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PARTITIONING STRONGLY REGULAR GRAPHS

Dedicated to Professor Hirosi Nagao on his sixtieth birthday

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1. Introduction

A strongly regular graph with parameters (n,a,c,d) is a graph with n vertices whish is regular of valency a and has the property that the number of vertices adjacent to p_1 and p_2 $(p_1 \neq p_2)$ is c or d acsordihg as p_1 and p_2 are adjacent or not.

In this paper we consider the strongly regular graphs which can be partitioned into two strongly regular subgraphs with equally many vertices. The strongly regular graphs with $d=0$ $(d=a)$ having this property are the disjoint unions of two complete graphs on $\frac{1}{2}n$ vertices (their complementary graphs). So we are interested in the strongly regular graphs with $0<d<a$.

In section 2 we first prove the following

Theorem 1. *Let a strongly regular graph Γ whose parameters are (n, a, c, d) with $0<d<a$ be partitioned into two strongly regular subgraphs Γ_1 and Γ_2 with parameters (n_1, a_1, c_1, d_1) and (n_2, a_2, c_2, d_2) where $n_1=n_2$. Assume that $a \geqq a_1+a_2$ by taking the complement of Γ if necessary. Then either of the following holds;*

(1) *The parameters of Γ and Γ_i $(i=1,2)$ are expressed in terms of two parameters $s(<0)$ and $t(>0)$ with $-s \geqq t^2+2t$ as follows.*

$$n = \frac{2(s-t)^2(2ts+s+t^2+2t)}{(t^2+s)(t^2+2t-s)}$$

$$a = \frac{s(2st-t^2+s)}{t^2+2t-s},$$

$$c = \frac{t(s+1)(s+t^2+2t)}{t^2+2t-s},$$

$$d = \frac{s(t+1)(t^2+s)}{t^2+2t-s},$$

$$n_1 = n_2 = \frac{(s-t)^2(2ts+s+t^2+2t)}{(t^2+s)(t^2+2t-s)},$$

$$a_1 = a_2 = \frac{st(s+1)}{t^2+2t-s},$$

$$c_1 = c_2 = \frac{(t^2+2t+s)(t^2+st+3t-s)}{2(t^2+2t-s)},$$

$$d_1 = d_2 = \frac{t(s+t+2)(s+t^2)}{2(t^2+2t-s)}.$$

Here s and t are the eigenvalues of the adjacency matrix of Γ different from a.

(2) *The parameters of Γ and Γ_i $(i=1,2)$ are expressed in terms of one parameter $t(>0)$ as follows.*

$$\begin{aligned}
n &= 2(2t^2+2t+1)\,,\\
a &= (2t+1)(t+1)\,,\\
c &= t(t+2)\,,\\
d &= (t+1)^2\,,\\
n_1 &= n_2 = 2t^2+2t+1\,,\\
a_1 &= a_2 = t(t+1)\,,\\
c_1 + c_2 &= c_1+d_1-1 = t^2+t-2\,,\\
d_1+d_2 &= d_1+c_1+1 = t^2+t\,.
\end{aligned}$$

Here t is the eigenvalue of the adjacency matrix of Γ different from a.

The strongly regular graph with the parameters of Theorem 1(1) is refered to as a Smith graph and has the property that its subconstituents are both strongly regular (see [2], Theorems 5.4, 6.1 and 6.6). The Higman Sims graph has a partition as in Theorem 1(1) with $(s,t)=(-8,2)$ (see [2], page 40). The strongly regular graphs with the parameters (112, 30, 2, 10) and (162, 56, 10, 24) which are the subconstituents of the Mclaughlin graph ([5], pp. 109–111) also have such partitions with $(s,t)=(-10, 2)$ and $(-16, 2)$. See a result of P.J. Cameron described in the last part of this section. It was proved in [1] that a generalized quadrungle with parameters (q,q^2) gives rise to a strongly regular graph as in Theorem 1(1) with $s=-q^2-1$ and $t=q-1$. The above graph with 112 vertices is associated with the generalized quadrungle with parameters (3, 9) (see [1]). The complement of the Petersen graph has a partition as in Theorem 1(2) with $t=1$.

In section 3 we consider the construction of the strongly regular graphs having partitions as Theorem 1(2) and prove the following.

