)$. Then $\sigma=\begin{pmatrix}A & B\\ C & zC\end{pmatrix}$, $A, B\in M_2(K)$, $C\in GL(2, K)$ by

(2.4.1) and (2.4.2). Since $(A+tC)^{-1}(B+tzC)=tE$ for $t\in\{a, b\}$, $a^2C+a(A-zC)-B=0$ and $b^2C+b(A-zC)-B=0$. Hence $A=(z-a-b)C$ and $B=-abC$ so $\sigma=\begin{pmatrix} cC & dC \\ C & zC \end{pmatrix}$, where

$$c=z-a-b \quad \text{and} \quad d=-ab\,. \tag{5.7}$$

Set $M(u, v)=L(x, y)\sigma$. Then $M(u, v)=C^{-1}(cE+M(x, y))^{-1}(dE+zM(x, y))C$. Assume $y\neq 0$. Then $v\neq 0$. Applying Lemma 4.1, $r(v)=(d+zt_1)/(c+t_1)+(d+zt_2)/(c+t_2)$ and $-s(v)=((d+zt_1)/(c+t_1))\,((d+zt_2)/(c+t_2))$, where $t_1+t_2=tr(M(x, y))=r(y)$ and $t_1t_2=\det(M(x, y))=-s(y)$. Hence $kv^n=(2cd+(cz+d)r-2zs)/(c^2+cr-s)$ and $-lv^{2n}=(d^2+dzr-z^2s)/(c^2+cr-s)$, where $r=ky^n$ and $s=ly^{2n}$. (5.8)

From this, $k^2(c^2+cr-s)\,(d^2+dzr-z^2s)+l(2cd+(cz+d)r-2zs)^2=0$. Substituting (5.7) and (5.8) into this equation gives $(k^2+4l)\,(z^2l^2y^{4n}+(ab+a+b-z^2)zkly^{3n}+(a+b-z)ab(k^2-2l)zy^{2n}+(a+b-z)\,(z^2-az-bz-ab)abky^n+(a+b-z)^2a^2b^2)=0$ for any $y\in K^\#$. (5.9)

By Lemma 5.1 (ii), $k^2+4l\neq 0$. Since $|K|>5$, $|\Psi-\{L(\infty)\}|\geq 2$. Hence $z^2l^2\neq 0$ for some z such that $L(z, 0)\in\Psi$. Applying Lemma 5.9 to (5.9), we have the lemma.

Lemma 5.11. *Assume* $n\notin\{e(q-1)\,|\,e=0,\ 1/2,\ 1/3,\ 2/3,\ 1/4,\ 3/4\}$ *and* $|K|>5$. *Then* $L(\infty)G\cup L(0, 0)G\subset\Delta$.

Proof. Let Ω' be the G-orbit on l_∞ which contains Ω. It suffices to show $\Omega'\cap\{L(\infty), L(0, 0)\}=\phi$. Suppose false.

First we argue that $L(\infty)\in\Omega'$. Assume $L(\infty)\notin\Omega'$ and $L(0, 0)\in\Omega'$. Then, by Lemma 5.6, $L(\infty)G_{L(0,0)}=\{L(\infty)\}$. Therefore $G_{L(0,0)}=G_{L(\infty)}$ by Lemma 5.2 so $|L(\infty)G|=|L(0, 0)G|=|\Omega'|>q^2-q$. This forces $L(\infty)\in\Omega'$, a contradiction. Thus $L(\infty)\in\Omega'$.

Set $\Delta'=\Delta\cup\Omega-\Omega'$. By Lemmas 5.4, 5.8 and 5.10, $|\Delta'|\geq 3$. Let W be the pointwise stabilizer of Δ' in G. Clearly W is a normal subgroup of G. Let $\sigma\in W$ and set $\sigma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A, B, C, D\in M_2(K)$. Then $(A+tC)^{-1}(B+tD)=tE$ for all $t\in\Lambda=\{t\,|\,L(t, 0)\in\Delta'\}$. Hence $t^2C+t(A-D)-B=O$ for all $t\in\Lambda$. Assume $\sigma\notin H$. Then $C\neq\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $|\Lambda|=|\Delta'|\geq 3$, there exist x and y in Λ such that $x\neq y$ and $(x+y)E\neq C^{-1}(D-A)$. As above $x^2C+x(A-D)-B=O$ and $y^2C+y(A-D)-B=O$. Hence $(x^2-y^2)C+(x-y)\,(A-D)=0$ so $(x+y)E=C^{-1}(D-A)$, contrary to the choice of x and y. Thus $\sigma\in H$ and so $W\subset H$. Using Lemma 5.3, a Sylow p-subgroup of H is normal in W and therefore normal in G. This implies that Δ is G-invariant and $\Omega'=\Omega$, a contradiction. Thus we have the lemma.

