Morimoto, K.
Osaka J. Math.
22 (1985), 277–287

KLEIN BOTTLES IN GENUS TWO 3-MANIFOLDS

KANZI MORIMOTO

(Received April 20, 1984)

Introduction

For a closed 3-manifold M, it is very interesting to study the relation between a Heegaard surface of M and an embedded surface in M. For this purpose W. Haken has shown in [2] that if a closed 3-manifold M is not irreducible, then there is an essential 2-sphere in M which intersects a fixed Heegaard surface of M in a single circle, and W. Jaco has given in [4] an alternative proof of it. M. Ochiai has shown in [8] that if a closed 3-manifold M contains a 2-sided projective plane, then there is a 2-sided projective plane in M which intersects a fixed Heegaard surface of M in a single circle, and moreover he has shown in [9] that if a closed 3-manifold M with a Heegaard splitting of genus two contains a 2-sided projective plane, then M is homeomorphic to $P^2 \times S^1$. Succesively T. Kobayashi has shown in [5] that if a closed 3-manifold M with a Heegaard splitting of genus two contains a 2-sided non-separating incompressible torus, then there is a 2-sided non-separating incompressible torus in M which intersects a fixed Heegaard surface in a single circle. In this paper we will show a similar result for a Klein bottle.

Theorem 1. *Let M be a closed connected orientable 3-manifold with a fixed Heegaard splitting $(V_1, V_2; F)$ of genus two. If M contains a Klein bottle, then there is a Klein bottle in M which intersects F in a single circle.*

By the way it is well known that a closed orientable 3-manifold M with a Heegaard splitting of genus one contains a Klein bottle if and only if M is homeomorphic to $L(4n, 2n+1)$ for some non-negative integer n (c.f. [1]). Using Theorem 1 we will give a necessary and sufficient condition for a closed orientable 3-manifold with a Heegaard splitting of genus two to contain a Klein bottle. Namely we will give three families of closed orientable 3-manifolds, and we will show that a closed orientable 3-manifold M with a Heegaard splitting of genus two contains a Klein bottle if and only if M belongs to one of the three families (Theorem 2).

I would like to express my gratitude to Prof. F. Hosokawa and Prof. S. Suzuki and the members of KOOK seminar for their helpful suggestions.

0. Preliminaries

Throughout this paper, we will work in the piecewise linear category. S^n and P^n means the n-sphere and the real n-dimensional projective space respactively. I means the unit interval [0, 1]. $Cl(\cdot)$, $Int(\cdot)$ and $\partial(\cdot)$ mean the closure, the interior and the boundary respectively. A handlebody of genus n is defined by disk sum of n-copies of $S^1 \times D^2$ where D^2 is a 2-disk, and we call a handlebody of genus one a solid torus. A Heegaard splitting of genus n of a closed orientable 3-manifold M is a pair $(V_1, V_2; F)$, where V_i is a handlebody of genus n ($i=1, 2$) and $M=V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = F$. Then F is called a Heegaard surface of M. According to J. Hempel [3] we call a closed orientable 3-manifold with a Heegaard splitting of genus one a lens space. A properly embedded surface F in a 3-manifold M is essential if F is incompressible in M and is not boundary parallel. $A \# B$ and $A \cong B$ mean the connected sum of A and B and that A is homeomorphic to B respectively. Furthermore for the definitions of standard terms in three dimensional topology and knot theory, we refer to [3], [4] and [9]. For the definition of a hierarchy for a 2-manifold and an isotopy of type A, we refer to [4].

1. Proof of Theorem 1

Lemma 1.1. *If a compact orientable 3-manifold M contains a compressible Klein bottle in $IntM$, then $M \cong S^2 \times S^1 \# M'$ or $M \cong P^3 \# P^3 \# M'$ for some compact orientable 3-manifold M'.*

Proof. Let K be a compressible Klein bottle in $IntM$, then there is a 2-disk D in $IntM$ such that $D \cap K = \partial D$ and ∂D is a 2-sided essential simple loop in K. And so there is an embedding $D \times I \subset IntM$ such that $D \times \{1/2\} = D$ and $(D \times I) \cap K = (\partial D \times I) \cap K = \partial D \times I$. By W. Lickorish [7] there are following two cases.

