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A NOTE ON A REPRESENTATION OF A TRANSITION BY PIVOTAL MEASURE

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It is well known that a pivotal measure for a statistical structure (or experiment) plays an important role in proving the Neyman factorization theorem ([2]). In this note we give a necessary and sufficient condition for an equivalent dominating measure of a weakly dominated statistical structure to be a pivotal measure for the structure. Motivated by a transition type of equation appeared in this characterization of pivotal measure, we consider to represent some transition, which gives an equivalency of the original experiment and its sub-experiment induced by a sufficienct subfield, by pivotal measure. This gives us another characterization of pivotal measure in terms of transition.

1. A characterization of pivotal measure

A triplet $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$ where $\mathcal{X}$ is a set, $\mathcal{A}$ a σ-field of subsets of $\mathcal{X}$ and $\mathcal{P}$ is a family of probability measures on $(\mathcal{X}, \mathcal{A})$ is refered to as a statistical structure or an experiment synonymously. An experiment $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is called a majorized experiment ([2] and [7]) if there exists a measure m on $(\mathcal{X}, \mathcal{A})$ such that each P in $\mathcal{P}$ has density w.r.t. m. Such a measure m will be refered to as a dominating measure for $\mathcal{E}$. If a majorized experiment has a dominating measure which is localizable it is called weakly dominated. For definition of a localizable measure we refer to [2]. A special dominating measure n for $\mathcal{E}$ is called a pivotal measure for $\mathcal{E}$ ([7]) if the following conditions are satisfied:

(a) $\mathcal{P}$ is equivalent to n, denoted by $\mathcal{P}\sim n$, namely $n(A)=0$ is equivalent to $P(A)=0$ for all $P\in\mathcal{P}$,

(b) a sub-σ-field (subfield for short) $\mathcal{B}$ of $\mathcal{A}$ is pairwise sufficient and contains supports ([2]) if and only if each P in $\mathcal{P}$ has a $\mathcal{B}$-measurable version of the density w.r.t. n.

Here let us note that if $\mathcal{E}$ is weakly dominated and n is an equivalent dominating measure for $\mathcal{E}$, n is pivotal for $\mathcal{E}$ if and only if for each sufficient subfield $\mathcal{B}$ for $\mathcal{E}$ each P in $\mathcal{P}$ has a $\mathcal{B}$-measurable version of the density w.r.t. n (Theorem 2 in [2]). And also we note that any equivalent dominating measure n has the finite subset property, that is, any $A\in\mathcal{A}$ such that $n(A)>0$ includes B satisfying

$0<n(B)<\infty$([5] or [2]).

In [9] we constructed a pivotal measure for weakly dominated experiment and gave a characterization of it. For reference we quote them as lemmas.

Lemma 1 ([9] Theorem 3.1). *Let* $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$ *be weakly dominated and* $\mathcal{B}_0$ *be the smallest sufficient subfield for* $\mathcal{E}$. *Let* n *be any equivalent dominating measure for the sub-experiment* $\mathcal{E}(\mathcal{B}_0)=(\mathcal{X}, \mathcal{B}_0, \mathcal{P}|\mathcal{B}_0)$ *of* $\mathcal{E}$. *Then the measure* $\tilde{n}$ *on* $(\mathcal{X}, \mathcal{A})$ *defined by*

$$\tilde{n}(A)=\int E(I_A|\mathcal{B}_0)\,dn, \qquad A\in\mathcal{A},$$

is a pivotal measure for $\mathcal{E}$. *Here* $E(I_A|\mathcal{B}_0)$ *is the conditional expectation of* I_A *given* $\mathcal{B}_0$ *which is common to all* P *in* $\mathcal{P}$ *and* I_A *is the indicator function of* A.

Existence of the smallest sufficient subfield in the case of weak domination is known ([2] for instance). Also we note that if $\mathcal{E}$ is weakly dominated and $\mathcal{B}$ is sufficient for $\mathcal{E}$ then the sub-experiment $\mathcal{E}(\mathcal{B})=(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ of $\mathcal{E}$ is also weakly dominated (Theorem 2.2 of [3]). So an equivalent dominating measure foı $\mathcal{E}(\mathcal{B}_0)$ exists and the measure n in Lemma 1 is a localizable measure on $(\mathcal{X}, \mathcal{B}_0)$.