Theorem 2. *Let B be the adjacency matrix of a strongly regular graph with parameters $(2t^2+2t+1, t(t+1), \frac{1}{2}t(t+1)-1, \frac{1}{2}t(t+1))$ for some positive integer t. Assume that there exists a $(0,1)$-matrix C such tnat $BC=CB$ and $C^TC={}^TCC=t^2E+\frac{1}{2}t(t-1)(J-E)$. Then*

$$A=\begin{pmatrix} B, & C \\ {}^{T}C, & J\text{-}B\text{-}E \end{pmatrix}$$

is the adjacency matrix of a strongly regular graph with the parameters $(2(2t^2+2t+1), t(2t+1), t^2-1, t^2)$. *Here* J *is the matrix whose entries are all one.*

We remark that the strongly regular graphs appearing in Theorem 2 are the complements of those appearing in Theorem 1(2).
As an application of Theorem 2 we have the following

Theorem 3. *Let* $q=2t^2+2t+1$ *be a prime power, and* N *the additive group of the field* $GF(q)$ *with* q *elements. Assume that* N *has a difference set* X *with* t^2 *elements. Then there exists a strongly regular graph with the parameters* $(2q, 2t^2+t, t^2-1, t^2)$ *admitting a partition into two strongly regular subgraphs with equally many vertices.*

In the cases $t=2$ and 3 N has a difference set X with t^2 elements (see [4], pp. 291–292). So there exist strongly regular graphs with parameters (26, 10, 3, 4) and (50, 21, 8, 9) which admit partitions into two strongly regular subgraphs.

We use the same notation and terminology for strongly regular graphs as in [3]. For difference sets the reader is refered to [4].

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Theorem (P.J. Cameron). *Let a strongly regular graph* Γ *have the parameters of Theorem* 1(1) *with* $-s=t^2(2t+3)$. *For a vertex* v *of* Γ *let* $\Delta(v)$ *and* $\Delta'(v)$ *denote the set of vertices adjacent and non-adjcaent to* v *respectively. Then the strongly regular graph* $\Delta(v)$ *and* $\Delta'(v)$ *both have partitions into strongly regular subgraphs with equally many vertices as follows;*

(1) $\Delta(v)=\{\Delta(v)\cap\Delta(w)\}\cup\{\Delta(v)\cap\Delta'(w)\}$,
where w *is an arbitrary vertex of* $\Delta'(v)$.

(2) $\Delta'(v)=\{\Delta'(v)\cap\Delta(u)\}\cup\{\Delta'(v)\cap\Delta'(u)\}$,
where u *is an arbitrary vertex of* $\Delta(v)$.

2. Proof of Theorem 1

We begin with elementary results on strongly regular graphs.

Lemma 2.0 *Let* Γ *be a strongly regular graph with parameters* (n, a, c, d) *and* A *the adjacency matrix of* Γ. *Then*

(1) $a(a-c-1)=(n-a-1)\,d$,

(2) $0<d<a$ *if and only if* Γ *and its complementary graph are connected.*

(3) *The eigenvalues of* A *consist of* a *and the solutions of* $x^2-(c-d)x+(d-$

$a)=0$.

(4) *If* $(s(\leqq 0)$ *and* $t(\geqq 0)$ *are the solutions of* $x^2-(c-d)x+(d-a)=0$ *then*

(i) $s+t=c-d$ *and* $st=d-a$.

(ii) $nd=(a-s)(a-t)$.

(iii) $s\leqq -1$ *and if* $s=-1$ *then* $d=0$.

(iv) $0\leqq t\leqq a$ *and if* $t=a$ *then* $d=0$.

For the proof of Lemma 2.0, see [2] and [3].

Lemma 2.1. *Under the assumptions of Theorem* 1

$$a_1 = a_2\,, \tag{2.1}$$

$$(3a_1-a)\,c = a_1(c_1+c_2)\,,\ \textit{and} \tag{2.2}$$

$$(a-a_1)(a-a_1-1) = a_1(c-c_1)+(n_1-a_1-1)(d-d_1)\,. \tag{2.3}$$

Proof. Counting in two ways the number of the edges joining the vertices of Γ_1 to those of Γ_2 we have

$$n_1(a-a_1) = n_2(a-a_2), \text{ and hence } a_1 = a_2\,.$$

The number of triples (u_1, v_1, w_2) of distinct vertices with u_1, v_1 in Γ_1 and w_2 in Γ_2 such that any two of them are adjacent is equal to $n_1 a_1(c-c_1)$. Also the number of triples (u_1, v_2, w_2) of distinct vertices with u_1 in Γ_1 and v_2, w_2 in Γ_2 such that any two of them are adjacent is equal to $n_2a_2(c-c_2)$. Countihg in another way the total number of these triples we have

$$n_1 a_1(c-c_1)+n_2 a_2(c-c_2) = n_1(a-a_1)\,c\,, \text{ and hence}$$
$$c(3a_1-a) = a_1(c_1+c_2)\,.$$