Case 1: ∂D cuts K into an annulus. Then $(K - \partial D \times I) \cup (D \times \{0, 1\}) = S$ is a non-separating 2-sphere in M, so $M \cong S^2 \times S^1 \# M'$, because K is one-sided in M.

Case 2: ∂D cuts K into two Möbius bands. Then $(K - \partial D \times I) \cup (D \times \{0, 1\}) = P_0 \cup P_1$ is a disjoint union of two one-sided projective planes in M, so $M \cong P^3 \# P^3 \# M'$.

Proof of Theorem 1.

Let M be a closed orientable 3-manifold with a Heegaard splitting $(V_1, V_2; F)$ of genus two. If M contains a compressible Klein bottle, then by Lemma 1.1 $M \cong S^2 \times S^1 \# L$ where L is a lens space or $M \cong P^3 \# P^3$. In the both cases it is clear that M contains a Klein bottle which intersects either V_1 or V_2 in a non-separating disk. Hence we may assume that M is neither homeomorphic to

$S^2 \times S^1 \# L$ nor to $P^3 \# P^3$. Therefore any Klein bottle in M is incompressible. For any Klein bottle in M by thinning V_1 enough we may assume that the Klein bottle intersects V_1 in disks. Let K be a Klein bottle in M such that among all Klein bottles in M which intersects V_1 in disks the number of the components of $K \cap V_1$ is minimal, and put $K_i = V_i \cap K$ $(i=1, 2)$. We may assume that K_2 is incompressible in V_2 because K is incompressible in M. Then as in W. Jaco [4] we have a hierarchy $(K_2^1, \alpha_1), (K_2^2, \alpha_2), \cdots, (K_2^n, \alpha_n)$ for $K_2^1 = K_2$ which gives rise to a sequence of isotopies in M where the i-th isotopy is an isotopy of type A at α_i $(i=1, 2, \cdots, n)$. In addition we may suppose that $\alpha_i \cap \alpha_j = \phi$ $(i \neq j)$, so we assume that each α_i is a properly embedded essential arc in K_2.

By W. Lickorish [7], each α_i is one of the following five types. We say that α_i is of type I if α_i meets two distinct components of ∂K_2, α_i is of type II if α_i meets only one compoent of ∂K_2 and α_i cuts K_2 into a planar surface and Klein bottle with hole(s), α_i is of type III if α_i meets only one component of ∂K_2 and α_i cuts K_2 into an annulus (with holes), α_i is of type IV if α_i meets only one component of ∂K_2 and α_i cuts K_2 into two Möbius bands (with holes), α_i is of type V if α_i meets only one component of ∂K_2 and α_i cuts K_2 into a Möbius band (with holes). (Fig. 1.1)

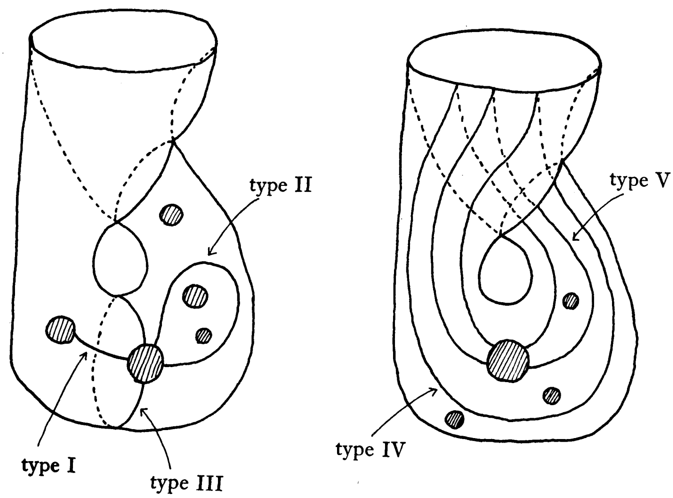

Fig. 1.1

In particular we say that α_i is a d-arc if α_i is of type I and there is a component C of ∂K_2 such that $\alpha_i \cap C \neq \phi$ and $\alpha_j \cap C = \phi$ for all $j < i$. Put $K_1 = D_1 \cup D_2 \cup \cdots \cup D_r$ where D_i is a disk and $C_i = \partial D_i$, so $\partial K_2 = \partial K_1 = C_1 \cup C_2 \cup \cdots \cup C_r$.

