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PARALLEL SUBMANIFOLDS IN A QUATERNION PROJECTIVE SPACE

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1. Introduction

We call submanifolds with parallel second fundamental form parallel submanifolds. It is well-known that parallel submanifolds in Riemannian locally symmetric spaces are locally symmetric. It is an interesting problem to classify the parallel submanifolds in a specific Riemannian symmetric space. In fact, these submanifolds have been classified by several authors when the ambient spaces are real space forms and complex space forms (for reference see remark after Theorem 3.10). In this paper we shall classify parallel submanifolds in a quaternion projective space and its non-compact dual.

A quaternion projective space and its non-compact dual are quaternionic Kaehler manifolds. After recalling these notions, we define in §2 three kinds of immersions of a Riemannian manifold into a quaternionic Kaehler manifold, namely, totally real, totally complex, and invariant immersions, and then study fundamental properties of these immersions. In §3, we shall show that a parallel but not totally geodesic submanifold in a quaternion projective space or its non-compact dual is one of the following (Theorem 3.10):

(*R–R*) totally real submanifold which is contained in a totally real totally geodesic submanifold,

(*R–C*) totally real submanifold which is contained in a totally complex totally geodesic submanifold,

(*C–C*) totally complex submanifold which is contained in a totally complex totally geodesic submanifold,

(*C–H*) totally complex submanifold which is contained in an invariant totally geodesic submanifold whose dimension is twice the dimension of the submanifold.

It is known that a totally real totally geodesic submanifold and a totally complex totally geodesic submanifold of a quaternion projective space or its non-compact dual are a real space form and a complex space form respectively. Therefore

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in order to determine all submanifolds in question we have only to classify the submanifolds of type $(C\text{–}H)$.

In §4, we give a characterization of a totally complex immersion into an n-dimensional quaternion projective space. Namely, associated with a totally complex immersion, there exists a Kaehler immersion into a $(2n+1)$-dimensional complex projective space whose composition with the projection of the complex projective space onto the quaternion projective space coincides with the given totally complex immersion (Theorem 4.1). In §6, we construct models of totally complex parallel immersions of Hermitian symmetric spaces into a quaternion projective space using the notion of the symplectic representations. For this purpose we review in §5 Kaehler immersions of Hermitian symmetric spaces into a complex projective space. In §7 we shall determine totally complex parallel immersions into a quaternion projective space and its non-compact dual. In fact, we show that there is no totally complex parallel immersion into the non-compact dual besides totally geodesic ones (Theorem 7.2) and that a totally complex parallel immersion of type $(C\text{–}H)$ into a quaternion projective space is one of the models constructed in §6 (Theorem 7.3).

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2. Quaternionic Kaehler manifolds and their submanifolds

Let $\boldsymbol{H}$ be the algebra of quaternions, i.e., $\boldsymbol{H}=\{\lambda=a+bi+cj+dk \mid a, b, c, d \in \boldsymbol{R},\ i^2=j^2=k^2=-1,\ ij=-ji=k,\ jk=-kj=i,\ ki=-ik=j\}$. Let V be an n-dimensional right vector space over $\boldsymbol{H}$ with a quaternion Hermitian inner product $(\ ,\ )$ such that $(u\lambda, v)=\bar{\lambda}(u, v)$ for $u, v\in V$, $\lambda\in\boldsymbol{H}$. We call this space a *quaternionic Hermitian vector space*. We shall treat this space also as a real vector space V endowed with an algebra A of linear transformations of V and a Euclidean inner product $\langle\ ,\ \rangle$ which satisfy the following condition:

(2.1) There is an algebra isomorphism of the algebra $\boldsymbol{H}$ of quaternions onto A such that the unit element of $\boldsymbol{H}$ corresponds to the identity transformation. Denote by A' the subspace of A spanned by the linear transformations which correspond to i, j, and k of $\boldsymbol{H}$ by this isomorphism. Then the Euclidean inner product $\langle , \rangle$ on V is invariant by any element L of A', i.e., $\langle Lu, v\rangle+\langle u, Lv\rangle=0$ for $u, v\in V$.

We note that the subspace A' of A defined in (2.1) does not depend on the choice of algebra isomorphisms of $\boldsymbol{H}$ onto A. In fact, it is easily verified that A' coincides with the subset consisting of elements $L\in A$ such that $L^2=-b\,id$ for some non-negative number b, where id denotes the identity transformation of V.

Lemma 2.1. *From a quaternionic Hermitian vector space, we can construct a real vector space with an algebra of real linear transformations and a Euclidean inner product which satisfy* (2.1). *Conversely given a triple* $(V, \langle,\rangle, A)$ *sfitisfying* (2.1), *we can define a structure of quaternionic Hermitian vector space on* V. *Moreover under this correspondence, the set of quaternion linear transformations which preserve the quaternicn Hermitian inner product* $(\, , \,)$ *coincides with that of real linear transformations which commute with each element of* A *and preserve the Euclidean inner product* $\langle,\rangle$.

Proof. Let V be a quaternionic Hermitian vector space with a quaternion Hermitian inner product $(\, , \,)$. We restrict the coefficient field to $\boldsymbol{R}$ and view V as a real vector space. Define real linear transformations $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$ by $\tilde{I}(u)=u(-i)$, $\tilde{J}(u)=u(-j)$, and $\tilde{K}(u)=u(-k)$ for $u\in V$ respectively, and define A to be the set of real linear transformations of V which are real linear combinations of $\tilde{I}$, $\tilde{J}$, $\tilde{K}$ and id. Putting $\langle u, v\rangle=$the real part of (u, v) for $u, v\in V$, we obtain a Euclidean inner product $\langle,\rangle$ on V. Then the set A and the Euclidean inner product $\langle,\rangle$ satisfy (2.1). Conversely let $(V, \langle,\rangle. A)$ be a triple which satisfies (2.1). Denote by $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$ the real linear transformations of V which correspond to i, j, and k respectively by an algebra isomorphism of $\boldsymbol{H}$ onto A. Define a quaternion scalar product by $u(a+bi+cj+dk)=(a-b\tilde{I}-c\tilde{J}-d\tilde{K})u$ for $u\in V$ and define a quaternion Hermitian inner product $(\, , \,)$ by $(u, v)=\langle u, v\rangle+\langle u, \tilde{I}v\rangle i+\langle u, \tilde{J}v\rangle j+\langle u, \tilde{K}v\rangle k$. Then V is a quaternionic Hermitian vector space with the quaternion Hermitian inner product $(\, , \,)$. The last statements are easily verified.

By this Lemma, we call a triple $(V, \langle,\rangle, A)$ satisfying (2.1) also a *quaternionic Hermitian vector space*.

DEFINITION 2.2. We call a basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of A' a *canonical basis* of A' if there is an algebra isomorphism of $\boldsymbol{H}$ onto A by which $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$ correspond to i, j, and k of $\boldsymbol{H}$ respectively.

Lemma 2.3. *Let* $(V, \langle,\rangle, A)$ *be a quaternionic Hermitian vector space with* $\dim_{\boldsymbol{R}} V=4n$ *and* A' *be the subspace of* A *defined in* (2.1). *Let* F *and* G *be mutually orthogonal elements of* A' *which have the same length* $\sqrt{4n}$ *with respect to the metric naturally induced from* $\langle,\rangle$. *Set* $H=FG$. *Then* $\{F, G, H\}$ *is a canonical basis of* A'.

Proof is easy and so is omitted.

Now we define quaternionic Kaehler manifolds and review their curvature properties (cf. Ishihara [8]).

DEFINITION 2.4. Let $\tilde{M}$ be a connected smooth manifold and $\tilde{g}$ and A' be a Riemannian metric of $\tilde{M}$ and a 3-dimensional subbundle of the vector bundle $\mathrm{Hom}(T\tilde{M}, T\tilde{M})$ over $\tilde{M}$ respectively, where $T\tilde{M}$ is the tangent vector bundle of $\tilde{M}$. Assume that $\tilde{g}$ and A' satisfy the following conditions:

(a) For an arbitrary point p of $\tilde{M}$ there is a neighborhood U of p over which there exists a local frame field $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ for A' satisfying

$$\begin{aligned}&\tilde{I}^2=\tilde{J}^2=\tilde{K}^2=-id,\ \tilde{I}\tilde{J}=-\tilde{J}\tilde{I}=\tilde{K},\\&\tilde{J}\tilde{K}=-\tilde{K}\tilde{J}=\tilde{I},\ \tilde{K}\tilde{I}=-\tilde{I}\tilde{K}=\tilde{J}.\end{aligned}$$

(b) For any element $L\in A'_p$, the metric $\tilde{g}_p$ is invariant by L, i.e., $\tilde{g}_p(Lu, v)+\tilde{g}_p(u, Lv)=0$ for $u, v\in T_p\tilde{M}$, $p\in\tilde{M}$.

(c) The vector bundle A' is parallel in $\mathrm{Hom}(T\tilde{M}, T\tilde{M})$ with respect to the Riemannian connection $\tilde{\nabla}$ associated with $\tilde{g}$. This means that if $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is a local frame field for A' over U which satisfies (a), then there exist local 1-forms α, β, and γ defined over U such that

$$\begin{aligned}\tilde{\nabla}_X\tilde{I}&= \qquad\qquad\ \gamma(X)\tilde{J}-\beta(X)\tilde{K}\\ \tilde{\nabla}_X\tilde{J}&=-\gamma(X)\tilde{I}\qquad\qquad\ +\alpha(X)\tilde{K}\\ \tilde{\nabla}_X\tilde{K}&=\ \ \beta(X)\tilde{I}-\alpha(X)\tilde{J}\,.\end{aligned}$$

Then the triple $(\tilde{M}, \tilde{g}, A')$ or simply $\tilde{M}$ is called a *quaternionic Kaehler manifold* and $(\tilde{g}, A')$ a *quaternionic Kaehler structure*. We call $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ in (a) a *local canonical basis* of A'.

Here we remark that, at any point $p\in\tilde{M}$, $(T_p\tilde{M}, \tilde{g}_p, A_p)$ is a quaternionic Hermitian vector space, where A_p denotes the algebra generated by A'_p and the identity transformation of $T_p\tilde{M}$.

A Riemannian manifold is a quaternionic Kaehler manifold if and only if its holonomy group is a subgroup of $Sp(1)\cdot Sp(n)$ (see [8], [20]). Wolf [20] classified symmetric spaces which have quaternionic Kaehler structures. An n-dimensional quaternion projective space and its non-compact dual are such spaces. Their quaternionic Kaehler structures will be given in §4.

We need following lemmas.

Lemma 2.5 ([8]). *The curvature tensor $\tilde{R}$ of a quaternion projective space or its non-compact dual is of the form*:

$$\begin{aligned}\tilde{R}(X, Y)Z=\frac{\tilde{c}}{4}\{&\tilde{g}(Y, Z)X-\tilde{g}(X, Z)Y+\tilde{g}(\tilde{I}Y, Z)\tilde{I}X-g(\tilde{I}X, Z)\tilde{I}Y+\\&\tilde{g}(\tilde{J}Y, Z)\tilde{J}X-\tilde{g}(\tilde{J}X, Z)\tilde{J}Y+\tilde{g}(\tilde{K}Y, Z)\tilde{K}X-\tilde{g}(\tilde{K}X, Z)\tilde{K}Y\\&-2\tilde{g}(\tilde{I}X, Y)\tilde{I}Z-2\tilde{g}(\tilde{J}X, Y)\tilde{J}Z-2\tilde{g}(\tilde{K}X, Y)\tilde{K}Z\}\,,\end{aligned}$$

where X, Y, Z are vector fields, $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is a local canonical basis and $\tilde{c}$ is a posi-

tive or negative constant according as the space is a quaternion projective space or its non-compact dual.

A complete and simply connected quaternionic Kaehler manifold whose curvature tensor is of the form given in Lemma 2.5 will be called a *quaternionic space form* and will be denoted by $\tilde{M}(\tilde{c})$.

Lemma 2.6 ([8]). *Let $\tilde{M}$ be a quaternionic Kaehler manifold with* $\dim_R \tilde{M} = 4m \geqq 8$. *Then $\tilde{M}$ is an Einstein space. Moreover if E, F, and G are the components of the curvature tensor $\tilde{R}$ acting on the vector bundle A' with respect to a local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$, i.e.,*

$$\begin{aligned}
\tilde{R}(X, Y)\tilde{I} &= \qquad\qquad\quad G(X, Y)\tilde{J} - F(X, Y)\tilde{K} \\
\tilde{R}(X, Y)\tilde{J} &= -G(X, Y)\tilde{I} \qquad\qquad\quad + E(X, Y)\tilde{K} \\
\tilde{R}(X, Y)\tilde{K} &= \quad F(X, Y)\tilde{I} - E(X, Y)\tilde{J},
\end{aligned}$$

then E, F, and G are as follows;

$$E = 2(d\alpha + \beta\wedge\gamma) = -\frac{\tilde{\tau}}{4m(m+2)}\Omega_{\tilde{I}}$$

$$F = 2(d\beta + \gamma\wedge\alpha) = -\frac{\tilde{\tau}}{4m(m+2)}\Omega_{\tilde{J}}$$

$$G = 2(d\gamma + \alpha\wedge\beta) = -\frac{\tilde{\tau}}{4m(m+2)}\Omega_{\tilde{K}}\,.$$

Here α, β, and γ are the connection forms with respect to this local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ (cf. Definition 2.4), $\tilde{\tau}$ denotes the scalar curvature of $\tilde{M}$, and $\Omega_{\tilde{I}}$, $\Omega_{\tilde{J}}$, and $\Omega_{\tilde{K}}$ denote local 2-forms defined by $\Omega_{\tilde{I}}(X, Y) = \tilde{g}(\tilde{I}X, Y)$, $\Omega_{\tilde{J}}(X, Y) = \tilde{g}(\tilde{J}X, Y)$, and $\Omega_{\tilde{K}}(X, Y) = \tilde{g}(\tilde{K}X, Y)$.

We shall define three kinds of immersions into a quaternionic Kaehler manifold. For this purpose, put.

DEFINITION 2.7 (Funabashi [6]). Let $(V, \langle,\rangle, A)$ be a quaternionic Hermitian vector space viewed as a real vector space and A' be the subspace of A spanned by $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$. Let W be a real subspace in V.

(i) W is called an *invariant* subspace if $L(W) \subset W$ for any $L \in A'$.

(ii) W is called a *totally complex* subspace if there exists a one-dimensional subspace A_0 of A' such that $L(W) \subset W$ for $L \in A_0$ and $L(W) \perp W$ for $L \in A'$ such that $L \perp A_0$.

(iii) W is called a *totally real* subspace if $L(W) \perp W$ for any $L \in A'$.

DEFINITION 2.8 ([6]). Let $\tilde{M}$ be a quaternionic Kaehler manifold and $f\colon M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold M into $\tilde{M}$.

(i) f is called an *invariant* immersion if $f_* T_p M$ is an invariant subspace of $T_{f(p)}\tilde{M}$ at any point $p\in M$.

(ii) f is called a *totally complex* immersion if f_*T_pM is a totally complex subspace of $T_{f(p)}\tilde{M}$ at any point $p\in M$.

(iii) f is called a *totally real* immersion if f_*T_pM is a totally real subspace of $T_{f(p)}\tilde{M}$ at any point $p\in M$.

REMARK. If f is invariant, then M is of dimension $4n$ and if f is totally complex, then M is of even dimension.

We shall discuss fundamental properties of these immersions. First the following fact is known.

Proposition 2.9 ([6]). *If f is an invariant immersion into a quaternionic Kaehler manifold, then f is totally geodesic.*

As for totally complex immersions, we have

Lemma 2.10. *Let $(\tilde{M}, \tilde{g}, A')$ be a quaternionic Kaehler manifold with $\dim_{\boldsymbol{R}}\tilde{M}=4m\geqq 8$ whose scalar curvature does not vanish. Let f be a totally complex immersion of a Riemannian manifold M with $\dim_{\boldsymbol{R}}M\geqq 4$ into $\tilde{M}$. Then for an arbitrary point $p\in M$, the induced bundle f^*A' has a local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ over some neighborhood U around p such that $\tilde{I}f_*T_qM=f_*T_qM$, $\tilde{J}f_*T_qM\perp f_*T_qM$, $\tilde{K}f_*T_qM\perp f_*T_qM$ at any point $q\in U$ and that $\tilde{\nabla}_X\tilde{I}=0$, $\tilde{\nabla}_X\tilde{J}=\alpha(X)\tilde{K}$, $\tilde{\nabla}_X\tilde{K}=-\alpha(X)\tilde{J}$. Here $\tilde{\nabla}$ denotes the connection on f^*A' induced from the connection on A', and X and α are a vector field and a 1-form on U respectively.*

Proof. By assumption we have the following orthogonal decomposition at any point $p\in M$;

$$A'_{f(p)} = A_0+A_1\,,$$

where A_0 is a one-dimensional subspace of $A'_{f(p)}$ and $Lf_*T_pM\subset f_*T_pM$ for $L\in A_0$, $Lf_*T_pM\perp f_*T_pM$ for $L\in A_1$. It is easily checked that A_0 and A_1 are subbundles of f^*A' and hence $f^*A'=A_0+A_1$ is an orthogonal decomposition of vector bundles. Over some neighborhood U of p we take a local section $\tilde{I}$ of A_0 which has the length $\sqrt{4m}$ and a local section $\tilde{J}$ of A_1 which has the length $\sqrt{4m}$. We set $\tilde{K}=\tilde{I}\tilde{J}$. Then by Lemma 2.3 $\tilde{K}$ is a local section of A_1 and $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is a local canonical basis of f^*A' over U. Moreover this basis satisfies $\tilde{I}f_*T_qM=f_*T_qM$, $\tilde{J}f_*T_qM\perp f_*T_qM$, $\tilde{K}f_*T_qM\perp f_*T_qM$ at any point $q\in U$. We denote by α, β, and γ the connection forms of the induced connection $\tilde{\nabla}$ on f^*A', and by E, F, and G the components of the curvature tensor $\tilde{R}$ with respect to this basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$. Then by Lemma 2.6 and by the above, we have

$$(2.2)\qquad E = 2(d\alpha+\beta\wedge\gamma) = -\frac{\tilde{\tau}}{4m(m+2)}\Omega_{\tilde{I}}$$

$$(2.3) \qquad F = 2(d\beta+\gamma\wedge\alpha) = 0$$

$$(2.4) \qquad G = 2(d\gamma+\alpha\wedge\beta) = 0\,.$$

Differentiating (2.4), we have $d\alpha\wedge\beta-\alpha\wedge d\beta=0$. Together with (2.2) and (2.3), it follows

$$\frac{\tilde{\tau}}{8m(m+2)}\Omega_{\tilde{I}}\wedge\beta = 0\,.$$

Since the scalar curvature $\tilde{\tau}$ of $\tilde{M}$ does not vanish, we have

$$(2.5) \qquad \Omega_{\tilde{I}}\wedge\beta = 0\,.$$

Now we shall show $\beta=0$ over U. Suppose that $\beta\neq0$ at a point $q\in U$. Since $\dim_{\boldsymbol{R}} M\geqq4$, there exist unit vectors X and Y of T_qM such that $X, \tilde{I}X\in\ker\beta$ and $Y\perp\ker\beta$, i.e., $\beta(Y)\neq0$. Substituting X, $\tilde{I}X$, and Y in (2.5), we have

$$\begin{aligned} 0 &= \Omega_{\tilde{I}}\wedge\beta(X, \tilde{I}X, Y) \\ &= \Omega_{\tilde{I}}(X, \tilde{I}X)\beta(Y)+\Omega_{\tilde{I}}(\tilde{I}X, Y)\beta(X)+\Omega_{\tilde{I}}(Y, X)\beta(\tilde{I}X) \\ &= \beta(Y)\,, \end{aligned}$$

which is a contradiction. Thus we have $\beta=0$ over U. Similarly we have $\gamma=0$. This proves Lemma.

REMARK (1) By Lemma 2.10, A_0 and A_1 are parallel subbundles of f^*A'. (2) This Lemma does not hold when $\dim_{\boldsymbol{R}} M=2$. Consider the Clifford torus of S^3. Regarding S^3 as a totally geodesic submanifold of S^4 and composing the totally geodesic imbedding of S^4 into a quaternion projective space, we have a totally complex immersion of the torus into a quaternion projective space. But the statements in Lemma 2.10 do not hold for this immersion.

Proposition 2.11. *Under the same assumptions as in Lemma* 2.10, *M admits locally a Kaehler structure I induced from the quaternionic Kaehler structure of $\tilde{M}$ and the fibres of the normal bundle $N(M)$ also admit locally a complex structure I such that $\nabla^{\perp}_X I=0$, where $\nabla^{\perp}$ denotes the connection of the normal bundle $N(M)$. Moreover if h is the second fundamental form of f, the equation $h(IX, Y)=h(X, IY)=Ih(X, Y)$ holds.*

Proof. Let $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ be a local canonical basis of f^*A' taken as in Lemma 2.10. Then the tensor field $\tilde{I}$ is an almost complex structure of the induced bundle $f^*T\tilde{M}$ on U and it is parallel with respect to the induced connection $\tilde{\nabla}$. Therefore $\tilde{I}$ induces complex structures in the fibres of the tangent bundle TM and of the normal bundle $N(M)$ over U, which we denote both by I. This almost complex structure I on M and the Riemannian metric define a Kaehler structure on U. The complex structure I of the normal bundle is parallel with respect to the normal connection $\nabla^{\perp}$. Moreover we have $h(IX, Y)=h(X, IY)=I\,h(X, Y)$.

In view of this proposition, we put the following.

DEFINITION 2.12. Let M be a Kaehler manifold with the complex structure I and $\tilde{M}$ be a quaternionic Kaehler manifold. An isometric immersion f of M into $\tilde{M}$ is called *a totally complex immersion of Kaehler type* if f is a totally complex immersion and at any point $p\in M$ there exists a canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of $A'_{f(p)}$ such that $f_*IX=\tilde{I}f_*X$ for $X\in T_pM$.

Under the same notation and assumptions as in Lemma 2.10, we define a tensor field T of (0, 3) over U by putting

$$T(x, y, z) = \langle h(x, y), \tilde{J}z\rangle \qquad \text{for } x, y, z\in T_pM \quad p\in U\,,$$

where $\langle , \rangle$ denotes the metric $\tilde{g}$ of $f^*T\tilde{M}$.

Lemma 2.13. (1) *T is a symmetric tensor field.*
(2) *$T(Ix, y, z)=T(x, Iy, z)=T(x, y, Iz)$ for any $x, y, z\in T_pM$.*

Proof. (1) For vector fields X, Y, and Z on U, we have

$$\begin{aligned} T(X, Y, Z) &= \langle h(X, Y), \tilde{J}Z\rangle = \langle \tilde{\nabla}_X Y, \tilde{J}Z\rangle = -\langle Y, \tilde{\nabla}_X(\tilde{J}Z)\rangle \\ &= -\langle Y, \tilde{J}\tilde{\nabla}_X Z+\alpha(X)\tilde{K}Z\rangle = -\langle Y, \tilde{J}\tilde{\nabla}_X Z\rangle \\ &= \langle \tilde{\nabla}_X Z, \tilde{J}Y\rangle = \langle h(X, Z), \tilde{J}Y\rangle = T(X, Z, Y)\,. \end{aligned}$$

Therefore T is symmetric. (2) By Proposition 2.11, we have $h(IX, Y)=h(X, IY)$. This, together with (1), implies that $T(IX, Y, Z)=T(X, IY, Z)=T(X, Y, IZ)$.

Now we discuss totally real immersions into a quaternionic Kaehler manifold.

Lemma 2.14. *Let f be a totally real immersion of a Riemannian manifold M into a quaternionic Kaehler manifold $(\tilde{M}, \tilde{g}, A')$ and h be the second fundamental form of f. Then for an arbitrary point $p\in M$, there exists a neighborhood U of p such that f^*A' has a parallel local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ over U. Moreover we have*

$$\begin{aligned} \langle h(x, y),\ \tilde{I}z\rangle &= \langle h(x, z), \tilde{I}y\rangle \\ \langle h(x, y),\ \tilde{J}z\rangle &= \langle h(x, z), \tilde{J}y\rangle \\ \langle h(x, y), \tilde{K}z\rangle &= \langle h(x, z), \tilde{K}y\rangle \end{aligned}$$

for any tangent vectors x, y, z at a point of U.

Proof. By Lemma 2.6, we see that the curvature tensor of f^*A' vanishes.

Therefore we can extend a canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of f^*A' at p over some neighborhood U of p so that $\tilde{\nabla}\tilde{I}=\tilde{\nabla}\tilde{J}=\tilde{\nabla}\tilde{K}=0$ on U. Since the connection $\tilde{\nabla}$ is a metric connection of f^*A', Lemma 2.3 implies that $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is a local canonical basis of f^*A' over U. The second part is proved by similar calculations as in Lemma 2.13. q.e.d.

We recall some results on curvature invariant subspaces of a quaternionic space form $\tilde{M}(\tilde{c})$. If a subspace W of the tangent space $T_p\tilde{M}$ of a Riemannian manifold $\tilde{M}$ at a point $p\in\tilde{M}$ satisfies $\tilde{R}(W, W)W\subset W$, where $\tilde{R}$ denotes the curvature tensor of $\tilde{M}$, then W is called a *curvature invariant subspace*.