Finally counting in two ways the number of triples (u_1, v_1, w_2) of distinct vertices with u_1, v_1 in Γ_1 and w_2 in Γ_2 such that w_2 is adjacent to both u_1 and v_1 we have

$$n_2(a-a_2)(a-a_2-1) = n_1 a_1(c-c_1)+n_1(n_1-a_1-1)(d-d_1)\,, \text{ hence}$$
$$(a-a_1)(a-a_1-1) = a_1(c-c_1)+(n_1-a_1-1)(d-d_1)\,.$$

Lemma 2.2. *In the situation of Theorem* 1 *put*

$$s = 2a_1-a\,. \tag{2.4}$$

Then s *is the negative eigenvalue of the adjacency matrix of* Γ.

Proof. Let A be the adjacency matrix of Γ. We may asume that A has the form,

$$A = \begin{pmatrix} A_1 & C \\ {}^TC & A_2 \end{pmatrix}, \tag{2.5}$$

where A_i are the adjacency matrix of Γ_i $(i=1, 2)$. Then it is easily checked that $(\overbrace{a-a_1, \cdots, a-a_1}^{n_1}, \overbrace{-(a-a_1) \cdots, -(a-a_1)}^{n_2})$ is an eigenvector of A with eigenvalue $2a_1-a$. By the assumption of Theorem 1 $s=2a_1-a\leqq 0$.

We proceed with the proof of Theorem 1. By (2.5) we have

$$A^2 = \begin{pmatrix} A_1^2+C^TC, & A_1C+CA_2 \\ {}^TCA_1+A_2{}^TC, & {}^TCC+A_2^2 \end{pmatrix}.$$

Then since $A^2=aE+cA+d(J-A-E)$ and $A_i^2=a_iE+c_iA_i+d_i(J-A_i-E)$ $(i=1, 2)$ we have

$$C^TC = (a-a_1)E+(c-c_1)A_1+(d-d_1)(J-A_1-E), \tag{2.6}$$

$$A_1C+CA_2 = cC+d(J-C), \text{ and} \tag{2.7}$$

$${}^TCC = (a-a_2)E+(c-c_2)A_2+(d-d_2)(J-A_2-E). \tag{2.8}$$

Let s_i and t_i be the eigenvalues of A_i different from $a_i(i=1, 2)$. First assume that $c-c_1+d_1-d\neq 0$. Then by (2.6) and (2.8) we may assume that s_1 and s_2, t_1 and t_2 have the same multiplicities respectively and that

$$\begin{aligned}(a-a_1)&+(c-c_1)s_1+(d-d_1)(-s_1-1)\\ &= (a-a_2)+(c-c_2)s_2+(d-d_2)(-s_2-1), \text{ and}\end{aligned} \tag{2.9}$$

$$\begin{aligned}(a-a_1)&+(c-c_1)t_1+(d-d_1)(-t_1-1)\\ &= (a-a_2)+(c-c_2)t_2+(d-d_2)(-t_2-1).\end{aligned} \tag{2.10}$$

By (2.1) and (2.9) we have

$$(c-c_1)s_1+(d-d_1)(-s_1-1) = (c-c_2)s_2+(d-d_2)(-s_2-1), \text{ hence}$$
$$s_1(c-c_1-d+d_1)+d_1 = s_2(c-c_2-d+d_2)+d_2.$$

Then since $c_i-d_i=s_i+t_i$ and $d_i=a_i+s_i t_i(i=1, 2)$, we have

$$s_1(c-d-s_1) = s_2(c-d-s_2), \text{ hence}$$
$$(s_1-s_2)(c-d-s_1-s_2) = 0. \tag{2.11}$$

Similarly by (2.10) we have

$$(t_1-t_2)(c-d-t_1-t_2) = 0. \tag{2.12}$$

Then by (2.11) and (2.12) we have either $s_1=s_2$ and $t_1=t_2$ or $s_1+s_2=t_1+t_2=c-d$. Note that $s_1=s_2$ implies $t_1=t_2$, and conversely because trace $A_i=0$. If $c-c_1+d-d_1=0$ then by (2.6) and (2.8) $c-c_2+d_2-d=0$, $a-a_1-d+d_1=a-a_2-d-d_2$, hence $c_1=c_2$, $d_1=d_2$ and we may assume that $s_1=s_2$ and $t_1=t_2$.