Before the proof of Theorem 1 we show some lemmas.

Lemma 1.2. *Any α_i is not a d-arc.*

Proof. If some α_i is a d-arc, then by using the argument of the inverse operation of an isotopy of type A defined in M. Ochiai [9] we can show that there is a Klein bottle K' in M such that each component of $K'\cap V_1$ is a disk and the number of the components of $K'\cap V_1$ is less that that of $K\cap V_1$. This is a contradiction.

Lemma 1.3. *Any α_i is not of type II.*

Proof. If some α_i is of type II, then by the definition of type II there is an arc β in ∂K_2 such that $\beta\cap\alpha_i=\partial\beta=\partial\alpha_i$ and $\beta\cup\alpha_i$ bounds a planar surface P in K_2. Since each α_j is an essential arc in K_2, some α_j in P is a d-arc. Hence the conclusion follows from Lemma 1.2.

Lemma 1.4. *If some α_i which is of type V meets C_j, then D_j is a non-separating disk in V_1.*

Proof. By performing an isotopy of type A at α_i, we obtain a Möbius band in V_1. Since V_1 is orientable a Möbius band in V_1 is one-sided, and so D_j is non-separating.

Lemma 1.5. *α_1 is of type III, IV or V. Moreover we may suppose without loss of generality that α_1 meets C_1, and D_1 is a non-separating disk in V_1.*

Proof. By lemma 1.2 and lemma 1.3 α_1 is of type III, IV or V. Suppose that α_1 meets C_1. If α_1 is of type V then by Lemma 1.4 D_1 is a non-separating disk in V_1. So we suppose that α_1 is of type III or IV and D_1 is a separating disk in V_1. Let A_1 be an annulus in V_1 obtained by performing an isotopy of type A at α_1 and K' be the image of K after the isotopy. Then $K'\cap V_1=A_1\cup D_2\cup\cdots\cup D_r$ and there is an annulus A' in ∂V_1 such that $K'\cap A'=A_1\cap A'=\partial A_1=\partial A'$. Let $K''=(K'-A_1)\cup A'$, then K'' is a Klein bottle in M and by pushing A' into V_2 we obtain a Klein bottle $\bar{K}$ from K'' such that each component of $\bar{K}\cap V_1$ is a disk and the number of the components of $\bar{K}\cap V_1$ is less than that of $K\cap V_1$. This is a contradiction. Therefore D_1 is a non-separating disk in V_1.

Now by Lemma 1.2 and Lemma 1.3 α_2 is of type III, IV or V.

Case 1: α_1 is of type III or IV.

At first let α_2 be of type III or IV. If α_2 also meets C_1, then there are two arcs β_1, β_2 in C_1 such that $\partial(\beta_1\cup\beta_2)=\partial(\alpha_1\cup\alpha_2)$ and $(\beta_1\cup\alpha_1)\cup(\beta_2\cup\alpha_2)$ bounds a planar surface in K_2, so there is a d-arc α_j for some $j\geq 3$. Therefore, by Lemma 1.2, α_2 meets only C_2. Let K^1 be the image of K after an isotopy of type A at α_1 and K^2 be the image of K^1 after an isotopy of type A at α_2. Then $K^2\cap V_1=A_1\cup A_2\cup D_3\cup\cdots\cup D_r$, where A_i is an essential annulus properly embedded in V_1 ($i=1, 2$). By cutting V_1 along a disk D parallel to D_2 missing $A_1\cup A_2$ we obtain a solid torus V containing $A_1\cup A_2$. (Fig. 1. 2).