Proposition 2.15 (Funabashi [6]). *Let $\tilde{M}(\tilde{c})$ be a quaternionic space form with $\tilde{c}\neq 0$ and with $\dim_{\mathbf{R}}\tilde{M}\geqq 8$. Then the subspace W of the tangent space $T_p\tilde{M}$ is a curvature invariant subspace if and only if W is one of the following*:
When $\dim_{\mathbf{R}} W\geqq 4$,

(1) *W is an invariant subspace*;
(2) *W is a totally complex subspace*;
(3) *W is a totally real subspace.*

When $\dim_{\mathbf{R}} W=3$,

(1) *W is a totally real subspace*;
(2) *$\{\tilde{I}, \tilde{J}, \tilde{K}\}$ being a canonical basis of A'_p, there exists a vector $X\in W$ such that W is linearly spanned by X, $\tilde{I}X$, and $\tilde{J}X$.*

When $\dim_{\mathbf{R}} W=2$,

(1) *W is a totally complex subspace*;
(2) *W is a totally real subspace.*

3. Reduction theorem

Let $\tilde{M}$ be an m-dimensional Riemannian manifold with the Riemannian connection $\tilde{\nabla}$ and M be an n-dimensional Riemannian manifold with the Riemannian connection ∇. We denote by $\tilde{R}$ and R the curvature tensor for $\tilde{\nabla}$ and ∇ respectively. Let f be an isometric immersion of M into $\tilde{M}$. We denote by h the second fundamental form of f, by $\nabla^{\perp}$ the normal connection on the normal bundle $N(M)$ of f, and by $R^{\perp}$ the curvature tensor for $\nabla^{\perp}$. For a point $p\in M$, we define the first normal space $N_p^1(M)$ and the first osculating space $O_p^1(M)$ as follows:

$$N_p^1(M) = \{h(X, Y)\in N_p(M),\ X, Y\in T_pM\}_{\mathbf{R}}$$
$$O_p^1(M) = T_pM+N_p^1(M)\,.$$

For the second fundamental form h, we define

$$\bar{\nabla}h(X, Y, Z) = \nabla_Z^{\perp}h(X, Y)-h(\nabla_Z X, Y)-h(X, \nabla_Z Y)\,,$$

where X, Y, and Z are vector fields on M. The isometric immersion f is called *parallel* if $\bar{\nabla} h=0$. If f is parallel and M is connected, the dimensions of $N_p^1(M)$ and $O_p^1(M)$ are constant on M. Therefore $N^1(M)=\bigcup_p N_p^1(M)$ and $O^1(M)=\bigcup_p O_p^1(M)$ are subbundles of the induced bundle $f^*T\tilde{M}$

Lemma 3.1 (Naitoh [11]). *If the immersion f is parallel and $\tilde{M}$ is locally symmetric, the following holds*:

(a) $\tilde{R}(X, Y)Z\in T_pM$

(b) $\tilde{R}(X, Y)h(T, Z)\in N_p^1(M)$

(c) $R^\perp(T, S)h(X, Y)=h(R(T, S)X, Y)+h(X, R(T, S)Y)$

(d) $h(T, \tilde{R}(X, Y)Z)=\tilde{R}(h(T, X), Y)Z+\tilde{R}(X, h(T, Y))Z+\tilde{R}(X, Y)h(T, Z)$

(e) $\tilde{R}(h(T, X), Y)h(S, Z)+\tilde{R}(X, h(T, Y))h(S, Z)\in O_p^1(M)$,

for $X, Y, Z, S, T\in T_pM$ and $p\in M$.

Proposition 3.2. *Let f be a parallel but not totally geodesic isometric immersion of a connected Riemannian manifold M* ($\dim M\geqq 2$) *into a quaternionic space form $\tilde{M}(\tilde{c})$, $\tilde{c}\neq 0$ with* $\dim_{\boldsymbol{R}}\tilde{M}(\tilde{c})\geqq 8$. *Then, for any point $p\in M$, the tangent space T_pM and the first osculating space $O_p^1(M)$ are curvature invariant subspaces in $T_p\tilde{M}(\tilde{c})$. Moreover the following cases occur*:

(a) *When* $\dim_{\boldsymbol{R}} M\geqq 4$,

	T_pM	$O_p^1(M)$
(*R–R*)	*totally real*	*totally real*
(*R–C*)	*totally real*	*totally complex*
(*C–C*)	*totally complex*	*totally complex*
(*C–H*)	*totally complex*	*invariant and* $\dim O_p^1(M)=2\dim T_pM$

(b) *When* $\dim_{\boldsymbol{R}} M=3$,

	T_pM	$O_p^1(M)$
(*R–R*)	*totally real*	*totally real*
(*R–C*)	*totally real*	*totally complex*
(*E–H*)	*E*	*invariant and* $\dim_{\boldsymbol{R}} O_p^1(M)=4$

(c) *When* $\dim_{\boldsymbol{R}} M=2$,

	T_pM	$O_p^1(M)$
(*R–R*)	*totally real*	*totally real*
(*R–C*)	*totally real*	*totally complex*
(*C–C*)	*totally complex*	*totally complex*
(*C–E*)	*totally complex*	*E*
(*C–H*)	*totally complex*	*invariant and* $\dim_{\boldsymbol{R}} O_p^1(M)=4$,

where E in (b) *means that for a fixed canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of A_p', there exists a vector $X\in T_pM$ such that T_pM is spanned by X, $\tilde{I}X$, and $\tilde{J}X$ and similarly E in* (c) *means that if we take a canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of A_p' such that $\tilde{I}\, T_pM$*

$=T_pM$, $\tilde{J}\,T_pM \perp T_pM$, *and* $\tilde{K}\,T_pM \perp T_pM$, *then there exists a vector* $X \in T_pM$ *such that* $O_p^1(M)$ *is spanned by* X, $\tilde{I}X$, *and* $\tilde{J}X$. *Moreover, each property ((R–R), (R–C), ⋯ etc) holds everywhere on* M *if it holds at one point of* M.

Proof. Note that the subsets of points $p \in M$ such that T_pM are totally real, totally complex, invariant, or E are respectively closed sets. Similarly the subsets of points $p \in M$ such that $O_p^1(M)$ are totally real, totally complex, invariant, or E are respectively closed sets. Thus the last claim follows.

By Lemma 3.1 (a), the tangent space T_pM is a curvature invariant subspace in $T_p\tilde{M}(\tilde{c})$. Curvature invariant subspaces are classified in Proposition 2.15 and by the above note, we see that each tangent space of M is a curvature invariant subspace of the same type. If T_pM is an invariant subspace in the quaternionic Hermitian vector space $T_p\tilde{M}(\tilde{c})$, the immersion f is invariant and thus totally geodesic by Proposition 2.9, which is the case excluded by assumption. Hence it is sufficient to consider the following four cases which may occur:

Case I: T_pM is a totally real subspace,
Case II: T_pM is a totally complex subspace with $\dim_{\mathbf{R}} T_pM \geqq 4$,
Case III: T_pM is a totally complex subspace with $\dim_{\mathbf{R}} T_pM = 2$,
Case IV: T_pM is a 3-dimensional subspace and for a fixed canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of A'_p, there exists a vector $X \in T_pM$ such that T_pM is spanned by X, $\tilde{I}X$, and $\tilde{J}X$. We shall treat these four cases separately.

Case I. We prepare three lemmas. Denote by $\langle , \rangle$ the Riemannian metric on $T_p\tilde{M}(\tilde{c})$ and let $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ be a canonical basis of A'_p.

Lemma 3.3. *In the Case I, the following equations hold*:

$$\begin{aligned}(3.1)\qquad &\langle \tilde{I}Y, H\rangle \tilde{I}X - \langle \tilde{I}X, H\rangle \tilde{I}Y + \langle \tilde{J}Y, H\rangle \tilde{J}X - \langle \tilde{J}X, H\rangle \tilde{J}Y + \\ &\langle \tilde{K}Y, H\rangle \tilde{K}X - \langle \tilde{K}X, H\rangle \tilde{K}Y \in N_p^1(M)\,,\end{aligned}$$

$$\begin{aligned}(3.2)\qquad &\langle \tilde{I}Y, H\rangle \tilde{I}h(T, X) - \langle \tilde{I}X, H\rangle \tilde{I}h(T, Y) + \langle \tilde{J}Y, H\rangle \tilde{J}h(T, X) - \\ &\langle \tilde{J}X, H\rangle \tilde{J}h(T, Y) + \langle \tilde{K}Y, H\rangle \tilde{K}h(T, X) - \langle \tilde{K}X, H\rangle \tilde{K}h(T, Y) - \\ &\langle \tilde{I}h(T, X), H\rangle \tilde{I}Y + \langle \tilde{I}h(T, Y), H\rangle \tilde{I}X - \langle \tilde{J}h(T, X), H\rangle \tilde{J}Y + \\ &\langle \tilde{J}h(T, Y), H\rangle \tilde{J}X - \langle \tilde{K}h(T, X), H\rangle \tilde{K}Y + \langle \tilde{K}h(T, Y), H\rangle \tilde{K}X \in O_p^1(M)\,,\end{aligned}$$

for X, Y, $T \in T_pM$ *and* $H \in N_p^1(M)$.

Proof. Applying Lemma 3.1 (b), we obtain (3.1). Applying Lemma 3.1 (e) and Lemma 2.14, we obtain (3.2). q.e.d.

Denote by $(N_p^1(M))^{\perp}$ the orthogonal complement of $N_p^1(M)$ in $N_p(M)$. For a normal vector $\xi \in N_p(M)$, we denote by ξ_a (resp. ξ_b) the $N_p^1(M)$-component (resp. $(N_p^1(M))^{\perp}$-component) of ξ.

Lemma 3.4. *For any element $L\in A'_p$, there exists a constant $k(L)$ such that $\langle (LX)_a, (LY)_a\rangle = k(L)\langle X, Y\rangle$ for $X, Y\in T_pM$.*

Proof. If $L=0$, then Lemma 3.4 holds trivially. If $L\neq 0$, for a suitable constant c, cL has the length $\sqrt{4n}$, where $\dim_R \tilde{M}(\tilde{c})=4n$. By Lemma 2.3, we can take a canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ such that $\tilde{I}=cL$. Therefore it is sufficient to prove Lemma 3.4 in the case of $L=\tilde{I}$ for an arbitrary canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$. By (3.1), we see that

$$\begin{aligned}&\langle(\tilde{I}Y)_a, H\rangle(\tilde{I}X)_b-\langle(\tilde{I}X)_a, H\rangle(\tilde{I}Y)_b+\langle(\tilde{J}Y)_a, H\rangle(\tilde{J}X)_b-\\ &\langle(\tilde{J}X)_a, H\rangle(\tilde{J}Y)_b+\langle(\tilde{K}Y)_a, H\rangle(\tilde{K}X)_b-\langle(\tilde{K}X)_a, H\rangle(\tilde{K}Y)_b=0\,.\end{aligned}$$

Putting $H=(\tilde{I}X)_a$ and taking the inner product of this equation with $(\tilde{I}X)_b$, we have

$$\begin{aligned}&\langle(\tilde{I}Y)_a,(\tilde{I}X)_a\rangle\langle(\tilde{I}X)_b,(\tilde{I}X)_b\rangle-\langle(\tilde{I}X)_a,(\tilde{I}X)_a\rangle\langle(\tilde{I}Y)_b,(\tilde{I}X)_b\rangle\\ &+\langle(\tilde{J}Y)_a,(\tilde{I}X)_a\rangle\langle(\tilde{J}X)_b,(\tilde{I}X)_b\rangle-\langle(\tilde{J}X)_a,(\tilde{I}X)_a\rangle\langle(\tilde{J}Y)_b,(\tilde{I}X)_b\rangle\\ &+\langle(\tilde{K}Y)_a,(\tilde{I}X)_a\rangle\langle(\tilde{K}X)_b,(\tilde{I}X)_b\rangle-\langle(\tilde{K}X)_a,(\tilde{I}X)_a\rangle\langle(\tilde{K}Y)_b,(\tilde{I}X)_b\rangle\\ =\;&\langle(\tilde{I}Y)_a,(\tilde{I}X)_a\rangle\{\langle X,X\rangle-\langle(\tilde{I}X)_a,(\tilde{I}X)_a\rangle\}-\langle(\tilde{I}X)_a,(\tilde{I}X)_a\rangle\{\langle Y,X\rangle-\langle\tilde{I}Y)_a,(\tilde{I}X)_a\rangle\}\\ &+\langle(\tilde{J}Y)_a,(\tilde{I}X)_a\rangle\{-\langle(\tilde{J}X)_a,(\tilde{I}X)_a\rangle\}-\langle(\tilde{J}X)_a,(\tilde{I}X)_a\rangle\{-\langle(\tilde{J}Y)_a,(\tilde{I}X)_a\rangle\}\\ &+\langle(\tilde{K}Y)_a,(\tilde{I}X)_a\rangle\{-\langle(\tilde{K}X)_a,(\tilde{I}X)_a\rangle\}-\langle(\tilde{K}X)_a,(\tilde{I}X)_a\rangle\{-\langle(\tilde{K}Y)_a,(\tilde{I}X)_a\rangle\}\\ =\;&\langle(\tilde{I}Y)_a,(\tilde{I}X)_a\rangle\langle X,X\rangle-\langle(\tilde{I}X)_a,(\tilde{I}X)_a\rangle\langle Y,X\rangle\\ =\;&0\,.\end{aligned}$$

If X is not zero, we get

$$\langle(\tilde{I}X)_a,(\tilde{I}Y)_a\rangle=\langle(\tilde{I}X)_a,(\tilde{I}X)_a\rangle\langle X,Y\rangle/\langle X,X\rangle$$

Since X and Y are arbitrary, the above equation implies that $(\tilde{I}X)_a$ and $(\tilde{I}Y)_a$ are orthogonal for any mutually orthogonal vectors X and Y. Therefore the value $\langle(\tilde{I}X)_a, (\tilde{I}X)_a\rangle/\langle X, X\rangle$ is independent on the choice of a non-zero vector X. Denote this value by $k(\tilde{I})$. Then we have $\langle(\tilde{I}X)_a, (\tilde{I}Y)_a\rangle=k(\tilde{I})\langle X, Y\rangle$ for $X, Y\in T_pM$. q.e.d.

Lemma 3.5. *The first normal space $N_p^1(M)$ is a totally real subspace in $T_p\tilde{M}(\tilde{c})$.*

Proof. Since T_pM is a totally real subspace, the immersion f is totally real on M. By Lemma 2.14, we can take a local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of f^*A' over some neighborhood U of p such that $\hat{\nabla}\tilde{I}=\hat{\nabla}\tilde{J}=\hat{\nabla}\tilde{K}=0$. Now we define a tensor field T of type (0, 3) over U by

$$T(X, Y, Z)=\langle h(X, Y), \tilde{I}Z\rangle\,.$$

Then by Lemma 2.14, T is symmetric. For vector field X, Y, Z, and V on

U, we have

$$
\begin{aligned}
(\nabla_V T)(X,Y,Z) &= V\,T(X,Y,Z) - T(\nabla_V X,Y,Z) - T(X,\nabla_V Y,Z) - T(X,Y,\nabla_V Z) \\
&= V\langle h(X,Y), \tilde{I}Z\rangle - \langle h(\nabla_V X,Y),\tilde{I}Z\rangle - \langle h(X,\nabla_V Y),\tilde{I}Z\rangle \\
&\quad - \langle h(X,Y),\tilde{I}(\nabla_V Z)\rangle \\
&= \langle \nabla_V^{\perp}(h(X,Y)),\tilde{I}Z\rangle + \langle h(X,Y),\nabla_V^{\perp}(\tilde{I}Z)\rangle \\
&\quad - \langle h(\nabla_V X,Y),\tilde{I}Z\rangle - \langle h(X,\nabla_V Y),\tilde{I}Z\rangle - \langle h(X,Y),\tilde{I}(\nabla_V Z)\rangle \\
&= \langle \bar{\nabla} h(X,Y,V),\tilde{I}Z\rangle + \langle h(X,Y),\tilde{\nabla}_V(\tilde{I}Z) - \tilde{I}(\nabla_V Z)\rangle \\
&= \langle h(X,Y),\tilde{I}(\tilde{\nabla}_V Z - \nabla_V Z)\rangle \\
&= \langle h(X,Y),\tilde{I}\,h(Z,V)\rangle .
\end{aligned}
$$

Thus we obtain $(\nabla_V T)(X,Y,Z)=\langle h(X,Y),\tilde{I}h(Z,V)\rangle$. Since $\langle h(X,Y),\tilde{I}h(Z,V)\rangle =\langle h(X,Y),\tilde{I}h(V,Z)\rangle$, we see that ∇T is a symmetric tensor field of type (0, 4). Moreover we have $(\nabla_X T)(X,X,X)=\langle h(X,X),\tilde{I}h(X,X)\rangle=0$ for $X\in T_pM$ and hence $\nabla T=0$. This implies that $\langle h(X,Y),\tilde{I}h(Z,V)\rangle=0$ for X, Y, Z, and $V\in T_pM$. Therefore we have $\tilde{I}\,N_p^1(M)$ is orthogonal to $N_p^1(M)$. Similarly we see that $\tilde{J}N_p^1(M)$ and $\tilde{K}\,N_p^1(M)$ are orthogonal to $N_p^1(M)$. Consequently $N_p^1(M)$ is totally real. q.e.d.

Now consider Case I. For a fixed unit vector $X\in T_pM$ we define $A_p'(X)$ by the subspace of $N_p(M)$ linearly spanned by LX, $L\in A_p'$. Since $\dim_{\mathbf{R}} A_p'(X) =3$, the following four cases may occur:

Case 1: $\dim A_p'(X)\cap N_p^1(M)=3$,
Case 2: $\dim A_p'(X)\cap N_p^1(M)=2$,
Case 3: $\dim A_p'(X)\cap N_p^1(M)=1$
Case 4: $\dim A_p'(X)\cap N_p^1(M)=0$.

Case 1 and Case 2. These cases do not occur.

Proof. We may take a canonical basis $\{\tilde{I},\tilde{J},\tilde{K}\}$ of A_p' such that $\tilde{I}X$ and $\tilde{J}X$ are included in $N_p^1(M)$. By Lemma 3.5 we see that $\tilde{J}X=\tilde{K}(\tilde{I}X)$ is orthogonal to $N_p^1(M)$, which is a contradiction.

Case 3. The first osculating space $O_p^1(M)$ is a totally complex subspace in $T_p\tilde{M}(\tilde{c})$.

Proof. We may take $\tilde{I}\in A_p'$ such that $\tilde{I}X\in N_p^1(M)$ and $\tilde{I}$ has the length $\sqrt{4n}$. Since $\langle(\tilde{I}X)_a,(\tilde{I}X)_a\rangle=\langle\tilde{I}X,\tilde{I}X\rangle=\langle X,X\rangle$, the constant $k(\tilde{I})$ defined in Lemma 3.4 is equal to 1. This implies that $\tilde{I}(T_pM)$ is contained in $N_p^1(M)$. We take $\tilde{J}\in A_p'$ which has the length $\sqrt{4n}$ and is orthogonal to $\tilde{I}$ and put $\tilde{K}$

The following proof of Case 3 and Case 4 is suggested by Dr. H. Naitoh. The original proof was more complicated.

$=\tilde{I}\tilde{J}$. Then by Lemma 2.3, $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is a canonical basis of A_p'. Since $\tilde{I}X \in N_p^1(M)$ and $N_p^1(M)$ is totally real, we have $(\tilde{J}X)_a=(\tilde{K}(\tilde{I}X))_a=0$ and $(\tilde{K}X)_a=(-\tilde{J}(\tilde{I}X))_a=0$. Hence by Lemma 3.4 we see that $\tilde{J}T_pM \perp N_p^1(M)$, $\tilde{K}T_pM \perp N_p^1(M)$. By (3.2) we have $\langle\tilde{I}Y, H\rangle\tilde{I}h(T, Z)-\langle\tilde{I}Z, H\rangle\tilde{I}h(T, Y)\in O_p^1(M)$. For an arbitrary vector $Z\in T_pM$, we take an orthogonal vector Y to Z. Putting $H=\tilde{I}Y$ in the above equation we get $\tilde{I}h(T, Z)\in O_p^1(M)$. Thus we see that $\tilde{I}O_p^1(M)\subset O_p^1(M)$, $\tilde{J}O_p^1(M)\perp O_p^1(M)$, and $\tilde{K}O_p^1(M)\perp O_p^1(M)$. This means that $O_p^1(M)$ is a totally complex subspace in $T_p\tilde{M}(\tilde{c})$.

Case 4. The first osculating space $O_p^1(M)$ is a totally real subspace in $T_p\tilde{M}(\tilde{c})$.

Proof. By (3.1), we have

$$\begin{aligned}(3.3)\qquad &\langle(\tilde{I}Y)_a, H\rangle(\tilde{I}X)_b-\langle(\tilde{I}X)_a, H\rangle(\tilde{I}Y)_b+\langle(\tilde{J}Y)_a, H\rangle(\tilde{J}X)_b\\ &-\langle(\tilde{J}X)_a, H\rangle(\tilde{J}Y)_b+\langle(\tilde{K}Y)_a, H\rangle(\tilde{K}X)_b-\langle(\tilde{K}X)_a, H\rangle(\tilde{K}Y)_b = 0\end{aligned}$$

for $X, Y\in T_pM$, $H\in N_p^1(M)$. Since $N_p^1(M)$ is totally real by Lemma 3.5, we apply $\tilde{I}$ to (3.1) and get

$$\begin{aligned}&-\langle(\tilde{I}Y)_a, H\rangle X+\langle(\tilde{I}X)_a, H\rangle Y+\langle(\tilde{J}Y)_a, H\rangle\tilde{K}X-\langle(\tilde{J}X)_a, H\rangle\tilde{K}Y\\ &-\langle(\tilde{K}Y)_a, H\rangle\tilde{J}X+\langle(\tilde{K}X)_a, H\rangle\tilde{J}Y\in T_pM+(N_p^1(M))^{\perp}\,.\end{aligned}$$

Therefore we have

$$\begin{aligned}(3.4)\qquad &\langle(\tilde{J}Y)_a, H\rangle(\tilde{K}X)_a-\langle(\tilde{J}X)_a, H\rangle(\tilde{K}Y)_a\\ &\qquad -\langle(\tilde{K}Y)_a, H\rangle(\tilde{J}X)_a+\langle(\tilde{K}X)_a, H\rangle(\tilde{J}Y)_a = 0\,.\end{aligned}$$

Similarly applying $\tilde{J}$ and $\tilde{K}$ to (3.1), we have

$$\begin{aligned}(3.5)\qquad &\langle(\tilde{K}Y)_a, H\rangle(\tilde{I}X)_a-\langle(\tilde{K}X)_a, H\rangle(\tilde{I}Y)_a\\ &\qquad -\langle(\tilde{I}Y)_a, H\rangle(\tilde{K}X)_a+\langle(\tilde{I}X)_a, H\rangle(\tilde{K}Y)_a = 0\end{aligned}$$

$$\begin{aligned}(3.6)\qquad &\langle(\tilde{I}Y)_a, H\rangle(\tilde{J}X)_a-\langle(\tilde{I}X)_a, H\rangle(\tilde{J}Y)_a\\ &\qquad -\langle(\tilde{J}Y)_a, H\rangle(\tilde{I}X)_a+\langle(\tilde{J}X)_a, H\rangle(\tilde{I}Y)_a = 0\,.\end{aligned}$$

We take an orthonormal system $\{X, Y\}$. Putting $H=(\tilde{J}X)_a$ in (3.4) and taking the inner product of its both sides with $(\tilde{J}X)_a$, we have

$$(3.7)\qquad \langle(\tilde{K}Y)_a, (\tilde{J}X)_a\rangle\langle(\tilde{J}X)_a, (\tilde{J}X)_a\rangle = 0\,.$$

Commuting X and Y in (3.7), we have

$$(3.8)\qquad \langle(\tilde{K}X)_a, (\tilde{J}Y)_a\rangle\langle(\tilde{J}Y)_a, (\tilde{J}Y)_a\rangle = 0\,.$$

Putting $H=(\tilde{I}X)_a$ in (3.5) and taking the inner product with $(\tilde{I}X)_a$, we have

$$(3.9)\qquad \langle(\tilde{K}Y)_a, (\tilde{I}X)_a\rangle\langle(\tilde{I}X)_a, (\tilde{I}X)_a\rangle = 0$$

and commuting X and Y, we get

$$(3.10)\qquad \langle(\tilde{K}X)_a, (\tilde{I}Y)_a\rangle\langle(\tilde{I}Y)_a, (\tilde{I}Y)_a\rangle = 0$$

Putting $H=(\tilde{I}X)_a$ in (3.6), we have similarly

$$(3.11)\qquad \langle(\tilde{J}Y)_a, (\tilde{I}X)_a\rangle\langle(\tilde{I}X)_a, (\tilde{I}X)_a\rangle = 0\,,$$

$$(3.12)\qquad \langle(\tilde{J}X)_a, (\tilde{I}Y)_a\rangle\langle(\tilde{I}Y)_a, (\tilde{I}Y)_a\rangle = 0\,.$$

If $\dim A'_p(X)\cap N^1_p(M)=0$ for some non-zero vector $X\in T_pM$, then $\dim A'_p(Y)\cap N^1_p(M)=0$ for any $Y\in T_pM$. In fact for an element $L\in A'_p$ of length $\sqrt{4n}$, we have $(LX)_b\neq 0$ and hence the constant $k(L)$ defined in Lemma 3.4 is not 1. Therefore in Case 4 $(\tilde{I}Y)_b$, $(\tilde{J}Y)_b$, and $(\tilde{K}Y)_b$ are linearly independent for any non-zero vector $Y\in T_pM$. Assume that $k(\tilde{I})\neq 0$ and $k(\tilde{J})\neq 0$. Then by (3.7)$\sim$(3.12) we have $\langle(\tilde{K}Y)_a, (\tilde{J}X)_a\rangle=\langle(\tilde{J}Y)_a, (\tilde{K}X)_a\rangle=\langle(\tilde{K}Y)_a, (\tilde{I}X)_a\rangle=\langle(\tilde{I}Y)_a, (\tilde{K}X)_a\rangle=\langle(\tilde{J}Y)_a, (\tilde{I}X)_a\rangle=\langle(\tilde{I}Y)_a, (\tilde{J}X)_a\rangle=0$ for an orthonormal system $\{X, Y\}$. Since $0=\langle\tilde{K}Y, \tilde{J}X\rangle=\langle(\tilde{K}Y)_a, (\tilde{J}X)_a\rangle+\langle(\tilde{K}Y)_b, (\tilde{J}X)_b\rangle$, we have $\langle(\tilde{K}Y)_b, (\tilde{J}X)_b\rangle=0$. Similarly we get $\langle(\tilde{J}Y)_b, (\tilde{K}X)_b\rangle=\langle(\tilde{K}Y)_b, (\tilde{I}X)_b\rangle=\langle(\tilde{I}Y)_b, (\tilde{K}X)_b\rangle=\langle(\tilde{J}Y)_b, (\tilde{I}X)_b\rangle=\langle(\tilde{I}Y)_b, (\tilde{J}X)_b\rangle=0$. Therefore the subspace in $N_p(M)$ spanned by $(\tilde{I}X)_b$, $(\tilde{J}X)_b$, and $(\tilde{K}X)_b$ is orthogonal to the subspace spanned by $(\tilde{I}Y)_b$, $(\tilde{J}Y)_b$, and $(\tilde{K}Y)_b$. Thus $(\tilde{I}X)_b$, $(\tilde{J}X)_b$, $(\tilde{K}X)_b$, $(\tilde{I}Y)_b$, $(\tilde{J}Y)_b$, and $(\tilde{K}Y)_b$ are linearly independent. By (3.3) it follows then $(\tilde{I}X)_a=0$, which is a contradiction. Next assume that $k(\tilde{I})\neq 0$, and $k(\tilde{J})=k(\tilde{K})=0$. Then by (3.3) we have $(\tilde{I}X)_a=0$, which is a contradiction. By these arguments in Case 4 LT_pM is orthogonal to $N^1_p(M)$ for any $L\in A'_p$. This, together with Lemma 3.5, implies that $O^1_p(M)$ is a totally real subspace in $T_p\tilde{M}(\tilde{c})$.