We now prove

Lemma 2.3. *In the above situation if* $s_1+s_2=t_1+t_2=c-d$ *then* $s_1+s_2=t_1+t_2=c-d=-1$ *and the case* 2 *of Theorem* 1 *occurs.*

Proof. Let f be the multiplicity of $s_i(i=1, 2)$. Then since trace $A_i=0$ we have

$$f=\frac{(n_1-1)\,t_1+a_1}{t_1-s_1}=\frac{(n_2-1)\,t_2+a_2}{t_2-s_2}\,.$$

Then since $s_1+s_2=t_1+t_2$ we have $(n_1-1)\,(t_1+t_2)+2a_1=0$. Then since $(n_1-1)>a_1$, and $t_1+t_2=c-d$ is integral we have $t_1+t_2=-1$, hence $s_1+s_2=c-d=-1$, and

$$n_1-1=2a_1\,. \tag{2.13}$$

Then it follows from the equality $a_i(a_i-c_i-1)=(n_i-a_i-1)\,d_i$ that $c_i+d_i+1=a_i(i=1, 2)$. Also we have

$$d_1=a_1+s_1\,t_1=a_2+(-s_2-1)\,(-t_2-1)=a_2+s_2+t_2+s_2\,t_2+1=c_2+1\,.$$

Therefore we have

$$c_1+c_2=c_1+d_1-1=a_1-2\,. \tag{2.14}$$

Then by (2.2), (2.4) and (2.14) we have

$$(3a_1-a)\,c=(a_1+s)\,c=a_1(c_1+c_2)=a_1(a_1-2)\,. \tag{2.15}$$

Also by (2.3), (2.4), (2.13) and (2.14) we have

$$(a_1-s)\,(a_1-s-1)=a_1(c-c_1+d-d_1)=a_1(2c-a_1+2)\,.$$

Then using (2.15) we have

$$\begin{aligned}(a_1-s)\,(a_1-s-1)\,(a_1+s)&=a_1\,\{2a_1(a_1-2)-(a_1-2)\,(a_1+s)\}\\&=a_1(a_1-2)\,(a_1-s)\,.\end{aligned}$$

Then since $a_1-s>0$ because of $s<0$ we have

$$\begin{aligned}&(a_1-s-1)\,(a_1+s)=a_1(a_1-2)\,,\ \text{and hence}\\&a_1=s(s+1)=(-t-1)\,(-t)=t(t+1)\,.\end{aligned}$$

Then it follows that

$$\begin{aligned}&a=2a_1-s=2s^2+s=2t^2+3t+1\,,\\&n_1=2a_1+1=2t(t+1)+1\\&d=a+st=(t+1)^2\,,\\&c=a+s+t+st=t^2+2t\,,\\&c_1+c_2=a_1-2=t^2+t-2\,,\ \text{and}\end{aligned}$$

$$d_1+d_2 = c_1+c_2+2 = t^2+t\,.$$

This completes the proof of Lemma 2.3.

By Lemma 2.3 we may assume for the remainder of this section that

$$s_1 = s_2,\ t_1 = t_2\,, \text{ hence } c_1 = c_2 \text{ and } d_1 = d_2\,.$$

By (2.7) we have

$$\begin{aligned} &A_1CA_2+CA_2^2 = (c-d)\,CA_2+dJA_2 = (c-d)\,CA_2+da_2J\,, \text{ and} \\ &A_1^2C+A_1CA_2 = (c-d)\,A_1C+dA_1J = (c-d)\,A_1C+da_1J\,, \text{ whence} \\ &CA_2^2-A_1^2C = (c-d)\,(CA_2-A_1C)\,. \end{aligned}$$

Then

$$\begin{aligned} &(c-d)\,(CA_2-A_1C) \\ &\quad = C((a_2-d_2)\,E+(c_2-d_2)\,A_2+d_2\,J)-((a_1-d_1)\,E+(c_1-d_1)\,A_1+d_1J)\,C \\ &\quad = (c_1-d_1)\,(CA_2-A_1C)+d_2(a-a_1)\,J-d_1\,(a-a_2)\,J\,, \end{aligned}$$

hence

$$(c_1-d_1-c+d)\,(CA_2-A_1C) = 0\,.$$

Then we have either $c-c_1=d-d_1$ or $CA_2-A_1C=0$. We treat these two cases separately.

Case 1.. $c-c_1=d-d_1$.