So we obtain an annulus A' in ∂V missing the image of D, so in ∂V_1, such

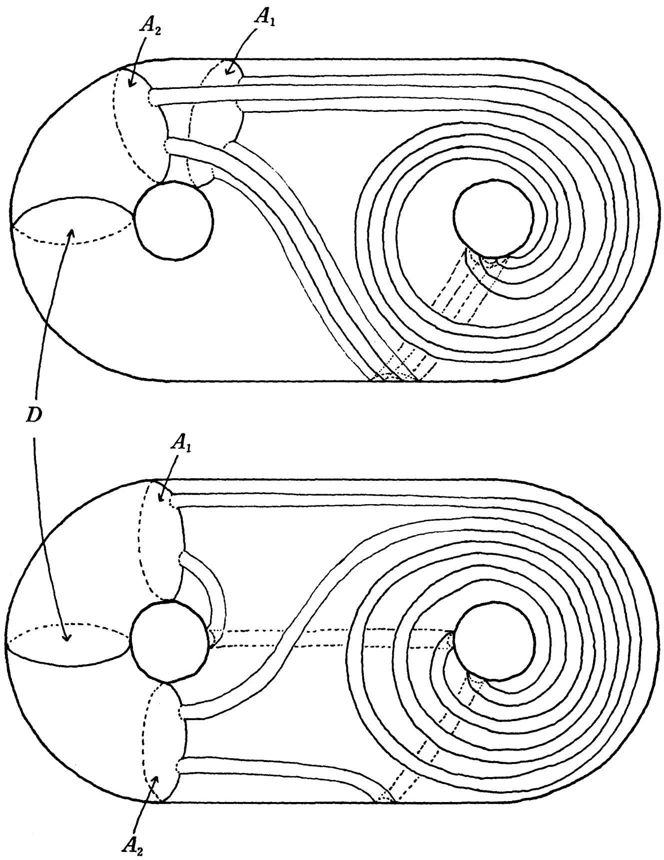

Fig. 1.2

that $A_i \cap A' = a$ component of $\partial A_i = a$ component of $\partial A'$ $(i=1, 2)$ and $K^2 \cap A' = \partial A'$. By cutting K along A' and pasting A' to the boundaries of the suitable component(s), we obtain a Klein bottle K' such that $K' \cap V_1 = A'' \cup D_{i1} \cup \cdots \cup D_{ip}$ $(p \leq r-2)$ where A'' is an annulus and $\{D_{i1}, \cdots, D_{ip}\}$ is a subset of $\{D_3, \cdots, D_r\}$. In the case that A'' is boundary parallel, then by pushing A'' into V_2 we obtain a Klein bottle which intersects V_1 in p disks. In the case that A'' is essential, then by performing an isotopy of type A we obtain a Klein bottle which intersects V_1 in $p+1$ disks. This is a contradiction. Therefore α_2 must be of type V. By Lemma 1.4 and Lemma 1.5 α_2 must meet C_1 and $r=1$. This completes the proof of Case 1.

Case 2: α_1 is of type V.

At first let α_2 be of type III or IV. If α_2 also meets C_1 and α_2 is of type III, then $\alpha_1 \cup \alpha_2$ cuts $Cl(K-D_1)$ into a disk, and so $r=1$ by Lemma 1.2. If α_2 also meets C_1 and α_2 is of type IV, then by Lemma 1.2 α_2 is an inessential arc in K_2^2 where K_2^2 is a surface obtained by cutting $K_2^1 = K \cap V_2$ along α_1. This is a