Lemma 5.12. *Assume* $n \notin \{e(q-1) \mid e=0, 1/2, 1/3, 2/3, 1/4, 3/4\}$ *and* $|K|>5$. *Then* $L(\infty)G \subset \{L(0, 0), L(\infty)\}$.

Proof. Suppose false. Then $G \neq H$. By Lemmas 5.2 and 5.11, $L(\infty)\sigma = L(w, 0)$ for some $\sigma \in G$ and $w \in K^{\#}$. Set $\sigma = \begin{pmatrix} A & B \\ C & wC \end{pmatrix}$. By (2.4.2), C is nonsingular. By Lemma 5.7, $L(0, 0)\sigma \neq L(\infty)$ and so, by (2.4.4), A is nonsingular. Applying Lemma 5.11, $L(0, 0)\sigma = L(i, 0)$ and $L(\infty)\sigma^{-1} = L(-j, 0)$ for some $i \in K$ and $j \in K^{\#}$. Hence $A^{-1}B = iE$ by (2.4.3) and $A - jC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ by (2.4.4). Therefore we have $\sigma = \begin{pmatrix} jC & ijC \\ C & wC \end{pmatrix}$.

Let $x \in K$ and $y \in K^{\#}$ and set $L(u, v) = L(x, y)\sigma$. Then $v \neq 0$ and $M(u, v) = C^{-1}(jE + M(x, y))^{-1}(ijE + wM(x, y))C$. Hence, by Lemma 4.1,

$$r(v) = (ij + wt_1)/(j + t_1) + (ij + wt_2)/(j + t_2) \tag{5.10}$$

and
$$-s(v) = ((ij + wt_1)/(j + t_1))\,((ij + wt_2)/(j + t_2))\,. \tag{5.11}$$

Here $t_1 + t_2 = tr(M(x, y)) = r(y)$ and $t_1 t_2 = \det(M(x, y)) = -s(y)$. Substituting these into (5.10) and (5.11) gives $kv^n = (2ij^2 + (wj + ij)r - 2ws)/(j^2 + jr - s)$ and $-lv^{2n} = (i^2j^2 + wijr - w^2s)/(j^2 + jr - s)$, where $r = ky^n$ and $s = ly^{2n}$. Hence $k^2(j^2 + jr - s)(i^2j^2 + wijr - w^2s) + l(2ij^2 + (wj + ij)r - 2ws)^2 = 0$. From this, $(k^2 + 4l)(w^2l^2y^{4n} - (jw^2 + wij)kly^{3n} + wij^2(k^2 - 2l)y^{2n} + (wij^3 + i^2j^3)ky^n + i^2j^4) = 0$ for any $y \in K^{\#}$. (5.12)

By Lemma 5.1 (ii), $k^2 + 4l \neq 0$ and by Lemma 2.4, $l \neq 0$. Applying Lemma 5.9 to (5.12), we have $n \in \{e(q-1) \mid e=0, 1/2, 1/3, 2/3, 1/4, 3/4\}$, contrary to the assumption. Thus we have the lemma.

Proposition 5.13. *Assume either r or s is not a constant function.*

(i) *If* $(q-1)/(n, q-1) \equiv 1 \pmod 2$ *and* $k=0$, *then* $\Delta - \{L(\infty), L(0, 0)\}$ *is divided into* $(n, q-1)/2$ *H-orbits of the same length.*

(ii) *If* $(q-1)/(n, q-1) \equiv 0 \pmod 2$ *or* $k \neq 0$, *then* $\Delta - \{L(\infty), L(0, 0)\}$ *is divided into* $(n, q-1)$ *H-orbits of the same length.*

(iii) *Assume* $|K|>5$ *and* $n \notin \{e(q-1) \mid e=0, 1/2, 1/3, 2/3, 1/4, 3/4\}$. *Then* $\{L(\infty), L(0, 0)\}$, Δ *and* Ω *are G-invariant.*

Proof. Let K_1 and K_2 be as defined in Lemma 5.4. We note that $-1 \notin K_1$ if and only if $p>2$ and $|K_1| \equiv 1 \pmod 2$.