Case II. Since T_pM is a totally complex subspace, the immersion f is totally complex on M. Let $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ be a local canonical basis of f^*A' over some neighborhood U of p taken as in Lemma 2.10. Note that $\tilde{I}N^1_p(M)=N^1_p(M)$ by Proposition 2.11.

Applying Lemma 3.1 (b) and (e), we have

Lemma 3.6. *In the Case II, the following equations hold*:

$$(3.13)\qquad \langle\tilde{J}Y, H\rangle\tilde{J}X-\langle\tilde{J}X, H\rangle\tilde{J}Y+\langle\tilde{K}Y, H\rangle\tilde{K}X-\langle\tilde{K}X, H\rangle\tilde{K}Y\in N^1_p(M)$$

$$\begin{aligned}(3.14)\qquad &\langle\tilde{J}Y, H\rangle\tilde{J}h(T, X)-\langle\tilde{J}h(T, X), H\rangle\tilde{J}Y+\langle\tilde{K}Y, H\rangle\tilde{K}h(T, X)\\ &-\langle\tilde{K}h(T, X), H\rangle\tilde{K}Y-2\langle\tilde{J}h(T, X), Y\rangle\tilde{J}H-2\langle\tilde{K}h(T, X), Y\rangle\tilde{K}H\\ &-\langle\tilde{J}X, H\rangle\tilde{J}h(T, Y)+\langle\tilde{J}h(T, Y), H\rangle\tilde{J}X-\langle\tilde{K}X, H\rangle\tilde{K}h(T, Y)\\ &+\langle\tilde{K}h(T, Y), H\rangle\tilde{K}X+2\langle\tilde{J}h(T, Y), X\rangle\tilde{J}H+2\langle\tilde{K}h(T, Y), X\rangle\tilde{K}H\in O^1_p(M)\end{aligned}$$

for X, Y, $T\in T_pM$, $H\in N^1_p(M)$.

Similarly to Lemma 3.4, we may prove

Lemma 3.7. *There exists a constant k such that $\langle(\tilde{J}X)_a, (\tilde{J}Y)_a\rangle=\langle(\tilde{K}X)_a, (\tilde{K}Y)_a\rangle=k\langle X, Y\rangle$ for $X, Y\in T_pM$.*

Lemma 3.8. *The first normal space $N_p^1(M)$ is a totally complex subspace, i.e., $\tilde{I}N_p^1(M)=N_p^1(M)$, $\tilde{J}N_p^1(M)\perp N_p^1(M)$, $\tilde{K}N_p^1(M)\perp N_p^1(M)$.*

Proof. We define a tensor field T of type (0, 3) as Lemma 2.13. Differentiating T, we obtain

$$(\nabla_V T)(X, Y, Z) = -\alpha(V)T(X, Y, IZ)+\langle h(X, Y), \tilde{J}h(Z, V)\rangle .$$

Using the same arguments as Lemma 3.5, we have $\langle h(X, Y), \tilde{J}h(Z, V)\rangle=0$. Therefore $\tilde{J}N_p^1(M)=\tilde{K}N_p^1(M)$ is orthogonal to $N_p^1(M)$. q.e.d.

By Lemma 3.7, the following two cases may occur;
Case 1: $k=1$,
Case 2: $k<1$.

Case 1. The first osculating space $O_p^1(M)$ is an invariant subspace in $T_p\tilde{M}(\tilde{c})$ and the dimension of $O_p^1(M)$ is twice that of T_pM.

Proof. By Lemma 3.7, the subspace $\tilde{J}T_pM=\tilde{K}T_pM$ of $N_p(M)$ is contained in $N_p^1(M)$. For an arbitrary vector $X\in T_pM$, we take a non-zero vector $Y\in T_pM$ which is orthogonal to X and $\tilde{I}X$. Putting $H=\tilde{J}Y$ in (3.14), we have

$$\langle Y, Y\rangle\tilde{J}h(T, X)\in O_p^1(M) .$$

Therefore $\tilde{J}N_p^1(M)=\tilde{K}N_p^1(M)$ is contained in $O_p^1(M)$. Consequently the first osculating space $O_p^1(M)$ is an invariant subspace. Moreover Lemma 3.8 implies that $\tilde{J}N_p^1(M)$ is contained in T_pM. Therefore the dimension of $O_p^1(M)$ is twice that of T_pM.

Case 2. The first osculating space $O_p^1(M)$ is a totally complex subspace in $T_p\tilde{M}(\tilde{c})$.

Proof. By Lemma 3.7, we have

$$\langle(\tilde{J}X)_b, (\tilde{J}Y)_b\rangle = \langle X, Y\rangle-\langle(\tilde{J}X)_a, (\tilde{J}Y)_a\rangle = (1-k)\langle X, Y\rangle .$$

Therefore $(\tilde{J}X)_b$, $(\tilde{J}Y)_b$, $(\tilde{K}X)_b=-(\tilde{J}(\tilde{I}X))_b$, and $(\tilde{K}Y)_b=-(\tilde{J}(\tilde{I}Y))_b$ are linearly independent for an orthonormal system $\{X, \tilde{I}X, Y, \tilde{I}Y\}$. By (3.13), we have

$$\langle\tilde{J}Y,H\rangle(\tilde{J}X)_b-\langle\tilde{J}X,H\rangle(\tilde{J}Y)_b+\langle\tilde{K}Y,H\rangle(\tilde{K}X)_b-\langle\tilde{K}X,H\rangle(\tilde{K}Y)_b = 0 .$$

Therefore we get $(\tilde{K}X)_a=0$ and hence $k=0$. This means that $\tilde{J}T_pM=\tilde{K}T_pM$

is orthogonal to $N_p^1(M)$. Thus $O_p^1(M)$ is a totally complex subspace.

Case III. Since T_pM is a totally complex subspace, the immersion f is totally complex on M. Then on some neighborhood U of p the induced bundle f^*A' has a local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ such that at any point $q \in U$, $\tilde{I}T_qM = T_qM$, $\tilde{J}T_qM \perp T_qM$, $\tilde{K}T_qM \perp T_qM$ and that

$$\begin{aligned}\tilde{\nabla}_X\tilde{I} &= \qquad\qquad \gamma(X)\tilde{J}-\beta(X)\tilde{K}\\ \tilde{\nabla}_X\tilde{J} &= -\gamma(X)\tilde{I} \qquad\qquad +\alpha(X)\tilde{K}\\ \tilde{\nabla}_X\tilde{K} &= \beta(X)\tilde{I}-\alpha(X)\tilde{J}.\end{aligned}$$

We remark that β and γ do not vanish in general when $\dim_{\boldsymbol{R}} M=2$. By usual computations, we obtain

$$h(X, \tilde{I}Y)-\tilde{I}h(X, Y) = \gamma(X)\tilde{J}Y-\beta(X)\tilde{K}Y \qquad \text{for } X, Y \in T_pM. \tag{3.15}$$

Applying Lemma 3.1 (b) and (e), we have

Lemma 3.9. *In Case III, we have*

$$\langle \tilde{J}Y, H\rangle \tilde{J}X-\langle \tilde{J}X, H\rangle \tilde{J}Y+\langle \tilde{K}Y, H\rangle \tilde{K}X-\langle \tilde{K}X, H\rangle \tilde{K}Y -2\langle \tilde{I}X, Y\rangle \tilde{I}H \in N_p^1(M), \tag{3.16}$$

$$\begin{aligned}&\langle \tilde{J}Y, H\rangle \tilde{J}h(T, X)-\langle \tilde{J}h(T, X), H\rangle \tilde{J}Y+\langle \tilde{K}Y, H\rangle \tilde{K}h(T, X)\\ &-\langle \tilde{K}h(T, X), H\rangle \tilde{K}Y-2\langle \tilde{J}h(T, X), Y\rangle \tilde{J}H-2\langle \tilde{K}h(T, X), Y\rangle \tilde{K}H\\ &-\langle \tilde{J}X, H\rangle \tilde{J}h(T, Y)+\langle \tilde{J}h(T, Y), H\rangle \tilde{J}X-\langle \tilde{K}X, H\rangle \tilde{K}h(T, Y)\\ &+\langle \tilde{K}h(T, Y), H\rangle \tilde{K}X+2\langle \tilde{J}h(T, Y), X\rangle \tilde{J}H+2\langle \tilde{K}h(T, Y), X\rangle \tilde{K}H \in O_p^1(M),\end{aligned} \tag{3.17}$$

for X, Y, $T \in T_pM$ and $H \in N_p^1(M)$.

We consider the following three cases:
Case 1: $\dim \tilde{J}T_pM \cap N_p^1(M)=2$,
Case 2: $\dim \tilde{J}T_pM \cap N_p^1(M)=1$,
Case 3: $\dim \tilde{J}T_pM \cap N_p^1(M)=0$.

Case 1. The first osculating space $O_p^1(M)$ is an invariant subspace and its dimension is 4.

Proof. In this case $\tilde{J}T_pM$ is contained in $N_p^1(M)$ and hence by (3.16) we have $\tilde{I}(N_p^1(M))=N_p^1(M)$. Since the dimension of $N_p^1(M)$ is even and is not greater than 3, it is just 2 and the first normal space $N_p^1(M)$ coincides with $\tilde{J}T_pM$. Thus $O_p^1(M)$ is an invariant subspace.

Case 2. There exists a vector $X \in T_pM$ such that $O_p^1(M)$ is spanned by X, $\tilde{I}X$, and $\tilde{J}X$.

Proof. We fix a unit vector $X \in T_pM$ such that $\tilde{J}X$ is an element of $N_p^1(M)$ and $\tilde{J}(\tilde{I}X) = -\tilde{K}X$ is not contained in $N_p^1(M)$. Putting $Y = \tilde{I}X$ in (3.16), we obtain

$$\langle \tilde{J}X, H \rangle \tilde{K}X - \tilde{I}H \in N_p^1(M) . \tag{3.18}$$

At first we shall show that $\tilde{K}X$ is orthogonal to $N_p^1(M)$. Assume that the $N_p^1(M)$-component $(\tilde{K}X)_a$ of $\tilde{K}X$ does not vanish. By (3.18), $\tilde{I}(\tilde{K}X)_a$ is contained in $N_p^1(M)$. Putting $H = \tilde{I}(\tilde{K}X)_a$ in (3.18), we see that $\langle \tilde{J}X, \tilde{I}(\tilde{K}X)_a \rangle \tilde{K}X = -\|(\tilde{K}X)_a\|^2 \tilde{K}X$ is an element of $N_p^1(M)$. This contradicts the assumption of Case 2.

Putting $Y = \tilde{I}X$ and $H = \tilde{J}X$ in (3.17), we obtain

$$\tilde{K}h(T, X) - \tilde{J}h(T, \tilde{I}X) \in O_p^1(M) . \tag{3.19}$$

Using (3.15), we have

$$\tilde{K}h(T, X) - \tilde{J}h(T, \tilde{I}X) = 2\tilde{K}h(T, X) + \gamma(T)X + \beta(T)\tilde{I}X$$

and hence $\tilde{K}h(T, X)$ is contained in $O_p^1(M)$ for any $T \in T_pM$. We shall prove that the first normal space $N_p^1(M)$ is spanned by $\tilde{J}X$. Assume that there exists a non-zero vector H in $N_p^1(M)$ which is orthogonal to $\tilde{J}X$. Then by (3.18), $\tilde{I}H$ is also contained in $N_p^1(M)$. Since $\tilde{J}X$, H, and $\tilde{I}H$ are mutually orthogonal, the dimension of $N_p^1(M)$ is equal to 3. Therefore there exists a vector $T \in T_pM$ such that $h(T, X) - \langle h(T, X), \tilde{J}X \rangle \tilde{J}X$ is not zero, which is denoted by H. Then H is an element of $N_p^1(M)$ which is orthogonal to $\tilde{J}X$. Since $\tilde{K}h(T, X)$ is contained in $O_p^1(M)$, so is $\tilde{K}H$. Moreover $\tilde{K}H$ is orthogonal to T_pM. Thus $\tilde{J}X$, H, $\tilde{I}H$, and $\tilde{K}H$ are mutually orthogonal vectors in $N_p^1(M)$. This is a contradiction. By the above arguments, we see that $O_p^1(M)$ is linearly spanned by X, $\tilde{I}X$, and $\tilde{J}X$.

Case 3. The first osculating space $O_p^1(M)$ is a totally complex subspace.

Proof. Putting $Y = \tilde{I}X$ in (3.16), we obtain

$$-\langle \tilde{K}X, H \rangle \tilde{J}X + \langle \tilde{J}X, H \rangle \tilde{K}X - \tilde{I}H \in N_p^1(M) \tag{3.20}$$

for any unit vector $X \in T_pM$ and $H \in N_p^1(M)$.
Putting $H = h(X, X)$ in (3.20) and using (3.15), we have

$$\{-\langle \tilde{K}X, h(X, X) \rangle + \gamma(X)\} \tilde{J}X + \{\langle \tilde{J}X, h(X, X) \rangle - \beta(X)\} \tilde{K}X \in N_p^1(M) .$$

Since $(\tilde{J}X)_b$ and $(\tilde{K}X)_b$ are linearly independent, we have

$$\langle \tilde{K}X, h(X, X) \rangle = \gamma(X) \quad \text{and} \quad \langle \tilde{J}X, h(X, X) \rangle = \beta(X) .$$

Putting $H = h(X, \tilde{I}X)$ in (3.20) and calculating similarly, we have

$$\langle \tilde{K}X, h(X, \tilde{I}X)\rangle = -\beta(X) \text{ and } \langle \tilde{J}X, h(X, \tilde{I}X)\rangle = \gamma(X).$$

These equations and (3.15) imply that $\beta(X)=\gamma(X)=0$ for any unit vector $X \in T_pM$. Therefore by (3.15), $N_p^1(M)$ is invariant by $\tilde{I}$. $\tilde{J}T_pM=\tilde{K}T_pM$ is orthogonal to $N_p^1(M)$. By (3.17), $\tilde{J}N_p^1(M)=\tilde{K}N_p^1(M)$ is orthogonal to $N_p^1(M)$. Thus $O_p^1(M)$ is a totally complex subspace.

Case IV. We fix a unit vector $X \in T_pM$ such that T_pM is spanned by X, $\tilde{I}X$, and $\tilde{J}X$. Applying Lemma 3.1 (b), we have

$$(3.21) \quad \langle \tilde{K}X, H\rangle \tilde{J}X + \tilde{I}H \in N_p^1(M)$$

$$(3.22) \quad \langle \tilde{K}X, H\rangle \tilde{I}X - \tilde{J}H \in N_p^1(M)$$

$$(3.23) \quad \langle \tilde{K}X, H\rangle X + \tilde{K}H \in N_p^1(M) \qquad \text{for } H \in N_p^1(M).$$

We consider the following two cases:
Case 1: $\tilde{K}X \in N_p^1(M)$,
Case 2: $\tilde{K}X \notin N_p^1(M)$.

Case 1. The first normal space $N_p^1(M)$ is linearly spanned by $\tilde{K}X$ and hence the first osculating space $O_p^1(M)$ is spanned by X, $\tilde{I}X$, $\tilde{J}X$, and $\tilde{K}X$.

Proof. Assume that there exists a non-zero vector H in $N_p^1(M)$ which is orthogonal to $\tilde{K}X$. Then by (3.21), (3.22), and (3.23), $\tilde{I}H$, $\tilde{J}H$, and $\tilde{K}H$ are also contained in $N_p^1(M)$. We can show that the equation of Lemma 3.1 (d) does not hold under this situation. Therefore $N_p^1(M)$ is linearly spanned by $\tilde{K}X$.

Case 2. This case does not occur.

Proof. At first we shall prove that $\tilde{K}X$ is orthogonal to $N_p^1(M)$. Assume that the $N_p^1(M)$-component $(\tilde{K}X)_a$ of $\tilde{K}X$ does not vanish. Let H be an arbitrary vector of $N_p^1(M)$ which is orthogonal to $(\tilde{K}X)_a$. Then by (3.21), $\tilde{I}H$ is contained in $N_p^1(M)$. Since $\langle \tilde{I}(\tilde{K}X)_a, H\rangle = -\langle (\tilde{K}X)_a, \tilde{I}H\rangle = \langle \tilde{K}X, \tilde{I}H\rangle = \langle \tilde{J}X, H\rangle = 0$, $\tilde{I}(\tilde{K}X)_a$ is orthogonal to $N_p^1(M)$. Then by (3.21), we have $\tilde{I}(\tilde{K}X)_a = -\|(\tilde{K}X)_a\|^2 \tilde{J}X$ and hence $\tilde{K}X = (\tilde{K}X)_a / \|(\tilde{K}X)_a\|^2$. This contradicts the assumption of Case 2. Therefore $\tilde{K}X$ is orthogonal to $N_p^1(M)$. By the same way as in Case 1, we can show that the equation of Lemma 3.1 (d) does not hold under this situation. Therefore this case does not occur.

Thus the proof of Proposition 3.2 is completely finished.

Theorem 3.10 (Reduction theorem). *Let f be a parallel isometric immersion of a connected Riemannian manifold M ($\dim M \geqq 2$) into a quaternionic space form $\tilde{M}(\tilde{c})$, $\tilde{c} \neq 0$ with $\dim_{\mathbf{R}} \tilde{M}(\tilde{c}) \geqq 8$. Then there exists a unique complete con-*

nected totally geodesic submanifold N *of* $\tilde{M}(\tilde{c})$ *such that the image* $f(M)$ *is contained in* N *and* $T_pN=O_p^1(M)$, $p\in M$. *Moreover the following cases occur*:

When $\dim_{\boldsymbol{R}} M\geqq 4$,

(R–R) *The submanifold* N *is a real projective space or a real hyperbolic space with sectional curvature* $\tilde{c}/4$ *according as* $\tilde{c}>0$ *or* $\tilde{c}<0$,

(R–C) *The submanifold* N *is a complex projective space or a complex hyperbolic space with the holomorphic sectional curvature* $\tilde{c}$ *according as* $\tilde{c}>0$ *or* $\tilde{c}<0$ *and* f *is totally real into* N,

(C–C) *The submanifold* N *is a complex projective space or a complex hyperbolic space with the holomorphic sectional curvature* $\tilde{c}$ *according as* $\tilde{c}>0$ *or* $\tilde{c}<0$ *and* f *is Kaehlerian into* N,

(C–H) *The submanifold* N *is a quaternionic space form and* f *is totally complex into* N. *Moreover the dimension of* N *is twice that of* M.

When $\dim_{\boldsymbol{R}} M=3$, *in addition to* ($R$–$R$) *and* ($R$–$C$) *the following occurs*:

(E–H) *The submanifold* N *is a* 4-*dimensional sphere or a* 4-*dimensional real hyperbolic space with sectional curvature* $\tilde{c}$ *according as* $\tilde{c}>0$ *or* $\tilde{c}<0$.

When $\dim_{\boldsymbol{R}} M=2$, *in addition to* ($R$–$R$), ($R$–$C$), *and* ($C$–$C$) *the following cases occur*:

(C–E) *The submanifold* N *is a* 3-*dimensional sphere or a* 3-*dimensional real hyperbolic space with the sectional curvature* $\tilde{c}$ *according as* $\tilde{c}>0$ *or* $\tilde{c}<0$,

(C–H) *The submanifold* N *is a* 4-*dimensional sphere or a* 4-*dimensional real hyperbolic space with the sectional curvature* $\tilde{c}$ *according as* $\tilde{c}>0$ *or* $\tilde{c}<0$.

Proof. Theorem 3.10 follows from Proposition 3.2 by the same way as Theorem 2.4 in Naitoh [13].

REMARK. Parallel submanifolds of a real space form with sectional curvature $\tilde{c}$ have been classified by Ferus [3], [4], [5] when $\tilde{c}\geqq 0$ and by Takeuchi [18] when $\tilde{c}<0$. Also parallel submanifolds of a complex space form with holomorphic sectional curvature $\tilde{c}$ have been classified. Kaehler parallel submanifolds of a complex space form have been classified by Nakagawa and Takagi [14] when $\tilde{c}>0$ and by Kon [10] when $\tilde{c}<0$. Totally real parallel submanifolds of a complex space form have been classified by Naitoh [12], [13]. Therefore in order to classify all parallel isometric immersions reduced by Theorem 3.10, we have only to classify the (C–H)-case, which will be done in § 7.

4. Hopf fibrations and totally complex immersions into a quaternion projective space

In this section, after recalling definitions of complex and quaternion projective spaces following Besse [1] Chapter 3, we shall give a characterization of totally complex immersions into a quaternion projective space.

Let $\boldsymbol{K}$ be either the field $\boldsymbol{C}$ of complex numbers or the algebra $\boldsymbol{H}$ of quaternions. The product space $\boldsymbol{K}^{n+1}$ is endowed with its right scalar product:

$$x\lambda = {}^t(x_0, \cdots, x_n)\lambda = {}^t(x_0\lambda, \cdots, x_n\lambda)\,, \qquad x\in \boldsymbol{K}^{n+1},\ \lambda\in \boldsymbol{K}\,,$$

its Hermitian inner product:

$$(x, y) = \sum_{i=0}^{n} \bar{x}_i y_i\,, \qquad x, y\in \boldsymbol{K}^{n+1}\,,$$

and its real scalar product:

$$\langle\ ,\ \rangle = \text{the real part of } (\ ,\ )\,.$$

We denote by $S\boldsymbol{K}^{n+1}$ the unit sphere in $\boldsymbol{K}^{n+1}$ defined by the equation $\langle x, x\rangle=1$. The projective space $P_n(\boldsymbol{K})$ is defined as the orbit space for the right action of the group $S\boldsymbol{K}$, where $S\boldsymbol{K}=U(1)=S^1$ for $\boldsymbol{K}=\boldsymbol{C}$ and $S\boldsymbol{K}=Sp(1)=S^3$ for $\boldsymbol{K}=\boldsymbol{H}$. We denote by $\pi_{\boldsymbol{K}}(x)$ the orbit of x. Then we get the principal fibre bundle $S\boldsymbol{K}^{n+1}(P_n(\boldsymbol{K}), S\boldsymbol{K})$ over the base manifold $P_n(\boldsymbol{K})$ with the structure group $S\boldsymbol{K}$. The tangent space $T_xS\boldsymbol{K}^{n+1}$ of $S\boldsymbol{K}^{n+1}$ at a point x may be identified with the real subspace of $\boldsymbol{K}^{n+1}$ as follows:

$$T_xSK^{n+1} = \{u\in \boldsymbol{K}^{n+1} \mid \langle x, u\rangle = 0\}\,.$$

The subspace tangent to the fibre at x in the principal fibre bundle $S\boldsymbol{K}^{n+1}(P_n(\boldsymbol{K}), S\boldsymbol{K})$ is then identified with $\{x\lambda \mid \lambda\in \boldsymbol{K}, \text{ real part of } \lambda=0\}$, which will be denoted by $V_xS\boldsymbol{K}^{n+1}$. Put $H_xS\boldsymbol{K}^{n+1}=\{u\in \boldsymbol{K}^{n+1} \mid (x, u)=0\}$. Then we have the decomposition orthogonal with respect to $\langle\ ,\ \rangle$:

$$T_xS\boldsymbol{K}^{n+1} = V_xS\boldsymbol{K}^{n+1}\oplus H_xS\boldsymbol{K}^{n+1}\,.$$

Moreover the distribution $\{H_xS\boldsymbol{K}^{n+1}; x\in S\boldsymbol{K}^{n+1}\}$ is invariant by the $S\boldsymbol{K}$-action, and hence defines a connection on the principal fibre bundle $S\boldsymbol{K}^{n+1}(P_n(\boldsymbol{K}), S\boldsymbol{K})$.