Lemma 2 ([9] Theorem 3.2). *Let* $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$ *be weakly dominated. A measure* n *on* $(\mathcal{X}, \mathcal{A})$ *is pivotal for* $\mathcal{E}$ *if and only if for each sufficient subfield* $\mathcal{B}$ *for* $\mathcal{E}$ *the following conditions are satisfied*:

(1) $\mathcal{E}(\mathcal{B})$ *is weakly dominated by* $n|\mathcal{B}$ *for which it holdes* $\mathcal{P}|\mathcal{B}\sim n|\mathcal{B}$.

(2) $n(A\cap B)=\int_B E(I_A|\mathcal{B})\,dn|\mathcal{B}$ *holds for all* $A\in\mathcal{A}$ *and* $B\in\mathcal{B}$ *satisfying* $n(B)<\infty$.

The conditions (1) *and* (2) *are equivalent to* (1) *and*

(2)′ $E(I_A|\mathcal{B})=E(I_A|\mathcal{B}, n)$ $[n|\mathcal{B}]$ *holds for all* $A\in\mathcal{A}$.

In Lemma 2 $E(I_A|\mathcal{B}, n)$ is the conditional n-expectation of I_A given $\mathcal{B}$. This notion is due to Mussmann ([5]) and the conditional n-expectation behaves like usual conditional expectation function w.r.t. a probability measure except that the equalities

$$\int_B f\,dn=\int_B E(f|\mathcal{B}, n)\,dn|\mathcal{B}$$

are satisfied only for all $B\in\mathcal{B}$ such that $n(B)<\infty$. Such a conditional n-expectation $E(f|\mathcal{B}, n)$ of f given $\mathcal{B}$ exists if f is an $\mathcal{A}$-measurable function for which $\int_B f\,dn$ exists and is finite for all $B\in\mathcal{B}$ for which it holds $n|\mathcal{B}(B)<\infty$, and if $n|\mathcal{B}$ is localizable and has the finite subset property. For details we refer to [5] or [9].

We give another characterization of pivotal measure similar to that in Lemma 2. And we change (2) in Lemma 2 to a more "transition type" equation which leads us to consider a role of pivotal measures in the theory of comparison of experiments.

Theorem 1. *Let n be an equivalent dominating measure of a weakly dominated statistical structure $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$. Let $\mathcal{B}_0$ be the smallest sufficient subfield for $\mathcal{E}$. Then n is pivotal for $\mathcal{E}$ if and only if the following condition is satisfied:*

$$n(A)=\int E(I_A|\mathcal{B}_0)\,dn|\mathcal{B}_0 \text{ holds for all } A\in\mathcal{A}.$$

Proof. "If" part. At first we shall show that $n|\mathcal{B}_0$ has the finite subset property. Take any B in $\mathcal{B}_0$ such that $n|\mathcal{B}_0(B)>0$. Since n has the finite subset property on $\mathcal{A}$ there exists $A\in\mathcal{A}$ such that $A\subset B$ and $0<n(A)<\infty$. Without loss of generality we can assume that $0\leqq E(I_A|\mathcal{B}_0)(x)\leqq 1$ for all $x\in\mathcal{X}$. We put $B_k=[(k+1)^{-1}<E(I_A|\mathcal{B}_0)\leqq k^{-1}]$ for all $k\geqq 1$. Then we have

$$\begin{aligned} n(A)=n(A\cap B)&=\int E(I_{A\cap B}|\mathcal{B}_0)\,dn|\mathcal{B}_0\\ &=\int_B E(I_A|\mathcal{B}_0)\,dn|\mathcal{B}_0=\sum_{k=1}^{\infty}\int_{B\cap B_k}E(I_A|\mathcal{B}_0)\,dn|\mathcal{B}_0 .\end{aligned}$$