Assume $(q-1)/(n, q-1) \equiv 1 \pmod 2$ and $k=0$. By Lemma 4.3 (ii), $p>2$. Since $|K_1| = (q-1)/(n, q-1) \equiv 1 \pmod 2$, $-1 \notin K_1$ so $|K_2| = 2|K_1| = 2(q-1)/(n, q-1)$. Applying Lemma 5.4, we have (i).

Assume $(q-1)/(n, q-1) \equiv 0 \pmod 2$ or $k \neq 0$. Applying Lemma 5.4, we have (ii).

Assume $|K|>5$ and $n \notin \{e(q-1) \mid e=0, 1/2, 1/3, 2/3, 1/4, 3/4\}$. By Lemmas

5.2 and 5.12, either (a) $G=H$ or (b) H is a normal subgroup of G of index 2 and $\{L(\infty), L(0, 0)\}$ is G-invariant. In particular a Sylow p-subgroup of H is normal in G. Hence Δ is G-invariant. Thus we have (iii).

We now set $N=\{a\begin{pmatrix} A & O \\ O & c^n A\end{pmatrix} \mid A=\begin{pmatrix} 1 & 0 \\ b & c^{1-n}\end{pmatrix}, a\ c\in K^* \ b\in K\}$ and

$$\tau=\begin{cases}\begin{pmatrix} J & O \\ O & -J\end{pmatrix} \quad J=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} & \text{if } k=0 \\ \begin{pmatrix} E & O \\ O & E\end{pmatrix} & \text{if } k\neq 0\end{cases}$$

$$\mu=\begin{cases}\begin{pmatrix} O & J \\ -l^{-1}J & O\end{pmatrix} & \text{if } 2n(n-1)\equiv 0 \pmod{q-1} \\ \begin{pmatrix} E & O \\ O & E\end{pmatrix} & \text{if } 2n(n-1)\not\equiv 0 \pmod{q-1}\,.\end{cases}$$

Then we have

Theorem 5.14. *Assume* $|K|>5$ *and* $n\notin\{e(q-1)\mid e=0, 1/2, 1/3, 2/3, 1/4, 3/4\}$. *Then* $G=\langle\tau, \mu\rangle N$.

Proof. By Lemma 5.3, $H=\langle\tau\rangle N$. Assume $2n(n-1)\equiv 0 \pmod{q-1}$. By Lemmas 5.5, 5.7 and 5.12, $G=\langle\mu\rangle H$ and so $G=\langle\tau, \mu\rangle N$. Assume $2n(n-1)\not\equiv 0 \pmod{q-1}$. By Lemmas 5.5, 5.7 and 5.12, $G=H=\langle\mu\rangle H=\langle\tau, \mu\rangle N$.

REMARK 5.15. (i) It follows from Proposition 5.13 and Theorem 5.14 that $\mathrm{Aut}(\pi_g)$ has no fixed point on l_∞ for any $\pi_g\in\Pi_K$, $|K|>5$. Therefore π_g is not a semifield plane for any $\pi_g\in\Pi_K$, $|K|>5$.

(ii) There exists a translation plane π_g $(g\in\Phi_K)$ which is not isomorphic to any plane in Π_K. For example, let $K=GF(7)$, $r(y)=4y^5+6y^4$ and $s(y)=6y^5+3y^4+6y^3+4y^2+3$. Then $g=(r, s)\in\Phi_K$ (See Remark 3.6 of [3]). It can be shown that $\mathrm{Aut}(\pi_g)=\langle\alpha, \beta, H\rangle$, where $H=\{\begin{pmatrix} A & O \\ O & A\end{pmatrix} \mid A=\begin{pmatrix} a & 0 \\ b & a\end{pmatrix}\ a, b\in K, a\neq 0\}$, $\alpha=\begin{pmatrix} O & 5E \\ E & 2E\end{pmatrix}$, $E=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ and $\beta=\begin{pmatrix} J & 4J \\ J & 6J\end{pmatrix}$, $J=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. Moreover $\mathrm{Aut}(\pi_g)$ has 4 orbits of lengths 7, 7, 8 and 28 on l_∞. Thus π_g is not isomorphic to any plane in Π_K by Theorem 5.14.

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Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560, Japan