We shall give the descriptions of projective spaces as Riemannian symmetric homogeneous spaces. In the case $\boldsymbol{K}=\boldsymbol{C}$, G denotes $SU(n+1)$ the Lie group of complex linear transformations of $\boldsymbol{C}^{n+1}$ with determinant 1 which leave the standard complex Hermitian inner product $(\ ,\ )$ invariant, and in the case $\boldsymbol{K}=\boldsymbol{H}, G$ denotes $Sp(n+1)$ the Lie group of quaternion linear transformations of $\boldsymbol{H}^{n+1}$ which leave the standard quaternion Hermitian inner product $(\ ,\ )$ invariant. Note that G acts as automorphisms of the principal fibre bundle $S\boldsymbol{K}^{n+1}(P_n(\boldsymbol{K}), S\boldsymbol{K})$ which preserve the connection and that G acts transitively on $S\boldsymbol{K}^{n+1}$ and hence transitively on $P_n(\boldsymbol{K})$. Let $\{e_0, e_1, \cdots, e_n\}$ be the canonical basis of $\boldsymbol{K}^{n+1}$ (i.e., e_i is the vector of $\boldsymbol{K}^{n+1}$ whose $(i+1)$-st component is one and the other components are zero), and K be the subgroup of G keeping the point $\pi_{\boldsymbol{K}}(e_0)$ fixed. Then $P_n(\boldsymbol{K})$ may be identified with G/K by the diffeomorphism ϕ: $G/K\to P_n(\boldsymbol{K})$ which is given by $\phi(AK)=\pi_{\boldsymbol{K}}(Ae_0)$ for $A\in G$. Every element $A\in K$ has the following form:

$$A=\begin{bmatrix} 1/\det B & 0 \\ 0 & B \end{bmatrix} \qquad \text{with } B\in U(n), \text{ when } \boldsymbol{K}=\boldsymbol{C}\,,$$

and

$$A=\begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} \qquad \text{with } \lambda\in Sp(1) \text{ and } B\in Sp(n), \text{ when } \boldsymbol{K}=\boldsymbol{H}\,.$$

Hence K is isomorphic to $S(U(1)\times U(n))$ or $Sp(1)\times Sp(n)$ according as $\boldsymbol{K}=\boldsymbol{C}$ or $\boldsymbol{K}=\boldsymbol{H}$. It is well-known that (G, K) is a symmetric pair. The Lie algebra $\mathfrak{su}(n+1)$ of $SU(n+1)$ is the set of $X\in M_{n+1}(\boldsymbol{C})$ such that ${}^tX+\bar{X}=0$ and trace $X=0$, and the Lie algebra $\mathfrak{sp}(n+1)$ of $Sp(n+1)$ is the set of $X\in M_{n+1}(\boldsymbol{H})$ such that ${}^tX+\bar{X}=0$, where $M_{n+1}(\boldsymbol{K})$ denotes the set of all matrices of degree $n+1$ with coefficients in $\boldsymbol{K}$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the canonical decomposition of $\mathfrak{g}$ for the symmetric pair(G, K). Then $\mathfrak{k}$ and $\mathfrak{p}$ are given as follows:

$$\mathfrak{k}=\begin{cases} \left\{\begin{bmatrix} -\operatorname{trace} Y & 0 \\ 0 & Y \end{bmatrix};\ Y\in M_n(\boldsymbol{C}),\ {}^tY+\bar{Y}=0\right\}, & \text{when } \boldsymbol{K}=\boldsymbol{C}\,, \\ \left\{\begin{bmatrix} \lambda & 0 \\ 0 & Y \end{bmatrix};\ \lambda\in\mathfrak{sp}(1),\ Y\in\mathfrak{sp}(n)\right\}, & \text{when } \boldsymbol{K}=\boldsymbol{H}\,, \end{cases}$$

$$\mathfrak{p}=\left\{\begin{bmatrix} 0 & -{}^t\bar{Z} \\ Z & 0 \end{bmatrix};\ Z\in\boldsymbol{K}^n\right\}.$$

Thus we may identify $\mathfrak{p}$ with $\boldsymbol{K}^n$ as real vector space. With this identification the adjoint representations of K and $\mathfrak{k}$ on $\mathfrak{p}$, denoted by $\mathrm{Ad}_{\mathfrak{p}}$ and $\mathrm{ad}_{\mathfrak{p}}$ respectively, are written as follows;

$$\mathrm{Ad}_{\mathfrak{p}}\begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}(Z)=(BZ)\bar{\lambda} \qquad \text{for}\begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}\in K$$

and

$$\mathrm{ad}_{\mathfrak{p}}\begin{bmatrix} \lambda & 0 \\ 0 & Y \end{bmatrix}(Z)=(YZ)-Z\lambda \qquad \text{for}\begin{bmatrix} \lambda & 0 \\ 0 & Y \end{bmatrix}\in\mathfrak{k}\,.$$

The vector space $\mathfrak{p}$ may be canonically identified with the tangent space of $P_n(\boldsymbol{K})=G/K$ at the point $o=\pi_{\boldsymbol{K}}(e_0)$. The standard real scalar product $\langle\ ,\ \rangle$ on $\mathfrak{p}\simeq\boldsymbol{K}^n$ is $\mathrm{Ad}_{\mathfrak{p}}K$-invariant and so defines a G-invariant Riemannian metric on the homogenenous space G/K by which $P_n(\boldsymbol{K})$ is a Riemannian symmetric space. Moreover the fibration $\pi_{\boldsymbol{K}}$: $S\boldsymbol{K}^{n+1}\to P_n(\boldsymbol{K})$ is a Riemannian submersion of the sphere $S\boldsymbol{K}^{n+1}$ with the Riemannian metric induced from the real scalar product $\langle\ ,\ \rangle$ in $\boldsymbol{K}^{n+1}$, and $H_xS\boldsymbol{K}^{n+1}$ $x\in S\boldsymbol{K}^{n+1}$ are horizontal subspaces with respect to this submersion. To see this, define the mapping q: $G\to S\boldsymbol{K}^{n+1}$ by $q(A)=Ae_0$, $A\in G$. Then it is easily seen that the differential of q at the identity of G defines a linear isometry of $\mathfrak{p}$ onto $H_{e_0}\boldsymbol{SK}^{n+1}$ Since $\pi_{\boldsymbol{K}}\circ q$ is the projection of G onto $P_n(\boldsymbol{K})=G/K$, the differential of $\pi_{\boldsymbol{K}}$ at e_0 is then a linear isometry of $H_{e_0}\boldsymbol{SK}^{n+1}$ onto $T_0P_n(\boldsymbol{K})$. Since the Riemannian metrics on $\boldsymbol{SK}^{n+1}$ and $P_n(\boldsymbol{K})$ and the distribution $\{H_x\boldsymbol{SK}^{n+1};\ x\in\boldsymbol{SK}^{n+1}\}$ are all G-invariant, it follows the assertion.

The complex structure of a complex projective space $P_n(\boldsymbol{C})$ is defined by the complex structure I on $\mathfrak{p}$ given by

$$I\begin{bmatrix}0 & -{}^t\bar{Z}\\ Z & 0\end{bmatrix}=\begin{bmatrix}0 & {}^t\bar{Z}\sqrt{-1}\\ Z\sqrt{-1} & 0\end{bmatrix}\qquad \text{for } Z\in\boldsymbol{C}^n.$$

Note that $q(IZ)=q(Z)\sqrt{-1}$ for $Z\in\mathfrak{p}$, i.e., q is a complex linear isomorphism of $\mathfrak{p}$ with the complex structure I onto $H_{e_0}\boldsymbol{SC}^{n+1}$. Here q is the identification mapping of $\mathfrak{p}$ with $H_{e_0}\boldsymbol{SC}^{n+1}$ defined above. This complex structure I on $\mathfrak{p}$ is invariant by the adjoint representation of K and so defines a G-invariant almost complex structure on $P_n(\boldsymbol{C})$. This almost complex structure and the Riemannian metric defined above give a Kaehler structure on $P_n(\boldsymbol{C})$.

Now we shall construct a quaternionic Kaehler structure on a quaternion projective space $P_n(\boldsymbol{H})$. We define the subspace A' of the algebra $\mathrm{Hom}(\mathfrak{p}, \mathfrak{p})$ consisting of all real linear endomorphisms of $\mathfrak{p}$ by $A'=\mathrm{ad}_{\mathfrak{p}}(\mathcal{S}p(1))$, where $\mathcal{S}p(1)$ denotes the first factor of the Lie algebra $\mathfrak{k}=\mathcal{S}p(1)+\mathcal{S}p(n)$. We set the basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of A' as follows:

$$\tilde{I}=\mathrm{ad}_{\mathfrak{p}}\begin{bmatrix}i & 0\\ 0 & 0\end{bmatrix},\quad \tilde{J}=\mathrm{ad}_{\mathfrak{p}}\begin{bmatrix}j & 0\\ 0 & 0\end{bmatrix},\quad \text{and}\quad \tilde{K}=\mathrm{ad}_{\mathfrak{p}}\begin{bmatrix}k & 0\\ 0 & 0\end{bmatrix}.$$

Then it follows that $\tilde{I}^2=\tilde{J}^2=\tilde{K}^2=-id$, $\tilde{I}\tilde{J}=-\tilde{J}\tilde{I}=\tilde{K}$, $\tilde{J}\tilde{K}=-\tilde{K}\tilde{J}=\tilde{I}$, $\tilde{K}\tilde{I}=-\tilde{I}\tilde{K}=\tilde{J}$. Since the real scalar product $\langle\ ,\ \rangle$ on $\mathfrak{p}$ is $\mathrm{ad}_{\mathfrak{p}}\mathfrak{k}$- and so $\mathrm{ad}_{\mathfrak{p}}\mathcal{S}p(1)$-invariant, the triple $(\mathfrak{p}, \langle\ ,\ \rangle, A)$ is a quaternionic Hermitian vector space, where A denotes the subalgebra of $\mathrm{Hom}(\mathfrak{p}, \mathfrak{p})$ generated by A' and the identity transformation. Note that $q(\tilde{I}Z)=q(Z)(-i)$, $q(\tilde{J}Z)=q(Z)(-j)$, and $q(\tilde{K}Z)=q(Z)(-k)$ for $Z\in\mathfrak{p}$, where q is the identification mapping of $\mathfrak{p}$ with $H_{e_0}\boldsymbol{SH}^{n+1}$ defined above. Since the subspace $\mathcal{S}p(1)$ in $\mathfrak{k}$ is invariant by the adjoint representation of K, we can define the vector bundle $V=G\times_K\mathcal{S}p(1)$ over $G/K=P_n(\boldsymbol{H})$ with the standard fibre $\mathcal{S}p(1)$ associated with the principal fibre bundle $G(G/K, K)$. This vector bundle V may be regarded as a 3-dimensional subbundle of the vector bundle $\mathrm{Hom}\,(T(G/K), T(G/K))$ consisting of tensors of type (1, 1) by the following bundle homomorphism ψ of V into $\mathrm{Hom}(T(G/K), T(G/K))$:

$$\psi((g, \lambda))=g_*\mathrm{ad}_{\mathfrak{p}}\lambda g_*^{-1}\in\mathrm{Hom}(T(G/K),\ T(G/K))$$

for $g\in G$, $\lambda\in\mathcal{S}p(1)$, where g_* denotes a linear isometry of $T_{eK}(G/K)$ onto $T_{gK}(G/K)$, e being the identity element of G. Moreover $\psi(V)$ is parallel in

$\mathrm{Hom}(T(G/K), T(G/K))$ with respect to the canonical connection in $G(G/K, K)$ defined by the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, which coincides with the Riemannian connection on $G/K=P_n(\boldsymbol{H})$. Denote this subbundle $\psi(V)$ by A'. Then A' and the Riemannian metric defined before give a quaternionic Kaehler structure on $P_n(\boldsymbol{H})$.

Return to the principal fibre bundle $\boldsymbol{SH}^{n+1}(P_n(\boldsymbol{H}), Sp(1))$. The set $\{\pm 1\}$ is a normal closed subgroup of $Sp(1)$ and hence acts on $\boldsymbol{SH}^{n+1}$ on the right. Let $\boldsymbol{SH}^{n+1}/\{\pm 1\}$ be the quotient space of $\boldsymbol{SH}^{n+1}$ by the action of $\{\pm 1\}$, which is denoted by $\boldsymbol{PSH}^{n+1}$. Then $\boldsymbol{PSH}^{n+1}(P_n(\boldsymbol{H}), SO(3))$ is a principal fibre bundle over the base manifold $P_n(\boldsymbol{H})$ with the structure group $SO(3)=Sp(1)/\{\pm 1\}$ (see Proposition 5.5 in [9] p. 57). Moreover it has the connection induced from that of the principal fibre bundle $\boldsymbol{SH}^{n+1}(P_n(\boldsymbol{H}), Sp(1))$. We shall show that the principal fibre bundle $\boldsymbol{PSH}^{n+1}(P_n(\boldsymbol{H}), SO(3))$ is viewed as the bundle of canonical bases of A' on $P_n(\boldsymbol{H})$. For each $g\in Sp(n+1)$, define a canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of A'_{gK} by

$$\tilde{I}=g_*\mathrm{ad}_{\mathfrak{p}}\begin{bmatrix} i & 0 \\ 0 & 0\end{bmatrix}g_*^{-1},\quad \tilde{J}=g_*\mathrm{ad}_{\mathfrak{p}}\begin{bmatrix} j & 0 \\ 0 & 0\end{bmatrix}g_*^{-1},\quad \text{and}$$

$$\tilde{K}=g_*\mathrm{ad}_{\mathfrak{p}}\begin{bmatrix} k & 0 \\ 0 & 0\end{bmatrix}g_*^{-1}.$$

The mapping $q\colon Sp(n+1)\to \boldsymbol{SH}^{n+1}$ defined by $q(g)=ge_0$ is a bundle homomorphism of the principal fibre bundle $G(G/K, K)$ onto $\boldsymbol{SH}^{n+1}(P_n(\boldsymbol{H}), Sp(1))$ which preserves the connections. Two elements g and g' of $Sp(n+1)$ such that $gK=g'K$ define the same canonical basis of A'_{gK} if and only if $q(g)=\pm q(g')$. Hence the principal fibre bundle $\boldsymbol{PSH}^{n+1}(P_n(\boldsymbol{H}), SO(3))$ may be regarded as the bundle consisting of canonical bases of A'. Moreover the connection of A' coincides with the one induced from the principal fibre bundle $\boldsymbol{PSH}^{n+1}(P_n(\boldsymbol{H}), SO(3))$.

The field $\boldsymbol{C}$ of complex numbers is included in the algebra $\boldsymbol{H}$ of quaternions in the standard way. Then $S^1=\{\lambda\in\boldsymbol{C};\ |\lambda|=1\}$ is a closed subgroup of $Sp(1)$. Consider the fibre bundle with standard fibre $Sp(1)/S^1$ associated with the principal fibre bundle $\boldsymbol{SH}^{n+1}(P_n(\boldsymbol{H}), Sp(1))$. Since the total space of the fibre bundle is identified with $\boldsymbol{SH}^{n+1}/S^1$, it is a $2n+1$-dimensional complex projective space $P_{2n+1}(\boldsymbol{C})$. Denote by μ the projection $\boldsymbol{SH}^{n+1}\to\boldsymbol{SH}^{n+1}/S^1 =P_{2n+1}(\boldsymbol{C})$. For each $w\in P_{2n+1}(\boldsymbol{C})$, define the horizontal subspace $H_wP_{2n+1}(\boldsymbol{C})$ in the tangent space $T_wP_{2n+1}(\boldsymbol{C})$ as follows (cf. [9] p. 87). Choose a point $u\in \boldsymbol{SH}^{n+1}$ such that $\mu(u)=w$. Then the horizontal subspace $H_wP_{2n+1}(\boldsymbol{C})$ is, by definition, the image of $H_u\boldsymbol{SH}^{n+1}$ by μ. The subspace $H_wP_{2n+1}(\mathrm{C})$ is independent of the choice of u. Next we introduce a complex structure on $H_wP_{2n+1}(\boldsymbol{C})$. For $u\in\boldsymbol{SH}^{n+1}$, $\mu(u)=w$, let $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ be a canonical basis of $A'_{\pi_{\boldsymbol{H}}(u)}$ defined by $g\in Sp(n+1)$ such that $q(g)=u$. Identifying $H_wP_{2n+1}(\boldsymbol{C})$ with $T_{\pi(w)}P_n(\boldsymbol{H})$ by the fibration mapping π: $P_{2n+1}(\boldsymbol{C}) \to P_n(\boldsymbol{H})$, define a complex structure I on $H_w P_{2n+1}(\boldsymbol{H})$ by $I=\tilde{I}$. Then the complex structure I on $H_w P_{2n+1}(\boldsymbol{C})$ is independent of the choice of u and g. Moreover the complex structure on $H_u P_{2n+1}(\boldsymbol{C})$ defined in this way coincides with the restriction of the complex structure on $P_{2n+1}(\boldsymbol{C})$ defined before.

Theorem 4.1. *Let M be a connected Riemannian manifold with $\dim_{\boldsymbol{R}} M \geqq 4$ and let f be a totally complex immersion of M into $P_n(\boldsymbol{H})$. Then there exists a Kaehler manifold $\hat{M}$ which is a Riemannian covering manifold of M of degree at most two such that $f\circ\hat{\pi}$ is a totally complex immersion of Kaehler type of M into $P_n(\boldsymbol{H})$ (see Definition 2.12), where $\hat{\pi}$ is the covering mapping of $\hat{M}$ onto M. Moreover there exists a Kaehler immersion $\hat{f}$ of $\hat{M}$ into $P_{2n+1}(\boldsymbol{C})$ such that the following commutative diagram holds:*

$$\begin{tikzcd} \hat{M} \arrow[r, "\hat{f}"] \arrow[d, "\hat{\pi}"'] & P_{2n+1}(\boldsymbol{C}) \arrow[d, "\pi"] \\ M \arrow[r, "f"] & P_n(\boldsymbol{H}). \end{tikzcd}$$

Proof. By the immersion f, the following four fibre bundles over M are induced from the fibre bundles over $P_n(\boldsymbol{H})$:

$$(1)\quad \begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d] & S\boldsymbol{H}^{n+1} \arrow[d, "Sp(1)"] \\ M \arrow[r, "f"] & P_n(\boldsymbol{H}) \end{tikzcd} \qquad (2)\quad \begin{tikzcd} P' \arrow[r, "\tilde{f}"] \arrow[d] & PS\boldsymbol{H}^{n+1} \arrow[d, "SO(3)"] \\ M \arrow[r, "f"] & P_n(\boldsymbol{H}) \end{tikzcd}$$

$$(3)\quad \begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & P_{2n+1}(\boldsymbol{C}) \arrow[d] \\ M \arrow[r, "f"] & P_n(\boldsymbol{H}) \end{tikzcd} \qquad (4)\quad \begin{tikzcd} f^*A' \arrow[r, "\tilde{f}"] \arrow[d] & A' \arrow[d] \\ M \arrow[r, "f"] & P_n(\boldsymbol{H}) \end{tikzcd}$$

Here note that the fibre bundle over M in (3) and (4) are associated with the principal fibre bundles in (1) and (2) respectively. By Lemma 2.10, f^*A' is orthogonally decomposed into the subbundles A_0 and A_1, where $A_0=\{L\in A';\ Lf_*TM=f_*TM\}$ and $A_1=\{L\in A';\ Lf_*TM\perp f_*TM\}$. Moreover A_0 and A_1 are parallel with respect to the connection induced on f^*A'. Now we may assume that there exists a global section I of A_0 over M such that $I^2=-id$; otherwise, we have only to replace M by a two-fold Riemannian covering $\hat{M}$ of M. By Proposition 2.11 the global section I of A_0 and the Riemannian metric give rise to a Kaehler structure on M. For $u\in P'$, let $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ be the canonical basis of $A'_{\pi_H\tilde{f}(u)}$ defined by $\tilde{f}(u)\in PS\boldsymbol{H}^{n+1}$. We put $Q'=\{u\in P';\ \tilde{I}=I$ as element of $A'_{\pi_H\tilde{f}(u)}\}$. Then Q' is the subbundle of P' with the structure group S^1. Denote by Q the subbundle of P with the structure group S^1

which is induced from Q' by the projection of P onto P'. Then the associated fibre bundle E admits a cross section s: $M \to E$ such that $s(x)=\mu(u)$ for $u \in Q$ and $x=\pi(u)$, where μ is a projection μ: $P \to E$ (Proposition 5.6, [9] p. 57). Since A_0 is parallel, the connection of P reduces to that of Q. The image of $T_x M$ by s is a horizontal subspace of $T_{s(x)}E$ for each $x \in M$ (Proposition 7.4, [9] p. 88). Therefore $(\tilde{f}\circ s)_* T_x M$ is included in $H_{\tilde{f}\circ s(x)} P_{2n+1}(\boldsymbol{C})$. Since I_x defines the complex structure on $H_{\tilde{f}\circ s(x)} P_{2n+1}(\boldsymbol{C})$ under the identification with $T_{\pi\circ\tilde{f}\circ s(x)} P_n(\boldsymbol{H}) = T_{f(x)} P_n(\boldsymbol{H})$, we have $I(\tilde{f}\circ s)_* = (\tilde{f}\circ s)_* I$ on $T_x M$. Set $\hat{f}=\tilde{f}\circ s$. Then $\hat{f}$ is a Kaehler immersion of M into $P_{2n+1}(\boldsymbol{C})$, and by this construction it is obvious that $\pi\circ\hat{f}=f$.

5. Kaehler immersions of Hermitian symmetric spaces into a complex projective space

In this section, following Nakagawa and Takagi [14], Takagi and Takeuchi [16], and Takeuchi [17], we describe the canonical imbeddings of Hermitian symmetric spaces into a complex projective space. We give another proof to Theorem 2 in [16] on degrees of canonical imbeddings (Theorem 5.2).

First we recall the construction of Kaehler C-spaces M such that $\dim H^2(M, \boldsymbol{R})=1$ (cf. [14]). Here by a C-space we mean a compact simply connected complex homogeneous space and by a Kaehler C-space a C-space M which admits a Kaehler metric such that a group of holomorphic isometries is transitive on M. All irreducible Hermitian symmetric spaces of compact type are Kaehler C-spaces. Let $\bar{\mathfrak{g}}$ be a complex simple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$. Put $l=\dim_{\boldsymbol{C}} \mathfrak{h}$. We denote by Δ the set of all non-zero roots of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{h}$. Then we have a direct sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{h} + \sum_{\alpha\in\Delta} \boldsymbol{C} E_\alpha ,$$

where E_α is a root vector of a root α. Let B be the Killing form of $\bar{\mathfrak{g}}$. For $\xi \in \mathfrak{h}^*$, let H_ξ be the vector such that $B(H, H_\xi)=\xi(H)$ for all $H \in \mathfrak{h}$. Put $\mathfrak{h}_0 = \sum_{\alpha\in\Delta} \boldsymbol{R} H_\alpha$. Then $\dim_{\boldsymbol{R}} \mathfrak{h}_0 = l$ and the dual space $\mathfrak{h}_0^*$ of $\mathfrak{h}_0$ can be regarded as a real subspace of $\mathfrak{h}^*$. We define an inner product (,) on $\mathfrak{h}_0^*$ by $(\xi, \eta) = B(H_\xi, H_\eta)$ for any $\xi, \eta \in \mathfrak{h}_0^*$. Let $\{\alpha_1, \cdots, \alpha_l\}$ be a fundamental root system of $\bar{\mathfrak{g}}$. We choose a lexicographic order in $\mathfrak{h}_0^*$ with respect to which $\{\alpha_1, \cdots, \alpha_l\}$ is the set of simple roots, and denote by Δ^+ and Δ^- the sets of positive and negative roots respectively. Let $\{\Lambda_1, \cdots, \Lambda_l\}$ be the fundamental weight system of $\bar{\mathfrak{g}}$ associated with $\{\alpha_1, \cdots, \alpha_l\}$ which is defined by

$$2(\Lambda_i, \alpha_j) = (\alpha_i, \alpha_j)\delta_{ij} \qquad (i, j = 1, \cdots, l) .$$

On the other hand, we may choose root vectors E_α $(\alpha \in \Delta)$ in the following way;

$$B(E_\alpha, E_{-\alpha}) = -1$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}, \quad N_{\alpha,\beta} = N_{-\alpha-\beta} \in \boldsymbol{R}.$$

Then, $\sqrt{-1}\,\mathfrak{h}_0 + \sum_{\alpha\in\Delta^-}(\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha)$, denoted by $\mathfrak{g}$, is a compact real form of $\tilde{\mathfrak{g}}$, where $A_\alpha = E_\alpha + E_{-\alpha}$, $B_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ $(\alpha\in\Delta^-)$.

We choose a simple root α_i $(i=1, \cdots, l)$ and denote it by γ. We define a subset Δ_γ of Δ^- to be the set of roots

$$\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l \in \Delta^-$$

such that the coefficient of γ in α is strictly negative. Define the subalgebra $\mathfrak{k}$ and the subspace $\mathfrak{p}$ of $\mathfrak{g}$ as follows:

$$\mathfrak{k} = \sqrt{-1}\,\mathfrak{h}_0 + \sum_{\alpha\in\Delta^- - \Delta_\gamma}(\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha)$$

$$\mathfrak{p} = \sum_{\alpha\in\Delta_\gamma}(\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha).$$

Let G and K be a simply connected Lie group and its connected (closed) Lie subgroup which correspond to $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Then the compact homogeneous space $M = G/K$ is known to be simply connected, and the complex structure I on $\mathfrak{p}$ defined by $IA_\alpha = B_\alpha$, $IB_\alpha = -A_\alpha$, $\alpha\in\Delta_\gamma$ gives rise to a G-invariant complex structure on M. Then M is a Kaehler C-space with $\dim H^2(M, \boldsymbol{R}) = 1$. Conversely, every Kaehler C-space M with $\dim H^2(M, \boldsymbol{R}) = 1$ can be obtained in this way from the pair $(\tilde{\mathfrak{g}}, \gamma)$ of a complex simple Lie algebra $\tilde{\mathfrak{g}}$ and a simple root γ. We note that the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ obtained from the pair $(\tilde{\mathfrak{g}}, \gamma)$ becomes a canonical decomposition of an orthogonal symmetric Lie algebra of Hermitian type if and only if the coefficient of γ in every $\alpha\in\Delta_\gamma$ is equal to -1.