By (2.2) and (2.4) we have $c_1=\dfrac{(3a_1-a)\,c}{2a_1}=\dfrac{(a+3s)\,c}{2(a+s)}$, and hence

$$c-c_1 = d-d_1 = c\left(1-\frac{a+3s}{2(a+s)}\right) = \frac{c\,(a-s)}{2(a+s)}\,. \tag{2.16}$$

Then (2.3), (2.4) and (2.16) yield

$$\begin{aligned} &(a-s)\,(a-s-2)\,(a+s) = c(a-s)\,(n-2), \text{ hence} \\ &(a+s)\,(a-s-2) = (n-2)\,c\,. \end{aligned} \tag{2.17}$$

On the other hand since Γ is strongly regular we have

$$nd = n(a+st) = (a-s)\,(a-t)\,, \tag{2.18}$$

where t is the eigenvalue of A different frcm a and s.

By (2.17) and (2.18) we have

$$(a+s)\,(a-s-2)\,(a+st) = \{(a-s)\,(a-t)-2\,(a+st)\}\,(a+s+t+st)\,.$$

Arranging this we have

$$t(s+t+2)(s+1)a = st(s+1)(s-t)\,, \text{ and hence}$$
$$a(s+t+2) = s(s-t)\,. \tag{2.19}$$

By (2.18) and (2.19) we have

$$(n-a-1)(s+t) = \left\{\frac{(a-s)(a-t)}{a+st} - a - 1\right\}(s+t) = (s+1)(t-s) \tag{2.20}$$

Since $s<-1$ and $t>0$ the right hand sides of (2.19) and (2.20) are positive and negative respectively, and hence we have $s+t+2>0$ and $s+t<0$, whence $s+t=-1$.
Then

$$a = \frac{s(s-t)}{s+t+2} = (t+1)(2t+1) \text{ by } (2.19)\,,$$
$$c = a+s+t+st = t(t+2)\,,$$
$$d = a+st = (t+1)^2\,,$$
$$n = 2(2t^2+2t+1) \text{ by } (2.18)\,,$$
$$n_1 = 2t^2+2t+1\,,$$
$$a_1 = \frac{1}{2}(a+s) = (t+1)t\,,$$
$$c_1 = \frac{1}{2}t(t+1)-1 \text{ by (2.2) and}$$
$$d_1 = \frac{1}{2}t(t+1)\,.$$

Case 2. $A_1C=CA_2$.
By (2.7) we have $2A_1C=(c-d)C+dJ$, and hence
$\{2A_1-(c-d)E\}\,C = dJ$.
If C is non singular then $2A_1-(c-d)E$ is of rank 1, so $2a_1=c-d=s+t$, and hence $a=2a_1-s=t$. This implies $d=0$, contrary to the assumption. Therefore C is singular. Then by (2.6) we may assume that

$$(a-a_1)+(c-c_1)t_1+(d-d_1)(-t_1-1) = 0\,.$$

This reduces

$$\begin{aligned}&(a-d)-(a_1-d_1)+(c-d-c_1+d_1)t_1\\ &\quad= -st+s_1t_1+(s+t-s_1-t_1)t_1\\ &\quad= -(t_1-s)(t_1-t) = 0\,.\end{aligned} \tag{2.21}$$

Assume that $t_1(=t_2)=s$. Then since Γ and Γ_1 are strongly regular

$$\frac{a_1-s_1}{a_1+s_1 s}=\frac{n_1}{a_1-s}=\frac{2n_1}{2a_1-2s}=\frac{n}{a-s}=\frac{a-t}{a+st}=\frac{2a_1-s-t}{2a_1-s+st}\text{, whence}$$
$$a_1(t-2s_1)=-ss_1\,. \tag{2.22}$$

On the other hand (2.2) and (2.4) give

$$(a_1+s)\,c=2a_1\,c_1\,,$$
$$(a_1+s)\,(2a_1+st+t)=2a_1(a_1+s_1+s+ss_1)\,,\ \text{and hence}$$
$$a_1(t-2s_1)=-st\,. \tag{2.23}$$

By (2.22) and (2.23) we have $s_1=t$ and then $a_1=s(<0)$ by (2.23), a contradiction. Therefore it follows from (2.21) that $t_1=t$. Then (2.2) and (2.4) yield

$$(a_1+s)\,(2a_1+st+t)=2a_1(a_1+s_1\,t+s_1+t)\,,\ \text{hence}$$
$$a_1(2s+st-t-2s_1\,t-2s_1)=-st(s+1)\,. \tag{2.24}$$