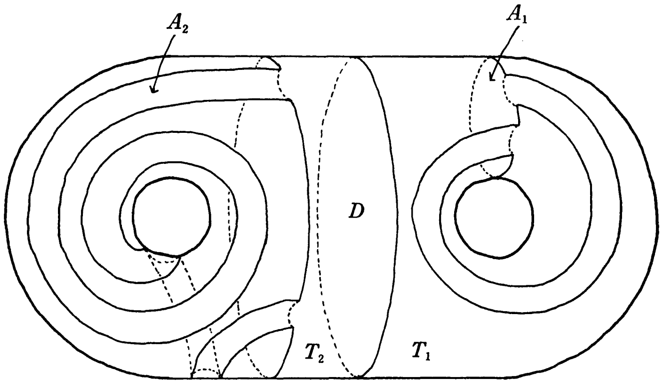

Fig. 1.3

contradiction. Therefore α_2 meets only C_2 and is of type IV. Let A_1 be a Möbius band obtained by an isotopy of type A at α_1, and A_2 be an annulus obtained by an isotopy of type A at α_2. If there is a properly embedded 2-disk D in V_1 such that D cuts V_1 into two solid tori T_1 and T_2 and A_i is properly embedded in T_i ($i=1, 2$). (Fig 1.3)

Then by the argument of Lemma 1.5 we obtain a Klein bottle K' such that each component of $K'\cap V_1$ is a disk and the number of the components of $K'\cap V_1$ is less than that of $K\cap V_1$. This is a contradiction. Hence there is a non-separating 2-disk D properly embedded in V_1 with $D\cap A_i=\phi$ ($i=1, 2$). (Fig. 1.4)

Let T be a solid torus obtained by cutting V_1 along D. Since ∂A_1 and ∂A_2 are mutually parallel simple loops in ∂T, there is an annulus A' in $\partial_2 T$ missing the image of D, so in ∂V_1, such that $A_1\cap A'=\partial A_1=$a component of $\partial A'$ and $A_2\cap A'=$a component of $\partial A_2=$a component of $\partial A'$. By cutting K along $\partial A'$ and pasting A' to the boundaries of the suitable components we obtain a Klein bottle K' such that $K'\cap V_1=S\cup D_{i1}\cup\cdots\cup D_{ip}$ ($p\leq r-2$) where S is a Möbius band and $\{D_{i1}, \cdots, D_{ip}\}$ is a subset of $\{D_3, \cdots, D_r\}$. Then by performing an isotopy of type A we obtain a Klein bottle which intersects V_1 in $p+1$ disks. This is a contradiction.

Secondly let α_2 be of type V. If α_2 also meets C_1 then we have the following two cases.

Case (a): Each component of $C_1-\partial\alpha_1$ contains one point of $\partial\alpha_2$.

Case (b): $\partial\alpha_2$ is contained in a component of $C_1-\partial\alpha_1$.

If Case (a) holds, then by Lemma 1.2 α_2 is an inessential arc in K_2^2 where K_2^2 is a surface obtained by cutting $K_2^1=K\cap V_2$ along α_1. This is a contradiction. If Case (b) holds, then $\alpha_1\cup\alpha_2$ cuts $Cl(K-D_1)$ into a disk, so $r=1$ by Lemma 1.2.

If α_2 meets only C_2, then α_3 meets C_1, C_2 or C_3. If α_3 meets only C_3, then α_3 must be of type IV. By a similar argument of the first case of Case 2, we get