Next we construct holomorphic imbeddings of a C-space M obtained from the pair $(\tilde{\mathfrak{g}}, \gamma)$ into a complex projective space. We put $\Lambda_\gamma = \Lambda_i$ if $\gamma = \alpha_i$. Let ρ be an irreducible complex representation of $\tilde{\mathfrak{g}}$ with the highest weight $p\Lambda_\gamma$ for a positive integer p. The representation ρ restricted to $\mathfrak{g}$ defines an irreducible representation of G, which will also be denoted by ρ. Since G is compact, we can choose a complex Hermitian inner product and a unitary frame $\{e_0, \cdots, e_N\}$ on the representation space such that e_0 is the highest weight vector and that $\rho(G) \subset SU(N+1)$. Then the representation ρ of $\mathfrak{g}$ induces a Lie algebra homomorphism ρ: $\mathfrak{g} \to \mathfrak{su}(N+1)$. Let $\mathfrak{su}(N+1) = \tilde{\mathfrak{k}} + \tilde{\mathfrak{p}}$ be the canonical decomposition of $\mathfrak{su}(N+1)$ which corresponds to an N-dimensional complex projective space $P_N(\boldsymbol{C})$ (cf. § 4). Recall that $\tilde{\mathfrak{k}}$ is given by

$$\tilde{\mathfrak{k}} = \{A \in \mathfrak{su}(N+1);\ Ae_0 = e_0\lambda,\ \lambda \in \sqrt{-1}\,\boldsymbol{R}\}.$$

Note that $\rho(A_\alpha)e_0 = \rho(E_\alpha + E_{-\alpha})e_0 = \rho(E_\alpha)e_0$, $\rho(B_\alpha)e_0 = \rho(\sqrt{-1}(E_\alpha - E_{-\alpha}))e_0 = (\rho(E_\alpha)e_0)\sqrt{-1}$, for $\alpha\in\Delta^-$ and $\rho(\sqrt{-1}H)e_0 = e_0(p\Lambda_\gamma(H)\sqrt{-1})$ for $H\in\mathfrak{h}_0$. Since $(\alpha, p\Lambda_\gamma)$ is zero for $\alpha\in\Delta^- - \Delta_\gamma$, $\alpha + p\Lambda_\gamma$ is not a weight and hence we have

$\rho(A_\alpha)e_0=\rho(B_\alpha)e_0=0$. Therefore $\rho(X)$ for $X\in\mathfrak{k}$ is contained in $\tilde{\mathfrak{k}}$. Since $(\alpha, p\Lambda_\gamma)$ is not zero for $\alpha\in\Delta_\gamma$, $\alpha+p\Lambda_\gamma$ is a weight and $\rho(A_\alpha)e_0$ and $\rho(B_\alpha)e_0$ are non-zero vectors in the weight space of weight $\alpha+p\Lambda_\gamma$. Let j be the projection of $\mathfrak{su}(N+1)$ onto $\tilde{\mathfrak{p}}$ and q be a complex linear isomorphism of $\tilde{\mathfrak{p}}$ onto $H_{e_0}SC^{N+1}$ defined in §4. Then $q\circ j\circ\rho(A_\alpha)=\rho(A_\alpha)e_0$ and $q\circ j\circ\rho(B_\alpha)=\rho(B_\alpha)e_0$ for $\alpha\in\Delta_\gamma$. Hence the linear mapping $q\circ j\circ\rho$ of $\mathfrak{p}$ into $H_{e_0}SC^{N+1}$ is injective and $j\circ\rho$ is also a linear injection of $\mathfrak{p}$ into $\tilde{\mathfrak{p}}$. Therefore the mapping $x\in G\rightarrow\pi(\rho(x)e_0)$ of G into $P_N(\boldsymbol{C})$ induces an immersion f of M into $P_N(\boldsymbol{C})$, where π denotes the Hopf fibration of SC^{N+1} onto $P_N(\boldsymbol{C})$ (cf. §4). Since $\rho(IA_\alpha)e_0=\rho(B_\alpha)e_0=(\rho(E_\alpha)e_0)\sqrt{-1}=(\rho(A_\alpha)e_0)\sqrt{-1}$ and $\rho(IB_\alpha)e_0=-\rho(A_\alpha)e_0=-\rho(E_\alpha)e_0=(\rho(B_\alpha)e_0)\sqrt{-1}$ for $\alpha\in\Delta_\gamma$, the mapping $q\circ j\circ\rho$ is a complex linear mapping of $\mathfrak{p}$ into $H_{e_0}SC^{N+1}$ and hence $j\circ\rho$ is a complex linear mapping of $\mathfrak{p}$ into $\tilde{\mathfrak{p}}$. Therefore the mapping f is holomorphic. It is known that f is a full imbedding, i.e., that $f(M)$ is not contained in any proper totally geodesic submanifold of $P_N(\boldsymbol{C})$. The imbedding f introduces a G-invariant Kaehler metric g on M. Thus (M, g) is a Kaehler C-space. Especially when the pair $(\bar{\mathfrak{g}}, \gamma)$ defines an orthogonal symmetric Lie algebra of Hermitian type, the Kaehler C-space (M, g) becomes an irreducible Hermitian symmetric space of compact type. The imbedding f constructed in this way is called the *p-th canonical imbedding* of M.

Now we shall construct a full Kaehler imbedding of the product manifold of a number of Kaehler C-spaces M such that $\dim H^2(M, \boldsymbol{R})=1$ into a complex projective space (cf. [16] and [17]). Let M_i $(1\leqq i\leqq s)$ be Kaehler C-spaces with dim $H^2(M_i, \boldsymbol{R})=1$ obtained from the pairs $(\bar{\mathfrak{g}}_i, \gamma_i)$ of complex simple Lie algebras $\bar{\mathfrak{g}}_i$ and simple roots γ_i. Let f_i: $(M_i, g_i)\rightarrow P_{N_i}(\boldsymbol{C})$ $(1\leqq i\leqq s)$ be the p_i-th canonical imbeddings of M_i constructed by the representations ρ_i of $\bar{\mathfrak{g}}_i$. Let $\mathfrak{h}_i$ and $(\mathfrak{h}_i)_0$ be Cartan subalgebras of $\bar{\mathfrak{g}}_i$ and their real parts respectively. Then the direct sum $\mathfrak{h}=\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_s$ is a Cartan subalgebra of the direct sum $\bar{\mathfrak{g}}=\bar{\mathfrak{g}}_1\oplus\cdots\oplus\bar{\mathfrak{g}}_s$. The dual spaces $\mathfrak{h}_i^*$ naturally become the subspaces of $\mathfrak{h}^*$ and the set of all roots of $\bar{\mathfrak{g}}$ coincides with the union of each set of all roots of $\bar{\mathfrak{g}}_i$. The direct sum $\mathfrak{h}_0=(\mathfrak{h}_1)_0\oplus\cdots\oplus(\mathfrak{h}_s)_0$ is a real part of $\mathfrak{h}$. We choose a lexicographic order in $\mathfrak{h}_0^*$ such that each fundamental root of $\bar{\mathfrak{g}}_i$ is a simple root. Let $\mathfrak{g}_i$ be compact real forms of $\bar{\mathfrak{g}}_i$. Then the direct sum $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_s$ is a compact real form of $\bar{\mathfrak{g}}$. We define a subalgebra $\mathfrak{k}$ and a subspace $\mathfrak{p}$ of $\mathfrak{g}$ by $\mathfrak{k}=\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_s$ and $\mathfrak{p}=\mathfrak{p}_1\oplus\cdots\oplus\mathfrak{p}_s$ respectively, where $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{p}_i$ $(1\leqq i\leqq s)$ are the decompositions of $\mathfrak{g}_i$ which correspond to M_i. Let G_i and K_i be compact simply connected Lie groups and their connected (closed) Lie subgroups which correspond to $\mathfrak{g}_i$ and $\mathfrak{k}_i$ respectively. Then the product Lie group $G=G_1\times\cdots\times G_s$ and $K=K_1\times\cdots\times K_s$ are a Lie group and its connected Lie subgroup which correspond to $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Since a homogeneous space $G/K=M$ coincides with $M_1\times\cdots\times M_s$, it is a Kaehler C-space. Let ρ be the external tensor product $\rho_1\boxtimes\cdots\boxtimes\rho_s$ of representations ρ_i

of $\mathfrak{g}_i$. Let v_i be the highest weight unit vectors with the highest weights $p_i\Lambda_{\gamma_i}$. Then $v_i\otimes\cdots\otimes v_s$ is the highest weight vector of ρ with the highest weight $\sum_{i=1}^{s} p_i\Lambda_{\gamma_i}$. The mapping $x\in G\to\pi(\rho(x)(v_1\otimes\cdots\otimes v_s))$ of G into $P_N(\boldsymbol{C})$ induces a full Kaehler imbedding of the product Kaehler manifold $(M, g)=(M_i\times\cdots\times M_s, g_1\times\cdots\times g_s)$ into $P_N(\boldsymbol{C})$, where $N=\prod_{i=1}^{s}(N_i+1)-1$. We call this Kaehler imbedding of M into $P_N(\boldsymbol{C})$ the *tensor product* of $f_1, \cdots, f_s$, and denote it by $f_1\boxtimes\cdots\boxtimes f_s$.

It is known that any full Kaehler immersion into a complex projective space of a product Kaehler manifold of some Kaehler C-spaces M with $\dim H^2(M, \boldsymbol{R})=1$ is obtained in this way (cf. [14], [17]).

We recall the notion of the degrees of Kaehler immersions (cf. [16]). Let M and $\tilde{M}$ be Kaehler manifolds and f: $M\to\tilde{M}$ be a Kaehler immersion of M into $\tilde{M}$. Then we have the following orthogonal decompositions:

$$\begin{aligned} f^*T\tilde{M} &= TM+N(M) \\ f^*T\tilde{M}^{\boldsymbol{C}} &= TM^{\boldsymbol{C}}+N(M)^{\boldsymbol{C}} \\ f^*T\tilde{M}^{\pm} &= TM^{\pm}+N(M)^{\pm}, \end{aligned}$$

where $f^*T\tilde{M}^{\boldsymbol{C}}$, $TM^{\boldsymbol{C}}$, and $N(M)^{\boldsymbol{C}}$ denote the complexifications of $f^*T\tilde{M}$, TM, and $N(M)$ respectively and $f^*T\tilde{M}^{\pm}$, $TM^{\pm}$, and $N(M)^{\pm}$ denote the $\pm\sqrt{-1}$-eigenspaces of $f^*T\tilde{M}^{\boldsymbol{C}}$, $TM^{\boldsymbol{C}}$, and $N(M)^{\boldsymbol{C}}$ by the action of the complex structures respectively. We shall define the *higher fundamental form* H^j $(j\geqq 2)$ of f as a smooth section of the complex vector bundle $\mathrm{Hom}(\otimes^j TM^+, N(M)^+)$. Let $h^2\in C^\infty(\mathrm{Hom}(\otimes^2 TM, N(M))$ be the second fundamental form of f. We define $h^j\in C^\infty(\mathrm{Hom}(\otimes^j TM, N(M)))$ $(j\geqq 3)$ inductively by

$$\begin{aligned} h^{j+1}(x_1, \cdots, x_j, x_{j+1}) &= \nabla^{\perp}_{x_{j+1}} h^j(X_1, \cdots, X_j) \\ &\quad -\sum_{k=1}^{j} h^j(x_1, \cdots, \nabla_{x_{j+1}}X_k, \cdots, x_j) \\ &\qquad \text{for } x_k\in T_pM\ (1\leqq k\leqq j+1), \end{aligned}$$

where X_k is a smooth local vector field on M around p with $(X_k)_p=x_k$. Extend h^j complex linearly to a smooth section of $\mathrm{Hom}(\otimes^j(TM)^{\boldsymbol{C}}, N(M)^{\boldsymbol{C}})$ and denote it by the same symbol h^j. Then we have

$$h^j(x_1, \cdots, x_j)\in N_p(M)^+, \qquad \text{for } x_k\in T_pM^+ .$$

Now we define $H^j\in C^\infty(\mathrm{Hom}(\otimes^j TM^+, N(M)^+))$ by

$$H^j(x_1, \cdots, x_j) = h^j(x_1, \cdots, x_j) \qquad \text{for } x_k\in T_pM^+ .$$

Then note that

$$
(5.1)\qquad
\begin{aligned}
H^2(X_1, X_2) &= \tilde{\nabla}_{X_2}X_1 - \nabla_{X_2}X_1\\
H^{j+1}(X_1, \cdots, X_j, X_{j+1}) &= \nabla^{\perp}_{X_{j+1}}H^j(X_1, \cdots, X_j)\\
&\quad -\sum_{k=1}^{j} H^j(X_1, \cdots, \nabla_{X_{j+1}}X_k, \cdots, X_j)\\
&\qquad (j \geqq 2) \text{ for } X_i \in C^\infty(TM^+)\,.
\end{aligned}
$$

Here we extend the connections $\tilde{\nabla}$, ∇, and $\nabla^{\perp}$ on $f^*T\tilde{M}$, TM, and $N(M)$ to the complexifications $f^*T\tilde{M}^C$, TM^C, and $N(M)^C$ respectively.

For an integer $j \geqq 1$ and $p \in M$, denote by $\mathcal{H}^j_p(M)$ the subspace of $T_{f(p)}\tilde{M}^+$ spanned by T_pM^+ and $H^k(\otimes^k T_pM^+)$, $2 \leqq k \leqq j$. For an integer $j \geqq 1$, we define a subset R_j of M inductively as follows. Define $R_1 = M$. For $j \geqq 2$, assume that R_{j-1} is already defined. Then we define

$$R_j = \{p \in R_{j-1};\ \dim_C \mathcal{H}^j_p(M) = \max_{p' \in R_{j-1}} \dim_C \mathcal{H}^j_{p'}(M)\}\,.$$

Then it is known ([16]) that there exists a unique integer d such that $\mathcal{H}^{d-1}_p(M) \subsetneqq \mathcal{H}^d_p(M)$ for some $p \in R_{d-1}$ and $\mathcal{H}^d_p(M) = \mathcal{H}^{d+1}_p(M) = \mathcal{H}^{d+2}_p(M) = \cdots$ for each $p \in R_d$, where $\mathcal{H}^0_p(M)$ is understood as $\{0\}$. This integer d is called the *degree* of f and denoted by $d(f)$.

From now on we assume that $\tilde{M} = P_N(C)$. The following lemma is given in Takagi and Takeuchi [16].

Lemma 5.1. *Let $f\colon (M, g) \to P_N(C)$ be a full Kaehler immersion of a Kaehler manifold (M, g). Then we have*

$$N = \dim_C \mathcal{H}^{d(f)}_p(M) \qquad \text{for } p \in R_{d(g)}\,.$$

Theorem 5.2 ([16]). *Let*

$$f_i\colon (M_i, g_i) \to P_{N_i}(C) \qquad (1 \leqq i \leqq s)$$

be the p_i-th canonical imbeddings of irreducible Hermitian symmetric spaces (M_i, g_i) of compact type with rank r_i $(1 \leqq i \leqq s)$ and

$$f\colon (M, g) \to P_N(C)$$

be the tensor product of f_i $(1 \leqq i \leqq s)$. Then the degree $d(f)$ of f is given by

$$d(f) = \sum_{i=1}^{s} r_i p_i\,.$$

We shall give another proof to this Theorem depending on Tits' tables [19]. For this we prepare some lemmas.

Let $\mathfrak{su}(N+1) = \tilde{\mathfrak{k}} + \tilde{\mathfrak{p}}$ be the canonical decomposition of $\mathfrak{su}(N+1)$ which corresponds to $P_N(C)$ (see §4). Then the complexification $\tilde{\mathfrak{p}}^C$ of $\tilde{\mathfrak{p}}$ is given by

$$\tilde{\mathfrak{p}}^C=\left\{\begin{bmatrix}0 & {}^tW\\ Z & 0\end{bmatrix};\quad Z,\, W\in \boldsymbol{C}^N\right\}.$$

Since the complex structure I acts on $\mathfrak{p}$ as follows (see §4):

$$I\begin{bmatrix}0 & -{}^t\bar{Z}\\ Z & 0\end{bmatrix}=\begin{bmatrix}0 & {}^t\bar{Z}\sqrt{-1}\\ Z\sqrt{-1} & 0\end{bmatrix}\qquad Z\in\boldsymbol{C}^N,$$

the $\pm\sqrt{-1}$-eigenspaces $\tilde{\mathfrak{p}}^{\pm}$ of $\tilde{\mathfrak{p}}^C$ by the action of I are given by

$$\tilde{\mathfrak{p}}^{+}=\left\{\begin{bmatrix}0 & 0\\ Z & 0\end{bmatrix};\quad Z\in\boldsymbol{C}^N\right\},\ \tilde{\mathfrak{p}}^{-}=\left\{\begin{bmatrix}0 & {}^tW\\ 0 & 0\end{bmatrix};\ W\in\boldsymbol{C}^N\right\}.$$

We extend the adjoint representation of $\tilde{\mathfrak{k}}$ on $\tilde{\mathfrak{p}}$ to the complex representation on $\tilde{\mathfrak{k}}^C$. Then the eigenspaces $\tilde{\mathfrak{p}}^{\pm}$ are invariant subspaces by this representation. Denote by

$$\mathrm{ad}_{\tilde{\mathfrak{p}}^{\pm}}\colon\ \tilde{\mathfrak{k}}\to\mathfrak{gl}(\tilde{\mathfrak{p}}^{\pm})$$

the representation of $\tilde{\mathfrak{k}}$ on $\tilde{\mathfrak{p}}^{\pm}$. These are written as follows:

$$\mathrm{ad}_{\tilde{\mathfrak{p}}^{+}}\begin{bmatrix}\lambda & 0\\ 0 & B\end{bmatrix}\begin{bmatrix}0 & 0\\ Z & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ BZ-Z\lambda & 0\end{bmatrix}$$
$$\mathrm{ad}_{\tilde{\mathfrak{p}}^{-}}\begin{bmatrix}\lambda & 0\\ 0 & B\end{bmatrix}\begin{bmatrix}0 & {}^tW\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & {}^tW\lambda-{}^tWB\\ 0 & 0\end{bmatrix}\qquad\text{for }\begin{bmatrix}\lambda & 0\\ 0 & B\end{bmatrix}\in\tilde{\mathfrak{k}}.$$

Define a mapping $q\colon\tilde{\mathfrak{p}}^{+}\to H_{e_0}S\boldsymbol{C}^{N+1}$ by

$$q(X)=Xe_0\qquad\text{for }X\in\tilde{\mathfrak{p}}^{+}.$$

Then q is a complex linear isomorphism. For $X\in\tilde{\mathfrak{k}}$ we define a complex linear homomorphism $\phi(X)\colon\boldsymbol{C}^{N+1}\to\boldsymbol{C}^{N+1}$ by the equation $\phi(X)v=Xv-v(e_0,Xe_0)$ for $v\in\boldsymbol{C}^{N+1}$. Evidently $\phi(X)$ induces a complex linear homomorphism of $H_{e_0}S\boldsymbol{C}^{N+1}$ into $H_{e_0}S\boldsymbol{C}^{N+1}$.

Lemma 5.3. *The homomorphism ϕ is a representation of $\tilde{\mathfrak{k}}$ on $H_{e_0}S\boldsymbol{C}^{N+1}$. Moreover the following diagram commutes*:

$$\begin{array}{ccc}\tilde{\mathfrak{p}}^{+} & \xrightarrow{\ q\ } & H_{e_0}S\boldsymbol{C}^{N+1}\\ {\scriptstyle\mathrm{ad}_{\tilde{\mathfrak{p}}^{+}}(X)}\downarrow & & \downarrow{\scriptstyle\phi(X)}\\ \tilde{\mathfrak{p}}^{+} & \xrightarrow{\ q\ } & H_{e_0}S\boldsymbol{C}^{N+1}\end{array}\qquad\textit{for }X\in\tilde{\mathfrak{k}}.$$

Let M_i be a Kaehler C-space obtained from the pair $(\bar{\mathfrak{g}}_i,\gamma_i)$ of a complex simple Lie algebra $\bar{\mathfrak{g}}_i$ and its simple root γ_i and $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{p}_i$ be the decomposition of the compact real form $\mathfrak{g}_i$ which corresponds to M_i. The $\pm\sqrt{-1}$

eigenspaces $\mathfrak{p}_i^\pm$ of the complexification $\mathfrak{p}_i^C$ by the action of the complex structure I are given by

$$\mathfrak{p}_i^+ = \sum_{\alpha\in\Delta_{\gamma_i}} \boldsymbol{C}E_\alpha \quad \text{and} \quad \tilde{\mathfrak{p}}_i^- = \sum_{\alpha\in\Delta_{\gamma_i}} \boldsymbol{C}E_{-\alpha}\,.$$

Moreover $\mathfrak{p}_i^\pm$ are invariant subspaces of $\mathfrak{p}_i^C$ by the complex extension of the adjoint representation of $\mathfrak{k}_i$. Let $M=M_1\times\cdots\times M_s$ be a product manifold of M_i $(1\leqq i\leqq s)$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the decomposition of the compact real form $\mathfrak{g}$ which corresponds to M. Then the $\pm\sqrt{-1}$-eigenspaces $\mathfrak{p}^\pm$ of $\mathfrak{p}^C$ are given by $\mathfrak{p}^\pm=\mathfrak{p}_1^\pm+\cdots+\mathfrak{p}_s^\pm$. The eigenspaces $\mathfrak{p}^\pm$ are invariant subspaces of $\mathfrak{p}^C$ by the complex extension of the adjoint representation of $\mathfrak{k}$. Denote by

$$\mathrm{ad}_{\mathfrak{p}^\pm}\colon \mathfrak{k}\to \mathfrak{gl}(\mathfrak{p}^\pm)$$

the representation of $\mathfrak{k}$ on $\mathfrak{p}^\pm$.
Let f_i be the p_i-th canonical imbeddings of M_i constructed by the representations ρ_i of $\tilde{\mathfrak{g}}_i$. Let ρ be the external tensor product of $\rho_1,\cdots,\rho_s$ and f be the tensor product of $f_1,\cdots,f_s$. Identify $\tilde{\mathfrak{p}}$ and $\mathfrak{p}$ with the tangent spaces at the origins of $P_N(\boldsymbol{C})$ and the product manifold M respectively. Then we can view the j-th fundamental form H^j of the Kaehler imbedding f at the origin as an element of $\mathrm{Hom}(\otimes^j\mathfrak{p}^+, \tilde{\mathfrak{p}}^+)$.

Lemma 5.4. *The following diagram commutes:*

$$\begin{array}{ccc} \otimes^j\mathfrak{p}^+ & \xrightarrow{H^j} & \tilde{\mathfrak{p}}^+ \\ {\scriptstyle \mathrm{ad}_{\mathfrak{p}^+}(X)}\downarrow & & \downarrow{\scriptstyle \mathrm{ad}_{\tilde{\mathfrak{p}}^+}(\rho(X))} \\ \otimes^j\mathfrak{p}^+ & \xrightarrow{H^j} & \tilde{\mathfrak{p}}^+ \end{array} \qquad X\in\mathfrak{k}\,.$$

Moreover combining this with the diagram in Lemma 5.3, *we obtain the following commutative diagram*:

$$\begin{array}{ccc} \otimes^j\mathfrak{p}^+ & \xrightarrow{q\circ H^j} & H_{e_0}S\boldsymbol{C}^{N+1} \\ {\scriptstyle \mathrm{ad}_{\mathfrak{p}^+}(X)}\downarrow & & \downarrow{\scriptstyle \phi(\rho(X))} \\ \otimes^j\mathfrak{p}^+ & \xrightarrow{q\circ H^j} & H_{e_0}S\boldsymbol{C}^{N+1} \end{array} \qquad X\in\mathfrak{k}\,.$$

Here $\phi(\rho(X))$ is given by

$$\phi(\rho(X))v = \rho(X)v - v(e_0, \rho(X)e_0) \qquad v\in H_{e_0}S\boldsymbol{C}^{N+1}\,.$$

Proof. Let G be a compact and simply connected Lie group which corresponds to $\mathfrak{g}$. Since the imbedding f is G-equivariant, the j-th fundamental form H^j is G-invariant, i.e.,

$$H^j(g_*x_1,\cdots,g_*x_j) = \rho(g)_*H^j(x_1,\cdots,x_j) \qquad \text{for } g\in G \text{ and } x_1,\cdots,x_j\in T_pM^+\,.$$

In fact we can prove this inductively using the equation (5.1). Therefore Lemma 5.4 holds.

From now on we consider exclusively Hermitian symmetric spaces of compact type. Let $(\bar{\mathfrak{g}}_i, \gamma_i)$ be the pair which defines an orthogonal symmetric Lie algebra of Hermitian type and M_i be an irreducible Hermitian symmetric space of compact type obtained from $(\bar{\mathfrak{g}}_i, \gamma_i)$. Let ρ_i be an irreducible complex representation of $\bar{\mathfrak{g}}_i$ with the highest weight $p_i\Lambda_{\gamma_i}$. We check each Hermitian symmetric space one by one using Tits' tables in [19] and obtain the following fact.

Lemma 5.5. *The coefficient of γ_i in "$p_i\Lambda_{\gamma_i}$- (the lowest weight of ρ_i)" is equal to (rank of M_i)$\times p_i$.*

Proof of Theorem 5.2. By the equivariance of the imbedding f, we see that $R_{d(f)}=M$ and so we discuss the higher fundamental forms at the origin.

Here we recall that if $\mathfrak{h}_0$ is the real part of the Cartan subalgebra $\mathfrak{h}$ of $\bar{\mathfrak{g}}$, then $\sqrt{-1}\ \mathfrak{h}_0$ is contained in $\mathfrak{k}$. For representations of $\mathfrak{k}$ we consider weights with respect to $\mathfrak{h}_0$ and when a weight ε is given, we write $\varepsilon=\sum_{\gamma} c(\varepsilon;\gamma)\gamma$, where γ runs over the fundamental root system and $c(\varepsilon;\gamma)$ denotes the coefficient of γ. The set of all weights of the representation $\mathrm{ad}_{\mathfrak{p}^+}\colon \mathfrak{k}\to\mathfrak{gl}(\mathfrak{p}^+)$ is equal to $\bigcup_{i=1}^{s}\Delta_{\gamma_i}$ and $c(\varepsilon;\gamma_i)=-1$ for $\varepsilon\in\Delta_{\gamma_i}$. The set of all weights of the representation $\mathrm{ad}_{\mathfrak{p}^+}\colon \mathfrak{k}\to\mathfrak{gl}\,(\otimes^j\mathfrak{p}^+)$ is given by $\{\varepsilon_1+\cdots+\varepsilon_j;\ \varepsilon_k\in\bigcup_{i=1}^{s}\Delta_{\gamma_i}\,(1\leqq k\leqq j)\}$. Hence we have $\sum_{i=1}^{s} c(\varepsilon;\gamma_i)=-j$ for any weight ε of that representation.