Also since Γ and Γ_1 are strongly regular we have

$$\frac{1}{a_1-s}\cdot\frac{(a_1-s_1)(a_1-t)}{a_1+s_1\,t}=\frac{n_1}{a_1-s}=\frac{2n_1}{2a_1-2s}=\frac{n}{a-s}=\frac{a-t}{a+st}$$
$$=\frac{2a_1-s-t}{2a_1-s+st}\,,\ \text{hence}$$
$$\{2s_1(t+1)-2s-st+t\}\,a_1^2-\{s_1(2st+2t+s+t^2)-st^2-s^2\}\,a_1$$
$$+s_1\,ts(s+1)=0\,. \tag{2.25}$$

Eliminating a_1 from (2.24) and (2.25) we have

$$st\,(s+1)\,(t+1)\,\{2s_1^2-(3s+t)\,s_1+s\,(s+t)\}=0\,,\ \text{whence}$$
$$(2s_1-s-t)\,(s_1-s)=0\,.$$

If $s_1=s$ then (2.24) yields $a_1=s\,(<0)$, a contradiction. So we have $2s_1=s+t$. Then

$$a_1=\frac{st(s+1)}{t^2+2t-s}\ \text{by (2.24)}\,,$$
$$a=2a_1-s=\frac{s\,(2st-t^2+s)}{t^2+2t-s}\,,$$
$$d=a+st=\frac{s(t+1)\,(t^2+s)}{t^2+2t-s}\,,$$
$$c=d+s+t=\frac{t(s+t^2+2t)\,(s+1)}{t^2+2t-s}\,,$$
$$n=\frac{2(s-t)^2\,(2st+s+t^2+2t)}{(t^2+s)\,(t^2+2t-s)}\,,$$

$$d_1 = a_1+s_1 t = \frac{t(s+t+2)(s+t^2)}{2(t^2+2t-s)}, \quad \text{and}$$

$$c_1 = d_1+s_1+t = \frac{(t^2+2t+s)(t^2+st+3t-s)}{2(t^2+2t-s)}.$$

Finally $c \geqq 0$ implies $-s \geqq t^2+2t$.
This completes the proof of Theorem 1.

3. Proofs of Theorems 2 and 3

Proof of Theorem 2. Put $\bar{B}=J-B-E$. Then by the assumption of Theorem 2

$$B^2 = (t^2+t)E+\left(\frac{1}{2}t(t+1)-1\right)B+\frac{1}{2}t(t+1)\bar{B},$$

$$C\,{}^TC = {}^TC\,C = t^2E+\frac{1}{2}t(t-1)(J-E),$$

$$BC+C\bar{B} = BC+CJ-CB-C = t^2J-C = (t^2-1)C+t^2(J-C), \text{ and}$$

$$\bar{B}^2 = (t^2+t)E+\left(\frac{1}{2}t(t+1)-1\right)\bar{B}+\frac{1}{2}t(t+1)B$$

Then

$$\begin{aligned} A^2 &= \begin{pmatrix} B^2+C\,{}^TC, & BC+C\bar{B} \\ {}^TCB+\bar{B}\,{}^TC, & {}^TCC+\bar{B}^2 \end{pmatrix} \\ &= \begin{pmatrix} (2t^2+t)E+(t^2-1)B+t^2\bar{B}, & (t^2-1)C+t^2(J-C) \\ (t^2-1)\,{}^TC+t^2\,{}^T(J-C), & (2t^2+t)E+(t^2-1)\bar{B}+t^2B \end{pmatrix} \\ &= (2t^2+t)E+(t^2-1)A+t^2(J-A-E). \end{aligned}$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Since $q=2t^2+2t+1$ is a prime power there exists the Paley graph $P(q)$ with parameters $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$ and N acts on it by addition as a group of automorphisms (see [3], pp. 19). On the other hand by the assumption on N a prir $(GF(q), \{X+h \mid h\in N\})$ forms a 2—$(q, t^2, \frac{1}{2}t(t-1))$ symmetric design $\mathcal{D}$. Let B be the adjacency matrix of the Paley graph $P(q)$ and C the incidence matrix of $\mathcal{D}$ whose rows has the same numbering as those of B and whose j-th column is numbered by $X+x_j$ if the j-th row of C is numbered by x_j $(o\leqq j\leqq q-1)$. Then C is the sum of the permutation matrices representing the elements of $-X$ acting on $P(q)$. Then C commutes with B. Hence Theorem 2 applies.

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