Hence there exists k such that $0<n|\mathcal{B}_0(B\cap B_k)<\infty$. So $n|\mathcal{B}_0$ has the finite subset property. Let m be an equivalent dominating measure of the weakly dominated experiment $\mathcal{E}(\mathcal{B}_0)$. Then m and $n|\mathcal{B}_0$ are equivalent and have the finite subset property. Since m is localizable, $n|\mathcal{B}_0$ is also localizable ([9] Lemma 2.1). And the totality of σ-finite sets in $\mathcal{B}_0$ w.r.t. m and $n|\mathcal{B}_0$ coincide with each other ([1] Lemma 3.1 or [2]). Then each $P|\mathcal{B}_0$ is concentrated on a σ-finite set in $\mathcal{B}_0$ w.r.t. $n|\mathcal{B}_0$, because $P|\mathcal{B}_0$ is concentrated on the σ-finite set $[dP|\mathcal{B}_0/dm<0]$ w.r.t. m. Hence each $P|\mathcal{B}_0$ has the density w.r.t. $n|\mathcal{B}_0$. By Lemma 1 the measure $\tilde{n}$ on $(\mathcal{X}, \mathcal{A})$ defined by $\tilde{n}(A)=\int E(I_A|\mathcal{B}_0)\,dn|\mathcal{B}_0$ is a pivotal measure for $\mathcal{E}$. Now $n=\tilde{n}$ follows by the assumption. So n itself is a pivotal measure for $\mathcal{E}$.

"Only if" part. Let $\mathcal{F}=\{F_\gamma;\ \gamma\in\Gamma\}$ be a maximal decomposition of the measure space $(\mathcal{X}, \mathcal{B}_0, n|\mathcal{B}_0)$. That is, $\mathcal{F}$ satisfies the following conditions; (i) $0<n|\mathcal{B}_0(F_\gamma)<\infty$ for all $\gamma\in\Gamma$, (ii) $n|\mathcal{B}_0(F_{\gamma_1}\cap F_{\gamma_2})=0$ for all γ_1, γ_2 such that $\gamma_1\neq\gamma_2$ and (iii) for $B\in\mathcal{B}_0$, $n|\mathcal{B}_0(B\cap F_\gamma)=0$ for all $\gamma\in\Gamma$ imply $n|\mathcal{B}_0(B)=0$. For existence of such an $\mathcal{F}$ we refer to [1]. Now we show that $\mathcal{F}$ is a maximal decomposition of $(\mathcal{X}, \mathcal{A}, n)$. Only the condition (iii) is to be proved. Suppose that $A\in\mathcal{A}$ and $n(A\cap F_\gamma)=0$ for all $\gamma\in\Gamma$. Since n is pivotal for $\mathcal{E}$, by Lemma 2, we have

$$0=n(A\cap F_\gamma)=\int_{F_\gamma}E(I_A|\mathcal{B}_0)\,dn|\mathcal{B}_0 .$$

Hence $n|\mathcal{B}_0(F_\gamma \cap [E(I_A|\mathcal{B}_0)>0])=0$, which implies that $n|\mathcal{B}_0([E(I_A|\mathcal{B}_0)>0])=0$ by (iii). So $P(A)=\int E(I_A|\mathcal{B}_0)\,dP|\mathcal{B}_0=0$ holds for all $P\in\mathcal{B}$. Therefore $\mathcal{F}$ is a maximal decomposition of $(\mathcal{X}, \mathcal{A}, n)$ because $\mathcal{P}\sim n$. Then we have, for all $A\in\mathcal{A}$,

$$\begin{aligned} n(A) &= \sum_{\gamma\in\Gamma} n(A\cap F_\gamma) \qquad (n \text{ has the finite subset property}) \\ &= \sum_{\gamma\in\Gamma}\int_{F_\gamma} E(I_A|\mathcal{B}_0)\,dn|\mathcal{B}_0 \text{ (by Lemma 2)} \\ &= \sum_{\gamma\in\Gamma}\int E(I_A|\mathcal{B}_0)\,dn_\gamma \qquad (dn_\gamma = I_{F_\gamma}dn|\mathcal{B}_0) \\ &= \int E(I_A|\mathcal{B}_0)\,d(\sum_{\gamma\in\Gamma} n_\gamma) \\ &= \int E(I_A|\mathcal{B}_0)\,dn|\mathcal{B}_0\,, \end{aligned}$$

which shows the condition.

REMARK 1. If n is pivotal for $\mathcal{E}$ then the condition in Theorem 1 is satisfied for any sufficient subfield.