Next we consider the representation $\phi\circ\rho$ of $\mathfrak{k}$ on $H_{e_0}SC^{N+1}$ defined in Lemma 5.4. Let V_j be the sum of all weight spaces of $\phi\circ\rho$ corresponding to weights λ such that $\sum_{i=1}^{s} c(\lambda;\gamma_i)=-j$. Lemma 5.4 implies that the image $q\circ H^j(\otimes^j\mathfrak{p}^+)$ is contained in V_j. Since the imbedding f is full, by Lemma 5.1 the image $q\circ H^j(\otimes^j\mathfrak{p}^+)$ coincides with V_j. Note that each weight of the representation $\phi\circ\rho$ has the form: $\lambda-\Lambda\in\mathfrak{h}_0^*$, where λ and Λ are a weight and the highest weight of the representation of ρ of $\bar{\mathfrak{g}}$ respectively. In fact we have, for $H\in\mathfrak{h}_0$ and a weight vector v_λ of ρ with weight λ,

$$\begin{aligned}\phi(\rho(\sqrt{-1}H))v_\lambda &= \rho(\sqrt{-1}H)v_\lambda - v_\lambda(e_0,\ \rho(\sqrt{-1}H)e_0)\\ &= v_\lambda(\lambda(H)\sqrt{-1})+v_\lambda(-\Lambda(H)\sqrt{-1})\\ &= v_\lambda((\lambda-\Lambda)\,(H)\sqrt{-1})\,.\end{aligned}$$

In our case the highest weight Λ is given by $\Lambda=\sum_{k=1}^{s} p_k\Lambda_{\gamma_k}$. Therefore V_j coincides with the sum of all weight spaces of the representation $\rho\colon\bar{\mathfrak{g}}\to\mathfrak{gl}(\boldsymbol{C}^{N+1})$

corresponding to weights λ such that $\sum_{i=1}^{s} c(\lambda-\sum_{k=1}^{s} p_k\Lambda_{\gamma_k};\gamma_i)=-j$. It is known that the coefficient of γ_i of the lowest weight of ρ is not larger than that of any weight of ρ. Therefore,

$$\begin{aligned}
&\sum_{i=1}^{s} c(\sum_{k=1}^{s} p_k\Lambda_{\gamma_k}-\lambda;\gamma_i)\\
&\quad\leqq \sum_{i=1}^{s} c(\sum_{k=1}^{s} p_k\Lambda_{\gamma_k}-\text{the lowest weight of } \rho;\gamma_i)\\
&\quad=\sum_{i=1}^{s} c(\sum_{k=1}^{s} p_k\Lambda_{\gamma_k}-\sum_{k=1}^{s}\text{ the lowest weight of } \rho_k;\gamma_i)\\
&\quad=\sum_{i=1}^{s} c(p_i\Lambda_{\gamma_i}-\text{the lowest weight of } \rho_i;\gamma_i)\\
&\quad=\sum_{i=1}^{s} (\text{rank of } M_i)\times p_i \qquad \text{for any weight } \lambda \text{ of } \rho\,.
\end{aligned}$$

The last equality in the above equation is due to Lemma 5.5. Then we have $d(f)\leqq\sum_{i=1}^{s}$ (rank of $M_i)\times p_i$. Since f is full, by Lemma 5.1 we obtain $d(f)=\sum_{i=1}^{s}$ (rank of $M_i)\times p_i$.

6. Totally complex immersions of Hermitian symmetric spaces into a quaternion projective space

First we shall recall the notion of symplectic and orthogonal representations. Let ρ: $\bar{\mathfrak{g}}\to\mathfrak{gl}(V)$ be a representation of a complex semi-simple Lie algebra $\bar{\mathfrak{g}}$ on a complex vector space V. A bilinear form Ω on V is called an invariant form for the representation ρ if $\Omega(\rho(X)u, v)+\Omega(u, \rho(X)v)=0$ for $X\in\bar{\mathfrak{g}}$ and $u, v\in V$.

DEFINITION 6.1. The representation ρ is called *orthogonal* or *symplectic* according as it has an invariant symmetric or skew-symmetric bilinear form.

Note that if ρ is irreducible and if there exists a non-zero invariant bilinear form for ρ, then it is non-degenerate and unique up to a constant multiple and moreover it is symmetric or skew-symmetric (cf. Tits [19]). Let $\mathfrak{g}$ be a compact real form of $\bar{\mathfrak{g}}$. We introduce a complex Hermitian inner product (,) on V such that $\rho(X)$ is skew-Hermitian for any $X\in\mathfrak{g}$. This inner product (,) is anti-linear with respect to the first factor.

Lemma 6.2. *Suppose that the representation ρ of $\bar{\mathfrak{g}}$ is irreducible and let Ω be an invariant form for ρ. We define a real linear endomorphism $\tilde{J}$ of V such that $\Omega(u, v)=(\tilde{J}u, v)$ for $u, v\in V$. Then,*

(i) *The real linear endomorphism $\tilde{J}$ is semi-linear, i.e.,*

$$\tilde{J}(u\lambda) = \tilde{J}(u)\bar{\lambda} \qquad \text{for } u \in V \text{ and } \lambda \in \boldsymbol{C}.$$

(ii) *By taking a suitable multiple of Ω if necessary, we have $\tilde{J}^2 = id$ or $-id$ according as ρ is orthogonal or symplectic, where id denotes an identity transformation.*

Proof. (i) Actually we have $(\tilde{J}(u\lambda), v) = \Omega(u\lambda, v) = \Omega(u, v)\lambda = (\tilde{J}u, v)\lambda = ((\tilde{J}u)\bar{\lambda}, v)$ for $u, v \in V$ and $\lambda \in \boldsymbol{C}$. Therefore $\tilde{J}(u\lambda) = \tilde{J}(u)\bar{\lambda}$ holds.

(ii) Since $(\tilde{J}\rho(X)u, v) = \Omega(\rho(X)u, v) = -\Omega(u, \rho(X)v) = -(\tilde{J}u, \rho(X)v) = (\rho(X)\tilde{J}u, v)$ for $X \in \mathfrak{g}$, $\rho(X)$ and $\tilde{J}$ commute. By (i) $\tilde{J}^2$ is a complex linear endomorphism of V, and moreover $\tilde{J}^2$ and $\rho(X)$ for $X \in \mathfrak{g}$ commute. Since ρ is irreducible, there exists a non-zero complex number c such that $\tilde{J}^2 = c\ id$. Here c is positive or negative according as ρ is orthogonal or symplectic. In fact if ρ is orthogonal, for a non-zero vector $v \in V$ we get $(\tilde{J}^2 v, v) = \Omega(\tilde{J}v, v) = \Omega(v, \tilde{J}v) = (\tilde{J}v, \tilde{J}v)$. On the other hand $(\tilde{J}^2 v, v) = \bar{c}(v, v)$ and hence $\bar{c} = (\tilde{J}v, \tilde{J}v)/(v, v)$ is positive. Thus c is positive. Replacing Ω by $\frac{1}{\sqrt{c}}\Omega$, we have $\tilde{J}^2 = id$. When ρ is symplectic, we can similarly prove $\tilde{J}^2 = -id$.

Let $\mathfrak{h}$ and $\mathfrak{h}_0$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$ and its real part respectively. Suppose that the compact real form $\mathfrak{g}$ contains $\sqrt{-1}\,\mathfrak{h}_0$ (cf. §5).

Lemma 6.3. *Let ρ: $\bar{\mathfrak{g}} \to \mathfrak{gl}(V)$ be an orthogonal or symplectic representation of $\bar{\mathfrak{g}}$ and $\tilde{J}$ be a real linear endomorphism defined in Lemma 6.2. If $\lambda \in \mathfrak{h}_0^*$ is a weight of ρ and V_λ is a weight space with the weight λ, then $-\lambda$ is also a weight and $\tilde{J}V_\lambda = V_{-\lambda}$.*

Proof. If v is a non-zero vector in V_λ, $\tilde{J}v$ is contained in $V_{-\lambda}$. In fact we have $\rho(\sqrt{-1}H)\,\tilde{J}v = \tilde{J}\,\rho(\sqrt{-1}H)\,v = \tilde{J}(v\lambda(H)\sqrt{-1}) = (\tilde{J}v)\,(-\lambda(H)\sqrt{-1})$ for any $H \in \mathfrak{h}_0$.

From now on we assume that ρ: $\bar{\mathfrak{g}} \to \mathfrak{gl}(V)$ is a symplectic representation. We fix a complex Hermitian inner product $(\ ,\ )$ and a semi-linear endomorphism $\tilde{J}$ of V defined in Lemma 6.2. We shall introduce the structure of a quaternionic Hermitian vector space on V. We restrict the coefficient field of V to $\boldsymbol{R}$ and define a real linear endomorphism $\tilde{I}$ of V by putting $\tilde{I}v = v\sqrt{-1}$ for $v \in V$. Then we have $\tilde{I}^2 = -id$ and $\tilde{I}\tilde{J} = -\tilde{J}\tilde{I}$. Put $\tilde{K} = \tilde{I}\tilde{J}$. Then it follows that $\tilde{I}^2 = \tilde{J}^2 = \tilde{K}^2 = -id$, $\tilde{I}\tilde{J} = -\tilde{J}\tilde{I} = \tilde{K}$, $\tilde{J}\tilde{K} = -\tilde{K}\tilde{J} = \tilde{I}$, $\tilde{K}\tilde{I} = -\tilde{I}\tilde{K} = \tilde{J}$. Let $\langle\ ,\ \rangle$ be the real part of the complex Hermitian inner product $(\ ,\ )$. Since $(\tilde{I}u, \tilde{I}v) = (u, v)$, we have $\langle \tilde{I}u, \tilde{I}v\rangle = \langle u, v\rangle$ and, since $(\tilde{J}u, \tilde{J}v) = \overline{(u, v)}$, $\langle \tilde{J}u, \tilde{J}v\rangle = \langle u, v\rangle$. Hence we have $\langle \tilde{K}u, \tilde{K}v\rangle = \langle u, v\rangle$. Denote by A the subspace of real linear endomorphisms of V spanned by $\tilde{I}$, $\tilde{J}$, $\tilde{K}$, and the identity transformation. Then by Lemma 2.1 $(V, \langle\ ,\ \rangle, A)$ is a quaternionic Hermitian vector space. Since $\rho(X)\tilde{I} = \tilde{I}\rho(X)$, $\rho(X)\tilde{J} = \tilde{J}\rho(X)$, and $\rho(X)\tilde{K} = \tilde{K}\rho(X)$ for $X \in \mathfrak{g}$, $\rho(X)$ is a quaternion linear endomorphism of V. Moreover since $\langle \rho(X)u, v\rangle +$

$\langle u, \rho(X)v\rangle=0$ for $X\in\mathfrak{g}$, $u, v\in V$, $\rho(X)$ is skew-Hermitian with respect to the quaternion Hermitian inner product.

Thus we have the following

Lemma 6.4. *Let ρ: $\bar{\mathfrak{g}}\to\mathfrak{gl}(V)$ be a symplectic representation of a complex semi-simple Lie algebra $\bar{\mathfrak{g}}$. Then we can introduce the structure of a quaternionic Hermitian vector space on V such that for any X in the compact real form $\mathfrak{g}$ of $\bar{\mathfrak{g}}$, $\rho(X)$ is skew-Hermitian with respect to this quaternion Hermitian inner product.*

The following theorem constructs totally complex immersions of Kaehler C-spaces into a quaternion projective space.

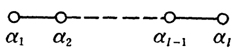
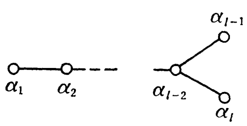
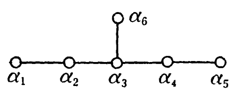
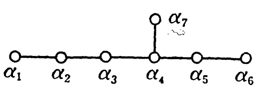
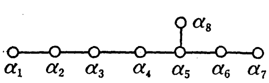
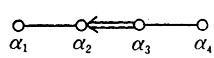
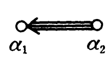
Theorem 6.5. *Let M_i $(1\leqq i\leqq s)$ be Kaehler C-spaces obtained from the pairs $(\bar{\mathfrak{g}}_i, \gamma_i)$ and f_i: $(M_i, g_i)\to P_{N_i}(\boldsymbol{C})$ be the p_i-th canonical imbeddings of M_i constructed by the representations ρ_i of $\bar{\mathfrak{g}}_i$. Let (M, g) be the product Kaehler manifold of (M_i, g_i) and f: $(M, g)\to P_N(\boldsymbol{C})$ be the tensor product of f_i constructed by the external tensor product ρ of ρ_i $(1\leqq i\leqq s)$. We assume that each ρ_i is eather orthogonal or symplectic and that the number of symplectic representations ρ_i is odd. Then ρ is a symplectic representation and in particular N is odd. Set $N=2n+1$. Except for the one case mentioned in Remark 6.7, there exists a totally complex immersion $\tilde{f}$ of (M, g) into $P_n(\boldsymbol{H})$ such that the following diagram commutes:*

$$\begin{tikzcd}
(M, g) \arrow[r, "f"] \arrow[dr, "\tilde{f}"'] & P_{2n+1}(\boldsymbol{C}) \arrow[d, "\pi"] \\
& P_n(\boldsymbol{H})
\end{tikzcd}$$

REMARK 6.6. Orthogonal or symplectic representations of a complex simple Lie algebra have been determined in Tits [19]. Let $\{\Lambda_1, \cdots, \Lambda_l\}$ be the fundamental weight system of a complex simple Lie algebra $\bar{\mathfrak{g}}$ and ρ_j be the irreducible complex representation of $\bar{\mathfrak{g}}$ with the highest weight Λ_j. The members of orthogonal or symplectic representations ρ_j are given in Table 1 due to Tits [19]. If ρ_j is an orthogonal representation, then the irreducible representation ρ with the highest weight $p\Lambda_j$ is also an orthogonal representation. If ρ_j is a symplectic representation, the representation ρ with the highest weight $p\Lambda_j$ is orthogonal or symplectic according as p is even or odd. In particular, we give in Table 2 an irreducible Hermitian symmetric space of compact type obtained from the pair $(\bar{\mathfrak{g}}, \alpha_j)$ of a complex simple Lie algebra $\bar{\mathfrak{g}}$ and its simple root α_j such that ρ_j is an orthogonal or symplectic representation.

REMARK 6.7. Let $M=Sp(2l)/Sp(2l-1)\times U(1)$ be a Kaehler C-space obtained from the pair (C_l, α_1) $(l\geqq 2)$ and f: $(M, g)\to P_N(\boldsymbol{C})$ be the p-th canonical imbedding into $P_N(\boldsymbol{C})$ with holomorphic sectional curvature c. It is known

Table 1

	Dynkin diagram	Orthogonal representation	Symplectic representation
A_l		ρ_{2k+2} when $l=4k+3$ $(k=0, 1, \cdots)$	ρ_{2k+1} when $l=4k+1$ $(k=0, 1, \cdots)$
B_l $(l\geqq 2)$		ρ_i $1\leqq i\leqq l-1$ for any l ρ_l when $l=4k-1$ or $4k$ $(k=1, 2, \cdots)$	ρ_l when $l=4k+1$ or $4k-2$ $(k=1, 2, \cdots)$
C_l $(l\geqq 3)$		ρ_{2i} $1\leqq i\leqq \left[\frac{l}{2}\right]$	ρ_{2i+1} $0\leqq i\leqq \left[\frac{l}{2}\right]$
D_l $(l\geqq 4)$		ρ_i $1\leqq i\leqq l-2$ for any l ρ_l and ρ_{l-1} when $l=4k$ $(k=1, 2, \cdots)$	ρ_l and ρ_{l-1} when $l=4k+2$ $(k=1, 2, \cdots)$
E_6		ρ_3, ρ_6	
E_7		ρ_2, ρ_4, ρ_5, ρ_6	ρ_1, ρ_3, ρ_7
E_8		ρ_i $1\leqq i\leqq 8$	
F_4		ρ_i $1\leqq i\leqq 4$	
G_2		ρ_1, ρ_2	

that (M, g) is a $2l-1$ dimensional complex projective space with holomorphic sectional curvature c/p (cf. Takeuchi [17] Remark 2.4). But $(Sp(2l), Sp(2l-1)\times U(1))$ is not a Riemannian symmetric pair. If p is odd, by Remark 6.6 and Table 1, f is a Kaehler imbedding constructed by a symplectic representation. When $p=1$, then $N=2l-1$ and hence there does not exist a totally complex immersion such that the diagram in Theorem 6.5 commutes. This is the exceptional case stated in Theorem 6.5.

Proof of Theorem 6.5. Denote by V the representation space of ρ. From the assumption it follows that $\rho\colon \bar{\mathfrak{g}}=\bar{\mathfrak{g}}_1\oplus\cdots\oplus\bar{\mathfrak{g}}_s\to\mathfrak{gl}(V)$ is a symplectic representation. By Lemma 6.4 we can introduce the structure of a quaternionic Heimitian vector space on V. Therefore we have $\dim_C V=$even. Set $\dim_C V=2(n+1)$. We fix a complex Hermitian inner product (,) and a semi-linear endomorphism $\tilde{J}$ of V as in Lemma 6.4, and choose a suitable quaternion unitary frame $\{e_0, \cdots, e_n\}$ such that e_0 is the highest weight vector of ρ. Then $\{e_0, \tilde{J}e_0, \cdots, e_k, \tilde{J}e_k, \cdots, e_n, \tilde{J}e_n\}$ is a unitary frame with respect to the complex Hermitian inner product (,). Let $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_s$ be a compact real from of $\bar{\mathfrak{g}}$, where $\mathfrak{g}_i$ denote compact real forms of $\bar{\mathfrak{g}}_i$ $(1\leqq i\leqq s)$. Then by Lemma 6.4 $\rho(\mathfrak{g})$ is contained in $\mathcal{S}p(n+1)$ and especially we obtain a Lie algebra homomorphism $\rho\colon \mathfrak{g}\to\mathcal{S}p(n+1)$.

Let $\mathcal{S}p(n+1)=\tilde{\mathfrak{k}}+\tilde{\mathfrak{p}}$ be the decomposition of $\mathcal{S}p(n+1)$ which corresponds to $P_n(\boldsymbol{H})$ (see §4) and $j\colon \mathcal{S}p(n+1)\to\tilde{\mathfrak{p}}$ be the projection with respect to this decomposition. Recall that $\tilde{\mathfrak{k}}$ is given by

$$\tilde{\mathfrak{k}}=\{A\in\mathcal{S}p(n+1);\ Ae_0=e_0\lambda,\ \lambda\in\mathcal{S}p(1)\}\ .$$

Let q be the identification mapping of $\tilde{\mathfrak{p}}$ with $H_{e_0}S\boldsymbol{H}^{n+1}$ defined in §4. Here q is given by $q(X)=Xe_0$ for $X\in\tilde{\mathfrak{p}}$ and q preserves the structure of a quaternionic Hermitian vector space (see §4). Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the decomposition of a compact real form $\mathfrak{g}$ which corresponds to M. First we shall show that $\rho(\mathfrak{k})$ is contained in $\tilde{\mathfrak{k}}$. By the arguments in §5 we have $\rho(A)e_0=e_0\lambda$, $\lambda\in\sqrt{-1}\boldsymbol{R}$ for $A\in\mathfrak{k}$. Therefore $\rho(\mathfrak{k})$ is contained in $\tilde{\mathfrak{k}}$ and especially $\rho(A)$ has the form:

$$\rho(A)=\begin{pmatrix} i\boldsymbol{R} & 0\\ 0 & Y\end{pmatrix}\quad Y\in\mathcal{S}p(n)\qquad \text{for } A\in\mathfrak{k}\,.$$

Next we shall show that the real linear mapping $q\circ j\circ\rho$ of $\mathfrak{p}$ into $H_{e_0}S\boldsymbol{H}^{n+1}$ is injective and hence $j\circ\rho$ is also a linear injection of $\mathfrak{p}$ into $\tilde{\mathfrak{p}}$. We denote by V_λ a weight space of ρ with a weight λ. For simplicity we use the notation Λ for the highest weight $\sum_{i=1}^{s} p_i\Lambda_{\gamma_i}$. Then we have $e_0\in V_\Lambda$ and, by Lemma 6.3, $\tilde{J}e_0\in V_{-\Lambda}$. Since Λ is the highest weight, it follows that $\dim_C V_\Lambda=1$ and hence, by Lemma 6.3, $\dim_C V_{-\Lambda}=1$. Therefore the quaternion subspace of V spanned by e_0 coincides with the sum $V_\Lambda+V_{-\Lambda}$. For $\alpha\in\Delta_{\gamma_i}$, we get $\rho(A_\alpha)\,e_0=\rho(E_\alpha+E_{-\alpha})\,e_0=\rho(E_\alpha)\,e_0\in V_{\alpha+\Lambda}$ and $\rho(B_\alpha)\,e_0=\rho(\sqrt{-1}(E_\alpha-E_{-\alpha}))\,e_0=(\rho(E_\alpha)\,e_0)\sqrt{-1}\in V_{\alpha+\Lambda}$. Here the weight $\alpha+\Lambda$ is not equal to $-\Lambda$. In fact if M is reducible, i.e., $s\geqq 2$, for any $\alpha\in\Delta_{\gamma_i}$ evidently $\alpha+\Lambda$ is not equal to $-\Lambda$. If M is irreducible, i.e., $s=1$, except the case when M is obtained from the pair (C_l, α_1) or (A_1, α_1) and $p=1$, we can see that for any $\alpha\in\Delta_\gamma$ the coefficient of γ in $\alpha+\Lambda$ is positive and hence $\alpha+\Lambda\neq-\Lambda$ by checking the tables in Tits [19] or Bourbaki [2]. Therefore we have $\rho(\mathfrak{p})e_0$ is contained in $H_{e_0}S\boldsymbol{H}^{n+1}$ and hence $q\circ j\circ\rho(X)=\rho(X)e_0$ for $X\in\mathfrak{p}$. The injectivity of $q\circ j\circ\rho$ is obvious by the arguments in §5.

Let G be a compact and simply connected Lie group which corresponds to $\mathfrak{g}$, and we extend ρ to the representation of G. Then we have $\rho(G)\subset Sp(n+1)$. Let π: $\boldsymbol{SH}^{n+1}\to P_n(\boldsymbol{H})$ be the Hopf fibration. Then by our arguments the mapping $x\in G\to\pi(\rho(x)e_0)$ of G into $P_n(\boldsymbol{H})$ induces an isometric immersion $\tilde{f}$ of (M, g) into $P_n(\boldsymbol{H})$.

Finally we shall show that the immersion $\tilde{f}$ is totally complex. By the equivariance of $\tilde{f}$, it is sufficient to show that the subspace $j\circ\rho(\mathfrak{p})$ is totally complex in $\tilde{\mathfrak{p}}$. Since $q\circ j\circ\rho(X)=\rho(X)e_0$ for $X\in\mathfrak{p}$, it follows that

$$q\circ j\circ\rho(\mathfrak{p})\subset\sum_{i=1}^{s}\sum_{\alpha\in\Delta_{\gamma_i}} V_{\alpha+\Lambda}\,.$$

By the arguments in §5, we have $\rho(IA_\alpha)e_0=\rho(B_\alpha)e_0=(\rho(A_\alpha)e_0)\sqrt{-1}=\tilde{I}\rho(A_\alpha)e_0$ and $\rho(IB_\alpha)e_0=-\rho(A_\alpha)e_0=(\rho(B_\alpha)e_0)\sqrt{-1}=\tilde{I}\rho(B_\alpha)e_0$, and hence $\tilde{I}\ q\circ j\circ\rho(\mathfrak{p})=q\circ j\circ\rho(\mathfrak{p})$.
For $\tilde{J}$, we get

$$\tilde{J}(q\circ j\circ\rho(\mathfrak{p}))\subset\tilde{J}(\sum_{i=1}^{s}\sum_{\alpha\in\Delta_{\gamma_i}} V_{\alpha+\Lambda})=\sum_{i=1}^{s}\sum_{\alpha\in\Delta_{\gamma_i}} V_{-\alpha-\Lambda}\,.$$

Define the sets Φ^+ and Φ^- of weights of ρ by

$$\Phi^+=\{\alpha+\Lambda;\,\alpha\in\Delta_{\gamma_i}\quad(i=1,\cdots,s)\}$$

and

$$\Phi^-=\{-\alpha-\Lambda;\,\alpha\in\Delta_{\gamma_i}\quad(i=1,\cdots,s)\}$$

respectively. We shall show that $\Phi^+\cap\Phi^-$ is empty. In the irreducible case, i.e., $s=1$, since the coefficient of γ in $\alpha+\Lambda$ is positive for any $\alpha\in\Delta_\gamma$, $\Phi^+\cap\Phi^-$ is empty. When M is reducible and $s\geqq 3$, for any $\alpha\in\bigcup_{j=1}^{s}\Delta_{\gamma_j}$ the number of i $(1\leqq i\leqq s)$ such that the coefficient of γ_i in $\alpha+\Lambda$ is positive is at least $s-1$. On the other hand, the number of i $(1\leqq i\leqq s)$ such that the coefficient of γ_i in $-\alpha-\Lambda$ is positive is at most 1. Therefore $\Phi^+\cap\Phi^-$ is empty. When M is reducible and $s=2$, we may assume that ρ_1 is a symplectic representation and ρ_2 is an orthogonal representation. Then we see

$$\begin{aligned}\Phi^+&=\{\alpha+p_1\Lambda_{\gamma_1}+p_2\Lambda_{\gamma_2}\,(\alpha\in\Delta_{\gamma_1}),\,p_1\Lambda_{\gamma_1}+\beta+p_2\Lambda_{\gamma_2}\,(\beta\in\Delta_{\gamma_2})\}\,,\\ \Phi^-&=\{-\alpha-p_1\Lambda_{\gamma_1}-p_2\Lambda_{\gamma_2}\,(\alpha\in\Delta_{\gamma_1}),\,-p_1\Lambda_{\gamma_1}-\beta-p_2\Lambda_{\gamma_2}\,(\beta\in\Delta_{\gamma_2})\}.\end{aligned}$$

Again checking tables in [2] and [19], we see that $\beta+p_2\Lambda_{\gamma_2}\neq-p_2\Lambda_{\gamma_2}$ for any $\beta\in\Delta_{\gamma_2}$. Therefore even if $\alpha+p_1\Lambda_{\gamma_1}=-p_1\Lambda_{\gamma_1}$ for some $\alpha\in\Delta_{\gamma_1}$, it follows that $-\alpha-p_1\Lambda_{\gamma_1}-p_2\Lambda_{\gamma_2}\neq p_1\Lambda_{\gamma_1}+\beta+p_2\Lambda_{\gamma_2}$ and $-p_1\Lambda_{\gamma_1}-\beta-p_2\Lambda_{\gamma_2}\neq\alpha+p_1\Lambda_{\gamma_1}+p_2\Lambda_{\gamma_2}$. Hence $\Phi^+\cap\Phi^-$ is empty. Consequently in all cases $\Phi^+\cap\Phi^-$ is empty. Therefore $\tilde{J}(\sum_{i=1}^{s}\sum_{\alpha\in\Delta_{\gamma_i}} V_{\alpha+\Lambda})$ is orthogonal to $\sum_{i=1}^{s}\sum_{\alpha\in\Delta_{\gamma_i}} V_{\alpha+\Lambda}$ with respect to the complex

Hermitian inner product (,). So $\tilde{J}(q\circ j\circ\rho(\mathfrak{p}))$ is orthogonal to $q\circ j\circ\rho(\mathfrak{p})$ with respect to (,) and also with respect to its real part $\langle\ ,\ \rangle$. Since $\tilde{K}=-\tilde{J}\tilde{I}$, $\tilde{K}(q\circ j\circ\rho(\mathfrak{p}))$ is orthogonal to $q\circ j\circ\rho(\mathfrak{p})$ with respect to the Euclidean inner product $\langle\ ,\ \rangle$. Thus we see that $j\circ\rho(\mathfrak{p})$ is totally complex in $\tilde{\mathfrak{p}}$. Hence the immersion $\tilde{f}$ is totally complex and it is obvious that $\tilde{f}$ satisfies the commutative diagram in Theorem 6.5.