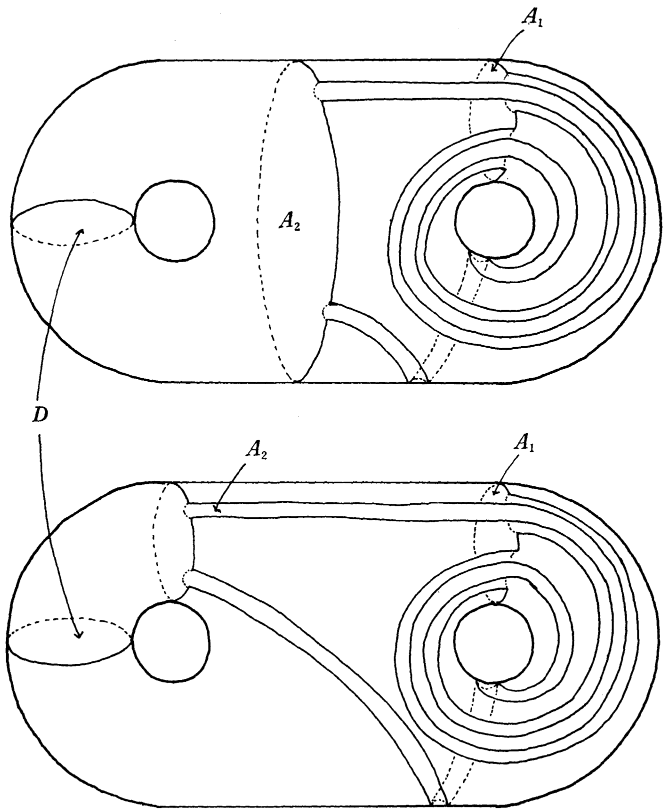

Fig. 1.4

a contradiction. If α_3 meets either only C_1 or only C_2, then α_3 is an inessential arc in K_2^2. Hence α_3 is of type I and meets both C_1 and C_2. Let K' be the image of K after a sequence of isotopies of type A at α_1, at α_2 and at α_3. Then $K' \cap V_2$ is a single disk. This completes the proof.

2. Statement and proof of Theorem 2

Let K be a Klein bottle and KI be the (orientable) twisted I-bundle over K. Then KI admits two Seifert fibrations $\mathcal{F}_1$, $\mathcal{F}_2$ where the orbit manifold of $\mathcal{F}_1$ is a disk with two exceptional points of each index 2, and the orbit manifold of $\mathcal{F}_2$ is a Möbius band without exceptional points. (see Ch. VI of W. Jaco [4]). Let α be a fiber of $\mathcal{F}_1$ in ∂KI and β be a fiber of $\mathcal{F}_2$ in ∂KI. In the following we give three families of closed orientable 3-manifolds containing a Klein bottle.

$C(1)$: Let $M(k)$ be a two bridge knot exterior in S^3 where k is a two bridge knot (possibly trivial) (c.f. Ch.4 of D. Rolfsen [10]). Let μ_1, μ_2 be two disjoint meridians of k in $\partial M(k)$ and $\bar{\mu}_1$, $\bar{\mu}_2$ be two disjoint simple loops in $IntM(k)$ obtained by pushing μ_1 and μ_2 into $IntM(k)$. Let M_1 be a 3-manifold obtained from $M(k)$ by performing arbitrary Dehn surgeries on $M(k)$ along $\bar{\mu}_1$ and $\bar{\mu}_2$. Then $C(1)$ is the family which consists of all 3-manifolds obtained from M_1 and KI by identifying ∂KI with ∂M_1 by a homeomorphism which takes β to μ_1.

$C(2)$: Let $M(k)$, μ_1 and $\bar{\mu}_1$ be a two bridge knot exterior, a meridian of k in $\partial M(k)$ and a simple loop in $IntM(k)$ as in $C(1)$ respectively. Let M_2 be a 3-manifold obtained from $M(k)$ by performing an arbitrary Dehn surgery on $M(k)$ along $\bar{\mu}_1$. Then $C(2)$ is the family which consists of all 3-manifolds obtained from M_2 and KI by identifying ∂KI with ∂M_2 by a homeomorphism which takes α to μ_1.

$C(3)$: Let $L=V_1\cup V_2$ be a lens space where V_i is a solid torus ($i=1, 2$) and $V_1\cap V_2=\partial V_1=\partial V_2$. Let $L(k)$ be a one bridge knot exterior in L (i.e. k is a simple loop in L and for $i=1, 2$ $(V_i, V_i\cap k)$ is homeomorphic to $(A\times I, \{p\}\times I)$ as pairs where A is an annulus and p is a point in $IntA$). Let μ be a meridian of k in $\partial L(k)$. Then $C(3)$ is the family which consists of all 3-manifolds obtained from $L(k)$ and KI by identifying ∂KI with $\partial L(k)$ by a homeomorphism which takes α to μ.

Theorem 2. *Let M be a closed connected orientable 3-manifold with a Heegaard splitting of genus two. Then M contains a Klein bottle if and only if M belongs to one of $C(1)$, $C(2)$ or $C(3)$.*

For the proof of Theorem 2 we prepare the following two Lemmas.