REMARK 2. We showed in the above proof that if $\mathcal{F}$ is a maximal decomposition of $(\mathcal{X}, \mathcal{B}_0, n|\mathcal{B}_0)$ and n is pivotal for $\mathcal{E}$ then $\mathcal{F}$ is a maximal decomposition of $(\mathcal{X}, \mathcal{A}, n)$. A problem, the existence of measurable cardinals, related to the commonality of maximal decomposition between a measure space and a "sub"-measure space of it is discussed in [6].

REMARK 3. For any localizable measure with the finite subset property there exists a family of probability measures forming a weakly dominated statistical structure for which the measure is pivotal. This can be proved by using uniform distributions in the wide sense in Example 2.2 of [2]. We omit the details.

2. Transition represented by pivotal measure

Let $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be weakly dominated and n be any pivotal measure for $\mathcal{E}$. Let $\mathcal{B}$ be any sufficient subfield for $\mathcal{E}$. By Theorem 1 and Remark 1 we have

$$n(A) = \int E(I_A|\mathcal{B})\,dn|\mathcal{B}\,, \qquad A\in\mathcal{A}\,. \tag{1}$$

On the other hand we have

$$P(A) = \int E(I_A|\mathcal{B})\,dP|\mathcal{B}\,, \qquad A\in\mathcal{A}\,,\ P\in\mathcal{P}\,, \tag{2}$$

which are satisfied by sufficiency of $\mathcal{B}$. Equation (2) could be regarded as "transition type" equation. And (1) has the same type as (2).

Motivated by this inspection we will consider a transition, which is defined in terms of pivotal measure, appearing in comparison of experiments $\mathcal{E}(\mathcal{B})$ and $\mathcal{E}$, both of which are generally recognized to be equivalent. We show in this section that using any pivotal measure n for $\mathcal{E}$ we can construct a transition from $\mathcal{E}(\mathcal{B})$ to $\mathcal{E}$ which makes the deficiency of $\mathcal{E}(\mathcal{B})$ w.r.t. $\mathcal{E}$ 0 and has the same type of representation as (2).

A transition from $\mathcal{E}(\mathcal{B})$ to $\mathcal{E}$ is a positive, linear and positively isometric mapping on $L(\mathcal{E}(\mathcal{B}))$ into $L(\mathcal{E})$ ([4]). Here $L(\mathcal{E})$ is the L-space of the experiment $\mathcal{E}$ which is defined as the smallest band containg $\mathcal{P}$ in the space of all bounded signed measures on $(\mathcal{X}, \mathcal{A})$ ([4] and [8]). The deficiency of $\mathcal{E}(\mathcal{B})$ w.r.t. $\mathcal{E}$ is defined by $\inf_T \sup\{\|TP|\mathcal{B}-P\|;\ P\in\mathcal{P}\}$, where the infimum is taken over the transitions from $\mathcal{E}(\mathcal{B})$ to $\mathcal{E}$ and $\|\cdot\|$ is the total variation norm ([4] and [8]).

By definition of a pivotal measure we have $P\sim n$. So, by Remark 3.3 of [8], it follows that $L(\mathcal{E}(\mathcal{B}))=\{f\cdot n|\mathcal{B};\ f\in L^1(\mathcal{X}, \mathcal{B}, n|\mathcal{B})\}$, $L(\mathcal{E})=\{f\cdot n;\ f\in L^1(\mathcal{X}, \mathcal{A}, n)\}$. Here $f\cdot n$ is the signed measure having the density f w.r.t. n. We define a mapping T_n on $L(\mathcal{E}(\mathcal{B}))$ into $L(\mathcal{E})$ by

$$T_n(f\cdot n|\mathcal{B})=f\cdot n,\ f\in L^1(X, \mathcal{B}, n|\mathcal{B})\,. \tag{3}$$

Then it is easy to check that T_n is positive, linear and positively isometric. Since n is pivotal, for each $P\in\mathcal{P}$, there exists a $\mathcal{B}$-measurable version of the density of P w.r.t. n. This version is clearly a version of the density of $P|\mathcal{B}$ w.r.t. $n|\mathcal{B}$. So we have $T_n(P|\mathcal{B})=P$ for all $P\in\mathcal{P}$. Hence the deficiency of $\mathcal{E}(\mathcal{B})$ w.r.t. $\mathcal{E}$ is 0. Next we show

$$T_n(f\cdot n|\mathcal{B})\,(A)=\int E(I_A|\mathcal{B})\,d(f\cdot n|\mathcal{B})\,, \tag{4}$$