We note that the almost complex structure $\tilde{I}$ can be defined globally on $\tilde{f}^*TP_n(\boldsymbol{H})$ such that $\tilde{I}_p\in A'_{\tilde{f}(p)}$ and $\tilde{f}_*Iv=\tilde{I}\tilde{f}_*v$ for $v\in T_pM$. In fact denote by $\tilde{I}$ the complex structure on the tangent space $T_oP_n(\boldsymbol{H})$ at the origin defined by $\mathrm{ad}_{\tilde{\mathfrak{p}}}\begin{pmatrix} i & 0\\ 0 & 0\end{pmatrix}$ under the identification of $T_oP_n(\boldsymbol{H})$ with $\tilde{\mathfrak{p}}$. Let K be a connected Lie subgroup of G which corresponds to $\mathfrak{k}$. Then by the form of $\rho(A)$ $A\in\mathfrak{k}$, we have $\mathrm{Ad}_{\tilde{\mathfrak{p}}}(\rho(x))\,\tilde{I}=\tilde{I}\ \mathrm{Ad}_{\tilde{\mathfrak{p}}}(\rho(x))$ for $x\in K$. So we can define the complex structure $\tilde{I}$ on $T_{\tilde{f}(xK)}P_n(\boldsymbol{H})$ by $\rho(x)_*\tilde{I}\rho(x)_*^{-1}$ for $x\in G$.

Since totally complex immersions are quite similar to Kaehler immersions (cf. Proposition 2.11), we can define the notion of higher fundamental forms and degrees of totally complex immersions by the same way as Kaehler immersions. On the degrees of totally complex immersions of Hermitian symmetric spaces into a quaternion projective space, the following holds.

Theorem 6.8. *In addition to the assumptions in Theorem* 6.5, *we assume that each pair* $(\mathfrak{g}_i, \gamma_i)$ *defines an orthogonal symmetric Lie algebra of Hermitian type. If* $\tilde{f}$ *is a totally complex immersion of* (M, g) *into* $P_n(\boldsymbol{H})$ *constructed in the proof of Theorem* 6.5, *then the degree* $d(\tilde{f})$ *of* $\tilde{f}$ *is given by*

$$d(\tilde{f})=d(f)-1=\sum_{i=1}^{s} r_i p_i-1\,,$$

where r_i *denote the rank of* M_i $(1\leqq i\leqq s)$.

REMARK 6.9. This Theorem does not hold if $(\mathfrak{g}_i, \gamma_i)$ dces not define an orthogonal symmetric Lie algebra of Hermitian type for some i. It does not hold even if (M, g) is a Hermitian symmetric space as a Riemannian manifold. For example, we consider a Kaehler C-space M obtained from the pair (C_l, α_1) $(l\geqq 2)$ and its p-th canonical imbedding $f\colon(M, g)\to P_N(\boldsymbol{C})$. Then $(M, g)=P_{2l-1}(\boldsymbol{C})$ (cf. Remark 6.7). If p is not less than 3 and odd, (M, g) has a totally complex immersion $\tilde{f}$ into a quaternicn projective space. But we have $d(\tilde{f})=d(f)=p$.

Proof of Theorem 6.8. For simplicity we denote by $\hat{M}=P_{2n+1}(\boldsymbol{C})$, $\tilde{M}=P_n(\boldsymbol{H})$. The vector bundle $f^*T\hat{M}$ over M has the Hermitian structure $(\hat{I}, \hat{g})$ and the Riemannian connection $\hat{\nabla}$ induced from the ones on $T\hat{M}$. Similarly the vector bundle $\tilde{f}^*T\tilde{M}$ over M has the Hermitian structure $(\tilde{I}, \tilde{g})$ constructed in the proof of Theorem 6.5 and the Riemannian connection $\tilde{\nabla}$ induced from the one on $T\tilde{M}$. The Riemannian submersion π: $\hat{M} \to \tilde{M}$ induces the vector bundle homomorphism $f^*T\hat{M} \to \tilde{f}^*T\tilde{M}$, which is denoted by the same notation π. (As for Riemannian submersions, see O'Neill [15]). By π we have the following orthogonal decomposition:

$$f^*T\hat{M} = \mathcal{H} + \mathcal{V},$$

where $\mathcal{H}_p$ and $\mathcal{V}_p$ for $p \in M$ denote the horizontal subspace and the vertical subspace of $T_{f(p)}\hat{M}$ with respect to the Riemannian submersion π respectively. Since $\mathcal{H}_p$ and $\mathcal{V}_p$ for $p \in M$ are complex subspaces in $T_{f(p)}\hat{M}$, $f^*T\hat{M}^\pm$ has the decomposition $f^*T\hat{M}^\pm = \mathcal{H}^\pm + \mathcal{V}^\pm$, where $\mathcal{H}^\pm$ and $\mathcal{V}^\pm$ denote the $\pm\sqrt{-1}$-eigenspaces of the complexifications $\mathcal{H}^C$ and $\mathcal{V}^C$ by the action of the complex structure $\tilde{I}$ respectively. If we restrict π to the subbundle $\mathcal{H}$, π: $\mathcal{H} \to \tilde{f}^*T\tilde{M}$ is a linear isomorphism which preserves Hermitian structures and hence π: $\mathcal{H}^\pm \to \tilde{f}^*T\tilde{M}^\pm$ is a complex linear isomorphism. We note that $f_*T_pM \subset \mathcal{H}_p$ and $\pi f_*T_pM = \tilde{f}_*T_pM$ for $p \in M$. By Lemma 3 in [15] for a local cross section ξ of $\mathcal{H}$ and a local vector field X on M $\pi(\hat{\nabla}_X\xi) = \tilde{\nabla}_X\pi(\xi)$ holds. Denote by $\nabla^\perp$ and $\tilde{\nabla}^\perp$ the connections of the normal bundles in $f^*T\hat{M}$ and $\tilde{f}^*T\tilde{M}$ respectively. Then, for a normal and horizontal vector field ξ of $f^*T\hat{M}$, $\pi(\xi)$ is a normal vector field of $\tilde{f}^*T\tilde{M}$ and $\pi(\nabla^\perp_X \xi) = \tilde{\nabla}^\perp_X \pi(\xi)$ holds. We denote by H^j and $\tilde{H}^j$ the j-th fundamental forms of f and $\tilde{f}$ respectively.

Lemma 6.10. *Let $d(f)$ be the degree of f. Then, for any $p \in M$ we have*

$$H^{d(f)}(\otimes^{d(f)} T_pM^+) = \mathcal{V}_p^+$$

and

$$f_*T_pM^+ + \sum_{k=2}^{d(f)-1} H^k(\otimes^k T_pM^+) = \mathcal{H}_p^+ .$$

Proof. We use the same notations as in Theorem 6.5. Let ρ be the representation of G which defines the Kaehler imbedding f of (M, g) into $P_{2n+1}(\boldsymbol{C})$. Since $\rho(G)$ is contained in $Sp(n+1)$, $\rho(x)$ for $x \in G$ is an automorphism of the Riemannian submersion π: $P_{2n+1}(\boldsymbol{C}) \to P_n(\boldsymbol{H})$, i.e., $\rho(x)$ is an automorphism of the Kaehler manifold $P_{2n+1}(\boldsymbol{C})$ and maps each fibre into a fibre and hence each horizontal subspace into a horizontal subspace. Since f is G-equivariant, it is sufficient to prove this Lemma at the origin of M.

Let V be the representation space of ρ. By Lemma 6.4, V has two structures of the complex Hermitian vector space and the quaternionic Hermitian vector space. Since $S\boldsymbol{H}^{n+1} = S\boldsymbol{C}^{2(n+1)}$, $T_{e_0}S\boldsymbol{H}^{n+1}$ coincides with $T_{e_0}S\boldsymbol{C}^{2(n+1)}$ as the subspaces in V. The vertical subspace $V_{e_0}S\boldsymbol{H}^{n+1}$ of $T_{e_0}S\boldsymbol{H}^{n+1}$ with respect to the fibration $\pi_{\boldsymbol{H}}$: $S\boldsymbol{H}^{n+1} \to P_n(\boldsymbol{H})$ is given by $\{(a\tilde{I} + b\tilde{J} + c\tilde{K})\, e_0;\ a, b, c \in \boldsymbol{R}\}$. Let $H_{e_0}S\boldsymbol{C}^{2(n+1)}$ be the horizontal subspace of $T_{e_0}S\boldsymbol{C}^{2(n+1)}$ with respect to the fibration π_C: $SC^{2(n+1)} \to P_{2n+1}(C)$. Then the intersection $H_{e_0}SC^{2(n+1)} \cap V_{e_0} SH^{n+1}$ is given by $\{(b\tilde{J}+c\tilde{K})\, e_0=(\tilde{J}e_0)\,(b+\sqrt{-1}c);\ b,\ c\in \boldsymbol{R}\}$. Therefore $H_{e_0} SC^{2(n+1)} \cap V_{e_0}SH^{n+1}$ coincides with $V_{-\Lambda}$. The image $\pi_C(H_{e_0}SC^{2(n+1)} \cap V_{e_0} SH^{n+1})$ is the vertical subspace $\mathcal{V}_o$ of $T_oP_{2n+1}(C)$ with respect to the submersion π: $P_{2n+1}(C) \to P_n(H)$. Let $\mathfrak{su}(2(n+1)) = \tilde{\mathfrak{k}}+\tilde{\mathfrak{p}}$ be the canonicald decomposition which corresponds to $P_{2n+1}(C)$ and $\tilde{\mathfrak{p}}=\mathcal{H}+\mathcal{V}$ be the decomposition which is induced from the decomposition $T_oP_{2n+1}(C)=\mathcal{H}_o+\mathcal{V}_o$. Then $q(\mathcal{V})$ coincides with $V_{-\Lambda}$ and hence $q(\mathcal{V}^+)=V_{-\Lambda}$. Recall that $q\circ H^j(\otimes^j \mathfrak{p}^+)$ coincides with the sum of all weight spaces of the representation ρ with weights λ such that $\sum_{i=1}^{s} c(\lambda-\Lambda;\gamma_i)=-j$ (see the proof of Theorem 5.2). Then we have $q\circ H^{d(f)}(\otimes^{d(f)} \mathfrak{p}^+)=V_{-\Lambda}$. In fact since $-\Lambda$ is the lowest weight of ρ, we have $\sum_{i=1}^{s} c(-\Lambda-\Lambda;\gamma_i)=-d(f)$. Moreover the representation theory of semi-simple Lie algebras implies that

$$\sum_{i=1}^{s} c(\lambda-\Lambda;\gamma_i) > \sum_{i=1}^{s} c(-\Lambda-\Lambda;\gamma_i) = -d(f)$$

for a weight λ ($\neq -\Lambda$) of ρ. Therefore we have $H^{d(f)}(\otimes^{d(f)}\mathfrak{p}^+)=\mathcal{V}^+$. Thus Lemma 6.10 is proved.

Owing to Lemma 6.10, it is sufficient to prove that

$$\pi H^j(x_1, \cdots, x_j) = \tilde{H}^j(x_1, \cdots, x_j) \text{ for } 2\leqq j\leqq d(f)-1$$

and $\quad x_1, \cdots, x_j\in T_pM^+$.

We shall prove this inductively. Since $H^2(X_1, X_2)=\hat{\nabla}_{X_2}X_1-\nabla_{X_2}X_1$ for $X_1, X_2 \in C^\infty(TM^+)$ and $\pi(\hat{\nabla}_{X_2}X_1)=\tilde{\nabla}_{X_2}X_1$, we have

$$\begin{aligned}\pi(H^2(X_1, X_2)) &= \pi(\hat{\nabla}_{X_2}X_1-\nabla_{X_2}X_1) = \tilde{\nabla}_{X_2}X_1-\nabla_{X_2}X_1 \\ &= \tilde{H}^2(X_1, X_2)\,.\end{aligned}$$

Here we assume that

$$\pi H^k(x_1, \cdots, x_k) = \tilde{H}^k(x_1, \cdots, x_k) \quad x_1, \cdots, x_k\in T_pM^+$$

holds for $2\leqq k\leqq d(f)-1$ at any point $p\in M$.
Since $H^k(x_1, \cdots, x_k)\in \mathcal{H}_p^+$ by Lemma 6.10, we have

$$\begin{aligned}&\pi H^{k+1}(X_1, \cdots X_{k+1}) \\ &= \pi(\nabla^{\perp}_{X_{k+1}}H^k(X_1 \cdots X_k)-\sum_{i=1}^{k} H^k(X_1, \cdots, \nabla_{X_{k+1}}X_i, \cdots, X_k)) \\ &= \tilde{\nabla}^{\perp}_{X_{k+1}}\pi H^k(X_1, \cdots X_k)-\sum_{i=1}^{k} \pi H^k(X_1, \cdots, \nabla_{X_{k+1}}, X_i, \cdots, X_k) \\ &= \tilde{\nabla}^{\perp}_{X_{k+1}}\tilde{H}^k(X_1, \cdots, X_k)-\sum_{i=1}^{k} \tilde{H}^k(X_1, \cdots, \nabla_{X_{k+1}}X_i, \cdots, X_k)\end{aligned}$$

$$= \tilde{H}^{k+1}(X_1, \cdots, X_{k+1}) \qquad X_1, \cdots, X_{k+1} \in C^\infty(TM^+).$$

Therefore we have $\pi H^{k+1}(x_1, \cdots, x_{k+1}) = \tilde{H}^{k+1}(x_1, \cdots, x_{k+1})$ for $x_1, \cdots, x_{k+1} \in T_pM^+$.

Corollary 6.11. *Let f: $(M, g) \to P_N(\boldsymbol{C})$ be one of the following Kaehler imbeddings (1)~(4) of Hermitian symmetric spaces. Then it satisfies the assumption of Theorem 6.5 and the totally complex immersion $\tilde{f}$ of (M, g) into $P_n(\boldsymbol{H})$ constructed in the proof of Theorem 6.5 has parallel second fundamental form. Moreover we have $\dim_C M = n$.*

(1) *$M = SU(6)/S(U(3) \times U(3))$, $Sp(3)/U(3)$, $SO(12)/U(6)$, or $E_7/E_6 \cdot T^1$ and f is the first canonical imbedding.*

(2) *$M = P_1(\boldsymbol{C}) \times P_1(\boldsymbol{C})$ and f is the tensor product of the first canonical imbedding of the former $P_1(\boldsymbol{C})$ and the second canonical imbedding of the latter $P_1(\boldsymbol{C})$.*

(3) *$M = P_1(\boldsymbol{C}) \times P_1(\boldsymbol{C}) \times P_1(\boldsymbol{C})$ and f is the tensor product of the first canonical imbeddings of $P_1(\boldsymbol{C})$.*

(4) *$M = P_1(\boldsymbol{C}) \times Q_m(\boldsymbol{C})$ $(m \geqq 3)$ and f is the tensor product of the first canonical imbedding of $P_1(\boldsymbol{C})$ and the first canonical imbedding of $Q_m(\boldsymbol{C})$.*

Proof. By checking Table 2 in Remark 6.6 we see that the Kaehler imbedding f: $(M, g) \to P_N(\boldsymbol{C})$ satisfies the assumption of Theorem 6.5. Also we know that $\dim_C M = n$ referring to Table 2 in Nakagawa and Takagi [14]. By

Table 2 Hermitian symmetric spaces with orthogonal representations.

	M	$\dim_C M$	rank M
$(A_{4k+3}, \alpha_{2k+2})$ $(k \geqq 0)$	$G_{2k+2,2k+2}(\boldsymbol{C})$ $= SU(4(k+1))/S(U(2k+2) \times U(2k+2))$	$(2k+2)^2$	$2k+2$
(B_l, α_1) $(l \geqq 2)$	$Q_{2l-1}(\boldsymbol{C}) = SO(2l+1)/SO(2l-1) \times SO(2)$	$2l-1$	2
(C_{2k}, α_{2k}) $(k \geqq 2)$	$Sp(2k)/U(2k)$	$k(2k+1)$	$2k$
(D_l, α_1) $(l \geqq 4)$	$Q_{2l-2}(\boldsymbol{C}) = SO(2l)/SO(2l-2) \times SO(2)$	$2l-2$	2
(D_{4k}, α_{4k}) $(k \geqq 2)$	$SO(8k)/U(4k)$	$2k(4k-1)$	$2k$

Hermitian symmetric spaces with symplectic representations.

	M	$\dim_C M$	rank M
(A_1, α)	$P_1(\boldsymbol{C})$	1	1
$(A_{4k+1}, \alpha_{2k+1})$ $(k \geqq 1)$	$G_{2k+1,2k+1}(\boldsymbol{C})$ $= SU(4k+2)/S(U(2k+1) \times U(2k+1))$	$(2k+1)^2$	$2k+1$
$(C_{2k+1}, \alpha_{2k+1})$ $(k \geqq 1)$	$Sp(2k+1)/U(2k+1)$	$(2k+1)(k+1)$	$2k+1$
$(D_{4k+2}, \alpha_{4k+2})$ $(k \geqq 1)$	$SO(2(4k+2))/U(4k+2)$	$(2k+1)(4k+1)$	$2k+1$
(E_7, α_1)	$E_7/E_6 \cdot T^1$	27	3

Theorem 6.8 we have $d(\tilde{f})=2$ and the image $H^2(\otimes^2 T_p M^+)$ spans the normal space $N_p(M)^+$ in $T_{\tilde{f}(p)}P_n(\boldsymbol{H})^+$ for any point $p\in M$. Hence the image $\tilde{h}^2(\otimes^2 T_p M)$ spans the normal space $N_p(M)$ in $T_{\tilde{f}(p)}P_n(\boldsymbol{H})$. In fact the mapping $v\in N_p(M)^+ \to v+\bar{v}\in N_p(M)$ is a real linear isomorphism of $N_p(M)^+$ onto $N_p(M)$ and we have

$$H^2(x, y)+\overline{H^2(x, y)} = H^2(x, y)+H^2(\bar{x}, \bar{y}) = \tilde{h}^2(x+\bar{x}, y+\bar{y})$$

for $\quad x, y\in T_p M^+$.

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the decomposition of the compact real form $\mathfrak{g}$ which corresponds to M. Then note that $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the canonical decomposition of an orthogonal symmetric Lie algebra. Let G be a compact and simply connected Lie group which corresponds to $\mathfrak{g}$. Then recall that the totally complex immersion $\tilde{f}$ is G-equivariant. We shall show that $\langle\nabla\tilde{h}^2(y, z, x), \tilde{h}^2(u, v)\rangle+\langle\tilde{h}^2(y, z), \nabla\tilde{h}^2(u, v, x)\rangle=0$ $x, y, z, u, v\in T_p M$. By the equivariance of $\tilde{f}$ it is sufficient to prove this at the origin of M. Let $X\in\mathfrak{p}$ be the vector which corresponds to $x\in T_o M$. Define a curve $c(t)$ by $c(t)=\mathrm{expt}X\cdot o$, where $\mathrm{expt}X$ denotes a one-parameter subgroup of G generated by X. Then $c(t)$ is a geodesic such that $\dot{c}(0)=x$. Set $Y=(\mathrm{expt}X)_* y$, $Z=(\mathrm{expt}X)_* z$, $U=(\mathrm{expt}X)_* u$, and $V=(\mathrm{expt}X)_* v$. Then it is known that Y, Z, U, and V are parallel vector fields along the geodesic c. We get

$$\begin{aligned}
&\langle\tilde{h}^2(Y, Z), \tilde{h}^2(U, V)\rangle \\
&= \langle\tilde{h}^2((\mathrm{expt}X)_* y, (\mathrm{expt}X)_* z), \tilde{h}^2((\mathrm{expt}X)_* u, (\mathrm{expt}X)_* v)\rangle \\
&= \langle\rho(\mathrm{expt}X)_*\tilde{h}^2(y, z), \rho(\mathrm{expt}X)_*\tilde{h}^2(u, v)\rangle \\
&= \langle\tilde{h}^2(y, z), \tilde{h}^2(u, v)\rangle
\end{aligned}$$

and hence $\langle\tilde{h}^2(Y, Z), \tilde{h}^2(U, V)\rangle$ is constant along c. Therefore we have $\langle\nabla\tilde{h}^2(y, z, x), \tilde{h}^2(u, v)\rangle+\langle\tilde{h}^2(y, z), \nabla\tilde{h}^2(u, v, x)\rangle=0$. Since $\tilde{f}$ is totally complex, $\nabla\tilde{h}^2(y, z, x)=\nabla\tilde{h}^2(y, x, z)$ holds by the equation of Codazzi. Using the above equations, we have $\langle\nabla\tilde{h}^2(x, y, z), \tilde{h}^2(u, v)\rangle=-\langle\tilde{h}^2(x, y), \nabla\tilde{h}^2(u, v, z)\rangle=\langle\nabla\tilde{h}^2(x, y, v), \tilde{h}^2(z, u)\rangle=-\langle\tilde{h}^2(v, x), \nabla\tilde{h}^2(z, u, y)\rangle=\langle\nabla\tilde{h}^2(v, x, u), \tilde{h}^2(y, z)\rangle=-\langle\tilde{h}^2(u, v), \nabla\tilde{h}^2(y, z, x)\rangle=-\langle\nabla\tilde{h}^2(x, y, z), \tilde{h}^2(u, v)\rangle$ and hence $\langle\nabla\tilde{h}^2(x, y, z), \tilde{h}^2(u, v)\rangle=0$. Since the image $\tilde{h}^2(\otimes^2 T_p M)$ spans the normal space $N_p(M)$, we obtain $\nabla\tilde{h}^2=0$.

7. Totally complex parallel immersions into a quaternionic space form

In this section we shall determine totally complex parallel immersions into a quaternionic space form.

Lemma 7.1. *Let f: $(M, g)\to\tilde{M}(\tilde{c})$ be a totally complex parallel immersion* *of a connected Riemannian manifold M with* $\dim_{\boldsymbol{R}} M \geqq 4$ *into a quaternionic space form* $\tilde{M}(\tilde{c})$ $\tilde{c} \neq 0$. *Then M is a Hermitian locally symmetric space with the local complex structure induced from the quaternionic Kaehler structure of* $\tilde{M}(\tilde{c})$.

Proof. Since the submanifolds with parallel second fundamental form in a Riemannian locally symmetric space are locally symmetric, M is locally symmetric. Moreover by Proposition 2.11, M with I is locally Kaehler and hence M is a Hermitian locally symmetric space.

Theorem 7.2. *If f is a totally complex parallel immersion of a connected Riemannian manifold M with* $\dim_{\boldsymbol{R}} M \geqq 4$ *into a quaternionic space form* $\tilde{M}(\tilde{c})$ $\tilde{c}<0$, *then f is totally geodesic.*

Proof. By Theorem 3.10, two cases $(C\text{–}C)$ and $(C\text{–}H)$ may occur. M. Kon has shown that there is no Kaehler parallel submanifold in a complex hyperbolic space besides totally geodesic one ([10]). Therefore we consider the case $(C\text{–}H)$. In this case the Gauss equation is given as follows:

$$\begin{aligned}\langle R(X, Y) Z, W\rangle = \frac{\tilde{c}}{4} \{&\langle Y, Z\rangle \langle X, W\rangle - \langle X, Z\rangle \langle Y, W\rangle \\ &+\langle IY, Z\rangle \langle IX, W\rangle - \langle IX, Z\rangle \langle IY, W\rangle - 2\langle IX, Y\rangle \langle IZ, W\rangle\} \\ &+\langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle \\ &\text{for} \quad X, Y, Z, W \in T_p M,\end{aligned}$$

where R denotes the curvature tensor of M. Note that it is quite similar to the Gauss equation of the Kaehler submanifold in a complex space form with holomorphic sectional curvature $\tilde{c}$. Nakagawa and Takagi showed that a Kaehler immersion of a Hermitian locally symmetric space into a complex hyperbolic space is totally geodesic (Theorem 3.2 [14]). If M is a Hermitian locally symmetric space different from a complex space form, we can prove similarly to their proof, using the Gauss equation, that there is no totally complex immersion of M into a quaternionic space form $\tilde{M}(\tilde{c})$, $\tilde{c}<0$. If f is a totally complex immersion of an n-dimensional complex space form M into a quaternionic space form and f is not totally geodesic, then we have $\dim_{\boldsymbol{C}} N_p^1(M) = n(n+1)/2$ (cf. Lemma 2.3 of [14]). On the other hand, the dimension of the first normal space is equal to that of the tangent space by Theorem 3.10, which is a contradiction. This proves Theorem 7.2.

Next we shall determine totally complex parallel immersions into a quaternionic space form $\tilde{M}(\tilde{c})$, $\tilde{c}>0$, i.e., a quaternion projective space $P_n(\boldsymbol{H})$.