Lemma 2.1 (Lemma 3.2 of T. Kobayashi [6]). *Let V be a handlebody of genus two and A be a non-separating essential annulus properly embedded in V. Then A cuts V into a handlebody V' of genus two and there is a complete system of meridian disks $\{D_1, D_2\}$ of V' such that $D_1\cap A$ is an essential arc of A.* (Fig. 2.1)

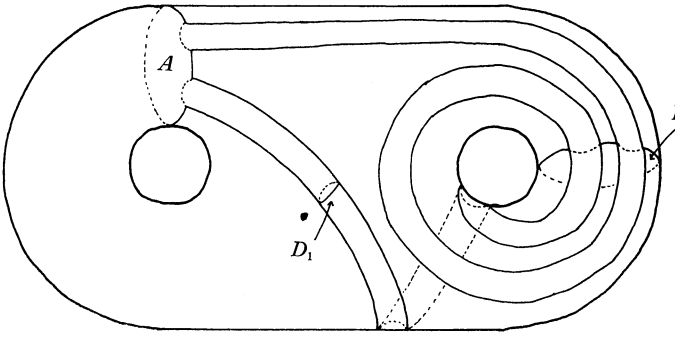

Fig. 2.1

Lemma 2.2. *Let S be a Möbius band properly embedded in a handlebody V of genus n. Then there is a 2-disk D properly embedded in V which cuts V into V_1 and V_2 where V_1 is a solid torus and V_2 is a handlebody of genus $n-1$ and S is* *properly embedded in* V_1.

Proof. Since Möbius band can not be properly embedded in a 3-ball, by using a complete system of meridian disks in V, we can find a non-separating disk D_1 properly embedded in V such that $D_1 \cap S \neq \phi$ and there is a component α of $D_1 \cap S$ which is an essential arc in S and is innermost in D_1. Therefore there is a 2-disk D_2 in D_1 such that $\partial D_1 \cap D_2 = \beta$ is an arc and $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial D_2$. Then there is a proper embedding $D_2 \times I \subset V$ such that $D_2 \times \{1/2\} = D_2$ and $(D_2 \times I) \cap S = \alpha \times I$. Let $D_3 = (S - (\alpha \times I)) \cup (D_2 \times \{0\}) \cup (D \times \{1\})$. Since S is one-sided in V, D_3 is a non-separating disk properly embedded in V. (Fig. 2.2)

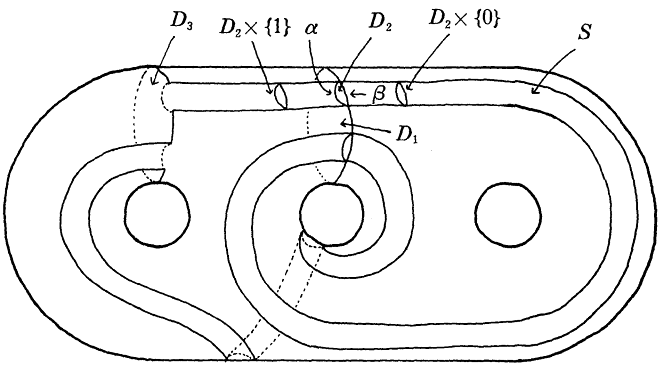

Fig. 2.2

Let $S_1 = D_3 \cup (\beta \times I)$, then S_1 is a Möbius band and S is obtained by pushing S_1 slightly into $Int V$. Let N be a regular neighborhood of S_1 in V, then N is a solid torus and S may be supposed to be properly embedded in N. Therefore $D = Cl(\partial N - \partial V)$ is the 2-disk satisfying the conditions of this Lemma.

Proof of Theorem 2.