$A\in\mathcal{A}$ and $f\in L^1(\mathcal{X}, \mathcal{B}, n|\mathcal{B})$, which is the same type of equation as (2). Take any $f\in L^1(\mathcal{X}, \mathcal{B}, n|\mathcal{B})$. Since $[|f|>0]$ is σ-finite w.r.t. $n|\mathcal{B}$ we can write $[|f|>0]=\sum_{k=1}^{\infty} B_k$, where B_k satisfies $n|\mathcal{B}(B_k)<\infty$ for each k. By Lemma 2, it follows that $E(I_A|\mathcal{B})=E(I_A|\mathcal{B}, n)$ $[n|\mathcal{B}]$, $A\in\mathcal{A}$. These imply that

$$\begin{aligned} T_n(f\cdot n|\mathcal{B})\,(A) &= \int_A f\,dn=\int_{[|f|>0]} I_A f\,dn \\ &= \sum_{k=1}^{\infty}\int_{B_k} I_A f\,dn \\ &= \sum_{k=1}^{\infty}\int_{B_k} E(I_A f\,|\,\mathcal{B}, n)\,dn|\mathcal{B} \\ &= \sum_{k=1}^{\infty}\int_{B_k} E(I_A|\mathcal{B}, n) f\,dn|\mathcal{B} \end{aligned}$$

$$= \sum_{k=1}^{\infty} \int_{B_k} E(I_A | \mathscr{B}) f \, dn | \mathscr{B}$$

$$= \int E(I_A | \mathscr{B}) \, d(f \cdot n | \mathscr{B}) ,$$

which shows (4). Summarizing the above we have

Theorem 2. *Let $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be weakly dominated and n be any pivotal measure for $\mathcal{E}$. For any sufficienct subfield $\mathscr{B}$ for $\mathcal{E}$ the transition T_n defined by* (3) *makes the deficiency of $\mathcal{E}(\mathscr{B})$ w.r.t. $\mathcal{E}$* 0 *and has the representation* (4).

Theorem 3. *Let $\mathcal{E}=(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a majorized experiment and $\mathscr{B}$ be any pairwise sufficient subfield which contains supports. Then the deficiency of $\mathcal{E}(\mathscr{B})$ w.r.t. $\mathcal{E}$ is* 0.

This can be proved just in the same way as the one showed in the proof of Theorem 2 using a pivotal measure for the majorized experiment E in the wide sense of the definition given in section 1.

Theorem 4. *Let $\mathcal{E}$ be weakly dominated and $\mathscr{B}_0$ be the smallest sufficient subfield for $\mathcal{E}$. And let n be an equivalent dominating measure for $\mathcal{E}$ such that $n | \mathscr{B}_0$ has the finite subset property. Then n is pivotal for $\mathcal{E}$ if and only if the mapping T_n on $L(\mathcal{E}(\mathscr{B}_0))$ into $L(\mathcal{E})$ defined by $T_n(f \cdot n | \mathscr{B}_0) = f \cdot n$, $f \in L^1(\mathcal{X}, \mathscr{B}_0, n | \mathscr{B}_0)$ satisfies $T_n(P | \mathscr{B}_0) = P$ for all $P \in \mathscr{B}$.*

This gives another characterization of pivotal measure in terms of transition. "Only if" part of the Theorem follows from the proof of Theorem 2. For the proof of "if" part we note that, just in the same way as the "if" part of the proof cf Theorem 1, $n | \mathscr{B}_0$ is an equivalent dominating measure for $\mathcal{E}(\mathscr{B}_0)$. Then it follows that

$$P = T_n(P | \mathscr{B}_0) = T_n(dP | \mathscr{B}_0 / dn | \mathscr{B}_0 \cdot n | \mathscr{B}_0) = dP | \mathscr{B}_0 / dn \cdot | \mathscr{B}_0 \cdot n ,$$

$P \in \mathcal{P}$, which imply each $P \in \mathcal{P}$ has a $\mathscr{B}_0$-measurable version of the density w.r.t. n.

REMARK 4. Using the pivotal measure in the sense of the definition given in section 1, we can extend Theorem 4 to the case of majorized experiment $\mathcal{E}$ if we replace the samllest sufficient subfield with the smallest pairwise sufficient subfield which contains supports whose existence is guaranteed in [2].

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