Let $(V_1, V_2; F)$ be a Heegaard splitting of genus two of M. If M contains a compressible Klein bottle, then by Lemma 1.1 $M \cong S^2 \times S^1 \# L$ where L is a lens space or $M \cong P^3 \# P^3$. If $M \cong S^2 \times S^1 \# L$, then M belongs to $C(3)$ because $S^2 \times S^1$ is obtained from KI and a solid torus by identifying their boundaries by some homeomorphism. If $M \cong P^3 \# P^3$, then M belongs to $C(2)$ by the same reason as above. If M contains an incompressible Klein bottle, then by Theorem 1 we can suppose without loss of generality that there exists a Klein bottle K in M which intersects V_1 in a non-separating disk. For $i=1, 2$ put $K_i = K \cap V_i$ then K_1 is a non-separating disk in V_1 and K_2 is a Klein bottle with one hole in V_2. Let $\bar{\alpha}$ be an essential arc in K_2 which gives rise to an isotopy of type A at $\bar{\alpha}$ and $\bar{K}$ be the image of K after an isotopy of type A at $\bar{\alpha}$ and put $\bar{K}_i = \bar{K} \cap V_i$ $(i=1, 2)$. Then we have the following three cases.

Case (1): $\bar{\alpha}$ is of type III. For $i=1, 2$ $\bar{K}_i$ is a non-separating essential annulus in V_i. So by using a similar argument of §4 of T. Kobayashi [5] and noting Lemma 2.1, we can show that M belongs to $C(1)$.

Case (2): $\bar{\alpha}$ is of type IV. $\bar{K}_1$ is a non-separating essential annulus in V_1 and $\bar{K}_2$ is a disjoint union of two Möbius bands in V_2. So by using a similar argument of §4 of T. Kobayashi [5] and noting Lemma 2.1 and Lemma 2.2, we can show that M belongs to $C(2)$.

Case (3): $\bar{\alpha}$ is of type V. For $i=1, 2$ $\bar{K}_i$ is a Möbius band in V_i. So by using a similar argument of §4 of T. Kobayashi [5] and noting Lemma 2.2, we can show that M belongs to $C(3)$.

Conversely if M belongs to one of $C(1)$, $C(2)$ or $C(3)$, then by tracing back the above procedure it is easy to see that M has a Heegaard splitting of genus two and contains a Klein bottle. This completes the proof.

Remarks.

(1) In the case that M is irreducible and has a non-trivial torus decomposition and has a Heegaard splitting of genus two, then M is completely characterized by T. Kobayashi [6].

(2) In the case that M is connected sum of two lens spaces L_1 and L_2 and contains a Klein bottle, then it is easily checked that either L_1 or L_2 is homeomorphic to $L(4n, 2n+1)$ for some non-negative integer n or both L_1 and L_2 are homeomorphic to P^3.

References

[1] G.E. Bredon and J.W. Wood: *Non-orientable surface in orientable 3-manifolds,* Invent. Math. **7** (1969), 83–110.

[2] W. Haken: *Some results on surface in 3-manifolds,* Studies in Modern Topology, Math. Assoc. Amer., Prentice-Hall, 1968.

[3] J. Hempel: 3-manifolds, Ann. of Math. Studies No. 86, Princeton N.J., Princeton University Press, 1976.

[4] W. Jaco: Lectures on three manifold topology, Conference board of Math. No. 43, 1980.

[5] T. Kobayashi: *Non-separating incompressible tori in 3-manifolds,* J. Math. Soc. Japan **36** (1984), 11–22.

[6] T. Kobayashi: *Structures of the Haken manifolds with Heegaard splittings of genus two,* Osaka J. Math. **21** (1984), 437–455.

[7] W.B.R. Lickorish: *Homeomorphisms of non-orientable two manifolds,* Proc. Cambridge Philos. Soc. **59** (1963), 307–317.

[8] M. Ochiai: *2-sided embedding of projective planes into 3-manifolds,* Trans. Amer. Math. Soc. **274** (1982), 641–650.

[9] M. Ochiai: *On Haken's theorem and its extension,* Osaka J. Math. **20** (1983), 461–468.

[10] D. Rolfsen: Knots and links, Mathematics Lecture Series 7, Publish or Perish Inc., Berkeley, Ca. 1976.

Department of Mathematics
Kobe University
Nada-ku, Kobe 657
Japan