Theorem 7.3. *Let M be a simply connected complete Riemannian manifold with* $\dim_{\boldsymbol{R}} M \geqq 4$ *and* $f\colon M \to P_n(\boldsymbol{H})$ *be a totally complex parallel immersion. We assume that* $O_p^1(M) = T_{f(p)} P_n(\boldsymbol{H})$ *for some point* $p \in M$. *Then such a pair*

(M, f) is one of the pairs consisting of the Hermitian symmetric spaces and their totally complex immersions given in Corollary 6.11.

The rest of this section is devoted to proving this Theorem. By Lemma 7.1 M is a Hermitian symmetric space. By Theorem 4.1 M admits a Kaehler immersion into $P_{2n+1}(\boldsymbol{C})$ and hence, by Theorem 4.1 (2) in Takeuchi [17], M is of compact type.

Following Helgason [7] we shall associate an orthogonal symmetric Lie algebra $\mathfrak{g}$ with M. Let g and R be the Riemannian metric and the curvature tensor of M respectively. We fix a point $o \in M$ and put $\mathfrak{p}=T_oM$. For $T\in \mathfrak{gl}(\mathfrak{p})$, we naturally extend T to a derivation of the mixed tensor algebra $\sum_{r,s}(\otimes^r\mathfrak{p})\otimes \otimes^s\mathfrak{p}^*)$ over $\mathfrak{p}$.
We put $\mathfrak{k}=\{T\in\mathfrak{gl}(\mathfrak{p});\ T\cdot g_o=0 \text{ and } T\cdot R_o=0\}$. Then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{p})$. Since M is a semi-simple Riemannian symmetric space, $\mathfrak{k}$ is spanned by the set $\{R_o(X, Y);\ X, Y\in T_oM\}$. Consider the direct sum $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and we introduce a bracket operation $[\ ,\ ]$ in $\mathfrak{g}$ as follows:

$$\begin{array}{lll} \text{For } X, Y\in\mathfrak{p} & [X, Y]=-R_o(X, Y) & \\ \text{For } X\in\mathfrak{p}\ T\in\mathfrak{k} & [T, X]=-[X, T]=TX & (T \text{ operating on } X) \\ \text{For } S, T\in\mathfrak{k} & [S, T]=ST-TS\,. & \end{array}$$

Then $\mathfrak{g}$ with this bracket operation is a real Lie algebra. Next define the mapping σ: $\mathfrak{g}\to\mathfrak{g}$ by $\sigma(T+X)=T-X$ for $T\in\mathfrak{k}$ $X\in\mathfrak{p}$. Then σ is an involutive automorphism of $\mathfrak{g}$. Consequently $(\mathfrak{g}, \sigma)$ is an effective orthogonal symmetric Lie algebra (Helgason [7] Lemma 5.4 p. 220). Moreover $(\mathfrak{g}, \sigma)$ is of Hermitian type and $\mathfrak{g}$ is isomorphic to the Lie algebra of $I_0(M)$, where $I_0(M)$ denotes the identity component of the full group of isometries of M. Let G and K be a simply connected Lie group and its connected Lie subgroup which correspond to $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Then we can reconstruct M as the quotient manifold G/K.

We recall the Lie algebra $\tilde{\mathfrak{g}}=\mathcal{S}p(n+1)$ and its canonical decomposition $\tilde{\mathfrak{g}}=\tilde{\mathfrak{k}}+\tilde{\mathfrak{p}}$ which corresponds to a quaternion projective space $P_n(\boldsymbol{H})$ (see §4). Let $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ be the canonical basis of A' on $\tilde{\mathfrak{p}}$ defined in §4. We identify $\tilde{\mathfrak{p}}$ with $T_{\pi_H(e_0)}P_n(\boldsymbol{H})$ as in §4. Moreover we identify $\tilde{\mathfrak{k}}$ with $\mathrm{ad}_{\tilde{\mathfrak{p}}}(\tilde{\mathfrak{k}})$. Then $\tilde{\mathfrak{k}}$ is given by

$$\begin{aligned} \tilde{\mathfrak{k}}=\{&T\in\mathfrak{gl}(\tilde{\mathfrak{p}});\ T \text{ is skew-symmetric with respect to } \langle\ ,\ \rangle \\ &\text{and } \ TA'\subset A'\}\,, \end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the standard real scalar product on $\tilde{\mathfrak{p}}$.

We may assume without loss of generality that $f(o)=\pi_H(e_0)$ and that $f_*I=\tilde{I}f_*$, $\tilde{J}f_*T_oM\perp f_*T_oM$, $\tilde{K}f_*T_oM\perp f_*T_oM$, where I denotes the complex structure on T_oM. We shall construct a Lie algebra homomorphism ρ: $\mathfrak{g}\to\tilde{\mathfrak{g}}$

in the same way as in Naitoh [12]. Noticing $\tilde{\mathfrak{p}}=T_{\pi_H(e_o)}P_n(\boldsymbol{H})=f_*T_oM+N_o^1(M)$, we define linear mappings s: $\mathfrak{p}\to\tilde{\mathfrak{p}}$, μ: $\mathfrak{p}\to\mathfrak{gl}(\tilde{\mathfrak{p}})$, and τ: $\mathfrak{k}\to\mathfrak{gl}(\tilde{\mathfrak{p}})$ as follows:

$$\begin{aligned}
&s(X)=f_*X\\
&\mu(X)\,(s(Y)+\xi)=-s(A_\xi(X))+h(X,\,Y)\\
&\tau(T)\,(s(X)+h(Y,\,Z))=s(TX)+h(TY,\,Z)+h(Y,\,TZ)\\
&\quad\text{for}\quad X,\,Y,\,Z\in\mathfrak{p},\ \xi\in N_0^1(M)\,,
\end{aligned}$$

where A_ξ denotes the shape operator of f.
We remark that $\tau(T)$ is well-defined for $T\in\mathfrak{k}$. In fact,

$$\begin{aligned}
\text{if}\qquad &T=\sum_i R(U_i,\,V_i)\quad\text{and}\quad \sum_j h(Y_j,\,Z_j)=0,\ \text{we have}\\
&\tau(T)\,(\sum_j h(Y_j,\,Z_j))=\sum_j\,\{h(TY_j,\,Z_j)+h(Y_j,\,TZ_j)\}\\
&\qquad=\sum_{i,j}\,\{h(R(U_i,\,V_i)\,Y_j,\,Z_j)+h(Y_j,\,R(U_i,\,V_i)\,Z_j)\}\\
&\qquad=\sum_{i,j}R^\perp(U_i,\,V_i)\,h(Y_j,\,Z_j)\\
&\qquad=\sum_i R^\perp(U_i,\,V_i)\,\sum_j h(Y_j,\,Z_j)\\
&\qquad=0\,.
\end{aligned}$$

We shall show that $\mu(\mathfrak{p})\subset\tilde{\mathfrak{k}}$ and $\tau(\mathfrak{k})\subset\tilde{\mathfrak{k}}$. For this we need the following

Lemma 7.4. *Let f: $M\to\tilde{M}(\tilde{c})$ be a totally complex immersion of M into a quaternionic space form $\tilde{M}(\tilde{c})$, $\tilde{c}\neq0$. We assume that* $\dim_R M=(1/2)\dim_R\tilde{M}(\tilde{c})$ ≥4. *Let $\{\tilde{I},\,\tilde{J},\,\tilde{K}\}$ be a canonical basis of $A'_{f(p)}$ at $p\in M$ such that $\tilde{I}f_*T_pM=f_*T_pM$, $\tilde{J}f_*T_pM\perp f_*T_pM$, $\tilde{K}f_*T_pM\perp f_*T_pM$. Then we have*

$$\begin{aligned}
&(1)\quad A_{\tilde{J}y}(x)=-\tilde{J}h(x,\,y),\ A_{\tilde{K}y}(x)=-\tilde{K}h(x,\,y)\,,\\
&(2)\quad R^\perp(x,\,y)\,(\tilde{J}z)=-\tilde{c}\Omega(x,\,y)\,\tilde{K}z+\tilde{J}(R(x,\,y)\,z)\,,\\
&\qquad\ R^\perp(x,\,y)\,(\tilde{K}z)=\tilde{c}\Omega(x,\,y)\,\tilde{J}z+\tilde{K}(R(x,\,y)\,z)
\end{aligned}$$

for $x,\,y,\,z\in T_pM$, where Ω is defined by $\Omega(x,\,y)=g(Ix,\,y)$.

Proof. Let $\{\tilde{I},\,\tilde{J},\,\tilde{K}\}$ be a local canonical basis of f^*A' over some neighborhood U around p taken in Lemma 2.10. Then we have

$$\begin{aligned}
\tilde{\nabla}_X(\tilde{J}Y)&=\alpha(X)\,\tilde{K}Y+\tilde{J}(\tilde{\nabla}_XY)\\
&=\alpha(X)\,\tilde{K}Y+\tilde{J}(\nabla_XY+h(X,\,Y))\,,
\end{aligned}$$

for vector fields X, Y over U.
On the other hand we have $\tilde{\nabla}_X(\tilde{J}Y)=\nabla^\perp_X(\tilde{J}Y)-A_{\tilde{J}Y}(X)$. Comparing the tangential components and the normal components of two equations, we get $A_{\tilde{J}Y}(X)=-\tilde{J}h(X,\,Y)$ and $\nabla^\perp_X(\tilde{J}Y)=\alpha(X)\,\tilde{K}Y+\tilde{J}(\nabla_XY)$ respectively. Similarly we get $A_{\tilde{K}Y}(X)=-\tilde{K}h(X,\,Y)$ and $\nabla^\perp_X(\tilde{K}Y)=-\alpha(X)\,\tilde{J}Y+\tilde{K}(\nabla_XY)$. Using these, we

obtain

$$R^{\perp}(X, Y)(\tilde{J}Z) = 2d\alpha(X, Y)\tilde{K}Z + \tilde{J}(R(X, Y)Z).$$

By Lemma 2.6, we have $R^{\perp}(X, Y)(\tilde{J}Z) = -\tilde{c}\Omega(X, Y)\tilde{K}Z + \tilde{J}(R(X, Y)Z)$. We have a similar expression for $R^{\perp}(X, Y)(\tilde{K}Z)$. q.e.d.

We shall show that $\mu(X) \in \tilde{\mathfrak{k}}$ for $X \in \mathfrak{p}$. First $\mu(X)$ is a skew-symmetric linear endomorphism of $\tilde{\mathfrak{p}}$. In fact, we have

$$\begin{aligned}
&\langle \mu(X)(s(Y)+\xi), s(Z)+\eta\rangle + \langle s(Y)+\xi, \mu(X)(s(Z)+\eta)\rangle \\
&\quad = \langle -s(A_{\xi}(X)) + h(X, Y), s(Z)+\eta\rangle + \langle s(Y)+\xi, -s(A_{\eta}(X)) \\
&\qquad + h(X, Z)\rangle = -\langle A_{\xi}(X), Z\rangle + \langle h(X, Y), \eta\rangle - \langle A_{\eta}(X), Y\rangle \\
&\qquad + \langle h(X, Z), \xi\rangle = -\langle h(X, Z), \xi\rangle + \langle h(X, Y), \eta\rangle \\
&\qquad - \langle h(X, Y), \eta\rangle + \langle h(X, Z), \xi\rangle = 0.
\end{aligned}$$

Note that $\tilde{J}T_oM = N_o^1(M)$. Using Lemma 7.4, we get

$$\begin{aligned}
&(\mu(X)\tilde{I})(s(Y)+\tilde{J}Z) \\
&\quad = \mu(X)(\tilde{I}(s(Y)+\tilde{J}Z)) - \tilde{I}(\mu(X)(s(Y)+\tilde{J}Z)) \\
&\quad = \mu(X)(s(IY)+\tilde{K}Z) - \tilde{I}(-s(A_{\tilde{J}Z}(X)) + h(X, Y)) \\
&\quad = -s(A_{\tilde{K}Z}(X)) + h(X, IY) - \tilde{I}(\tilde{J}h(X, Z)) - \tilde{I}h(X, Y) \\
&\quad = \tilde{K}h(X, Z) + h(X, IY) - \tilde{K}h(X, Z) - h(X, IY) \\
&\quad = 0.
\end{aligned}$$

Similarly we obtain $\mu(X)\tilde{J} = \mu(X)\tilde{K} = 0$. Consequently $\mu(X)$ is contained in $\tilde{\mathfrak{k}}$ and has the following form:

$$\begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix}, \qquad Y \in \mathcal{S}p(n).$$

We shall show that $\tau(T) \in \tilde{\mathfrak{k}}$ for $T \in \mathfrak{k}$. It is sufficient to prove this when $T = R(U, V)$. Note that $\tau(R(U, V))\xi = R^{\perp}(U, V)\xi$ for $\xi \in N_o^1(M)$. Since $R(U, V)$ and $R^{\perp}(U, V)$ are skew-symmetric linear endomorphisms of T_oM and $N_o^1(M)$ respectively, it is evident that $\tau(R(U, V))$ is skew-symmetric on $\tilde{\mathfrak{p}}$. We prove $\tau(R(U, V))\tilde{I} = 0$, $\tau(R(U, V))\tilde{J} = -\tilde{c}\Omega(U, V)\tilde{K} = -\tilde{c}\Omega(U, V)\tilde{I}\tilde{J}$, and $\tau(R(U, V))\tilde{K} = \tilde{c}\Omega(U, V)\tilde{J} = -\tilde{c}\Omega(U, V)\tilde{I}\tilde{K}$. Using Lemma 7.4 (2), we get

$$\begin{aligned}
&(\tau(R(U, V))\tilde{I})(s(X)+\tilde{J}Y) \\
&\quad = \tau(R(U, V))(\tilde{I}(s(X)+\tilde{J}Y)) - \tilde{I}(\tau(R(U, V))(s(X)+\tilde{J}Y)) \\
&\quad = \tau(R(U, V))(s(IX)+\tilde{K}Y) - \tilde{I}(s(R(U, V)X) + R^{\perp}(U, V)(\tilde{J}Y)) \\
&\quad = s(R(U, V)(IX)) + R^{\perp}(U, V)(\tilde{K}Y) - s(I(R(U, V)X)) \\
&\qquad - \tilde{I}(R^{\perp}(U, V)(\tilde{J}Y))
\end{aligned}$$

$$= \tilde{c}\Omega(U, V)\tilde{J}Y + \tilde{K}(R(U, V)Y) - \tilde{c}\Omega(U, V)\tilde{J}Y - \tilde{K}(R(U, V)Y)$$
$$= 0$$

and get

$$\begin{aligned}
&(\tau(R(U, V))\tilde{J})(s(X)+\tilde{J}Y) \\
&\quad = \tau(R(U, V))(-s(Y)+\tilde{J}X) - \tilde{J}(s(R(U, V)X) + R^{\perp}(U, V)(\tilde{J}Y)) \\
&\quad = -s(R(U, V)Y) + R^{\perp}(U, V)(\tilde{J}X) - \tilde{J}(R(U, V)X) \\
&\qquad -\tilde{J}(-\tilde{c}\Omega(U, V)\tilde{K}Y + \tilde{J}(R(U, V)Y)) \\
&\quad = -\tilde{c}\Omega(U, V)\tilde{K}X + \tilde{J}(R(U, V)X) - \tilde{J}(R(U, V)X) + \tilde{c}\Omega(U, V)\tilde{J}\tilde{K}Y \\
&\quad = -\tilde{c}\Omega(U, V)\tilde{K}(s(X)+\tilde{J}Y)\,.
\end{aligned}$$

We get a similar expression for $\tau(R(U, V))\tilde{K}$. Consequently $\tau(R(U, V))$ is contained in $\tilde{\mathfrak{k}}$. Moreover $\tau(R(U, V))$ has the form:

$$\tau(R(U, V)) = \begin{bmatrix} i\lambda & 0 \\ 0 & Y \end{bmatrix}, \qquad Y \in \mathcal{S}p(n),\ \lambda \in \boldsymbol{R}\,.$$

Now define a linear mapping ρ: $\mathfrak{g} \to \tilde{\mathfrak{g}}$ by the equation $\rho(X+T) = s(X) + \mu(X) + \tau(T)$, $X \in \mathfrak{p}$, $T \in \mathfrak{k}$. Then we have

Proposition 7.5. *The linear mapping ρ is a Lie algebra homomorphism.*

Proof. The proof depends on the following three equations:

$$\tilde{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}(Y) - A_{h(Y,Z)}(X) \tag{7.1}$$
$$\tilde{R}(X, Y)\xi = R^{\perp}(X, Y)\xi + h(A_{\xi}(X), Y) - h(A_{\xi}(Y), X) \tag{7.2}$$
$$T(A_{\xi}(X)) = A_{\tau(T)\xi}(X) + A_{\xi}(TX) \tag{7.3}$$

for $X, Y, Z \in \mathfrak{p}$ $\xi \in N_0^1(M)$ and $T \in \mathfrak{k}$.

Here the equations (7.1) and (7.2) are the ones of Gauss and Ricci respectively. For $T \in \mathfrak{k}$, we have

$$\begin{aligned}
\langle T(A_{\xi}(X)), V\rangle &= -\langle A_{\xi}(X), TV\rangle = \langle \xi, h(X, TV)\rangle \\
&= -\langle \xi, \tau(T)h(X, V) - h(TX, V)\rangle \\
&= \langle \tau(T)\xi, h(X, V)\rangle + \langle \xi, h(TX, V)\rangle \\
&= \langle A_{\tau(T)\xi}(X) + A_{\xi}(TX), V\rangle
\end{aligned}$$

and hence we obtain (7.3).

We prove the following three formulas, which imply immediately that ρ is a Lie algebra homomorphism.

$$\tau([T, S]) = [\tau(T), \tau(S)] \tag{7.4}$$
$$\rho([T, X]) = [\rho(T), \rho(X)] \tag{7.5}$$

(7.6) $\rho([X, Y]) = [\rho(X), \rho(Y)]$ for $T, S \in \mathfrak{k}$ and $X, Y \in \mathfrak{p}$.

By the definition of τ, we have

$$\begin{aligned}
&[\tau(T), \tau(S)](s(X)+h(Y, Z)) \\
&\quad = \tau(T)\tau(S)\,(s(X)+h(Y, Z)) - \tau(S)\tau(T)(s(X)+h(Y, Z)) \\
&\quad = \tau(T)\{s(SX)+h(SY, Z)+h(Y, SZ)\} \\
&\qquad - \tau(S)\{s(TX)+h(TY, Z)+h(Y, TZ)\} \\
&\quad = s(TSX)+h(TSY, Z)+h(SY, TZ)+h(TY, SZ)+h(Y, TSZ) \\
&\qquad - \{s(STX)+h(STY, Z)+h(TY, SZ)+h(SY, TZ)+h(Y, STZ)\} \\
&\quad = s([T, S](X))+h([T, S](Y), Z)+h(Y, [T, S](Z)) \\
&\quad = \tau([T, S])(s(X)+h(Y, Z))
\end{aligned}$$

and hence (7.4) is proved.

Next we have

$$\rho([T, X]) = \rho(TX) = s(TX)+\mu(TX)$$

and

$$\begin{aligned}
[\rho(T), \rho(X)] &= [\tau(T), s(X)+\mu(X)] = \tau(T)s(X)+[\tau(T), \mu(X)] \\
&= s(TX)+[\tau(T), \mu(X)]\,.
\end{aligned}$$

Therefore we need only to prove $\mu(TX)=[\tau(T), \mu(X)]$. Using (7.3), we have

$$\begin{aligned}
&[\tau(T), \mu(X)](s(Y)+\xi) \\
&\quad = \tau(T)\mu(X)\,(s(Y)+\xi) - \mu(X)\tau(T)\,(s(Y)+\xi) \\
&\quad = \tau(T)\{h(X, Y)-s(A_\xi(X))\} - \mu(X)\{s(TY)+\tau(T)\xi\} \\
&\quad = h(TX, Y)+h(X, TY)-s(T(A_\xi(X))) - \{h(X, TY)-s(A_{\tau(T)\xi}(X))\} \\
&\quad = h(TX, Y)-s(A_\xi(TX)) \\
&\quad = \mu(TX)(s(Y)+\xi)\,.
\end{aligned}$$

Consequently the equation (7.5) follows.

For $X, Y \in \mathfrak{p}$, we have

$$\rho([X, Y]) = \rho(-R(X, Y)) = -\tau(R(X, Y))$$

and

$$\begin{aligned}
[\rho(X), \rho(Y)] &= [s(X)+\mu(X), s(Y)+\mu(Y)] \\
&= [s(X), s(Y)]+[s(X), \mu(Y)]+[\mu(X), s(Y)] \\
&\quad +[\mu(X), \mu(Y)] \\
&= -\tilde{R}(X, Y)-h(X, Y)+h(X, Y)+[\mu(X), \mu(Y)]
\end{aligned}$$

$$= -\tilde{R}(X, Y)+[\mu(X), \mu(Y)].$$

By (7.1) and the definition of μ, we have

$$R(X, Y)Z = \tilde{R}(X, Y)Z-[\mu(X), \mu(Y)](Z).$$

By (7.2) and the definition of μ, we have

$$R(X, Y)\xi = \tilde{R}(X, Y)\xi-[\mu(X), \mu(Y)](\xi).$$

Thus we obtain $\tau(R(X, Y))=\tilde{R}(X, Y)-[\mu(X), \mu(Y)]$ and hence we get the formula (7.6). q.e.d.

We extend the Lie algebra homomorphism ρ to the Lie group homomorphism ρ: $G\to\tilde{G}=Sp(n+1)$. Since $\rho(\mathfrak{k})\subset\tilde{\mathfrak{k}}$ and K is connected, $\rho(K)$ is contained in $\tilde{K}=Sp(1)\times Sp(n)$. Therefore we can define a G-equivariant C^∞-mapping $\tilde{f}$ of M into $P_n(\boldsymbol{H})$ by $\tilde{f}(xK)=\pi_{\boldsymbol{H}}(\rho(x)e_0)$.

Proposition 7.6. *The mapping $\tilde{f}$ is a totally complex parallel immersion of M into $P_n(\boldsymbol{H})$. Let $\tilde{h}$ be a second fundamental form of $\tilde{f}$. Then we have $\tilde{f}(o)=f(o)=\pi_{\boldsymbol{H}}(e_0)$, $(\tilde{f}_*)_o=(f_*)_o$ and $\tilde{h}(X, Y)=h(X, Y)$, X, $Y\in T_oM$.*

Proof. By the construction of $\tilde{f}$, it is obvious that $(\tilde{f}_*)_o=s=(f_*)_o$. Since f_* is an isometry and $f_*I=\tilde{I}f_*$, $\tilde{J}f_*T_oM\perp f_*T_oM$, $\tilde{K}f_*T_oM\perp f_*T_oM$ and since $\tilde{f}$ is G-equivariant, $\tilde{f}$ is a totally complex immersion of M into $P_n(\boldsymbol{H})$.

The second fundamental form $\tilde{h}$ and its covariant differentiation $\tilde{\nabla}\tilde{h}$ at $o\in M$ are given by the following formulas (see Naitoh [11] Proposition 5.1 and Proposition 5.2):

$$\tilde{h}(Y, X) = [\rho(X)_{\tilde{\mathfrak{k}}}, \rho(Y)_{\tilde{\mathfrak{p}}}]$$
$$\tilde{\nabla}\tilde{h}(Z, Y, X) = [\rho(X)_{\tilde{\mathfrak{k}}}, [\rho(Y)_{\tilde{\mathfrak{k}}}, \rho(Z)_{\tilde{\mathfrak{p}}}]]+\tilde{f}_*(A_{\tilde{h}(Z,Y)}(X))$$

for X, Y, $Z\in\mathfrak{p}$, where $\rho(X)_{\tilde{\mathfrak{k}}}$ and $\rho(X)_{\tilde{\mathfrak{p}}}$ denote the $\tilde{\mathfrak{k}}$-component and the $\tilde{\mathfrak{p}}$-component of $\rho(X)$ with respect to the decomposition $\tilde{\mathfrak{g}}=\tilde{\mathfrak{k}}+\tilde{\mathfrak{p}}$ respectively. Therefore by the definition of ρ we have

$$\tilde{h}(Y, X) = [\rho(X)_{\tilde{\mathfrak{k}}}, \rho(Y)_{\tilde{\mathfrak{p}}}] = [\mu(X), s(Y)] = \mu(X)s(Y) = h(X, Y)$$

and

$$\begin{aligned}\tilde{\nabla}\tilde{h}(Z, Y, X) &= [\rho(X)_{\tilde{\mathfrak{k}}}, [\rho(Y)_{\tilde{\mathfrak{k}}}, \rho(Z)_{\tilde{\mathfrak{p}}}]]+\tilde{f}_*(A_{\tilde{h}(Z,Y)}(X)) \\ &= [\mu(X), [\mu(Y), s(Z)]]+s(A_{h(Z,Y)}(X)) \\ &= -s(A_{h(Y\ Z)}(X))+s(A_{h(Y,Z)}(X)) \\ &= 0.\end{aligned}$$

Thus we see that $\tilde{h}(X, Y)=h(X, Y)$ and $\tilde{\nabla}\tilde{h}=0$ at $o\in M$. By the G-equivariance, the second fundamental form of $\tilde{f}$ is parallel over M. q.e.d.

By the following Lemma and Proposition 7.6 it is obvious that $\tilde{f}$ and f coincide on M and hence we can reconstruct f as a G-equivariant immersion.

Lemma 7.7 (Naitoh [12] Lemma 3.2). *Let $\tilde{f}$ and f be parallel immersions of a complete connected Riemannian manifold M into another Riemannian manifold $\tilde{M}$, and $\tilde{h}$ and h be their second fundamental forms respectively. If there exists a point $o\in M$ such that $\tilde{f}(o)=f(o)=\bar{o}$, $(\tilde{f}_*)_o=(f_*)_o$: $T_oM\rightarrow T_{\bar{o}}\tilde{M}$, $\tilde{h}_o=h_o$, then $\tilde{f}$ and f coincide on M.*

Next we shall construct a Kaehler immersion $\hat{f}$ of M into $P_{2n+1}(\boldsymbol{C})$ making use of the Lie algebra homomorphism ρ. At first we introduce the structure of a complex Hermitian vector space on $\boldsymbol{H}^{n+1}$. Let $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$ be the real linear endomorphisms of $\boldsymbol{H}^{n+1}$ defined in Lemma 2.1 and $(\ ,\ )_{\boldsymbol{H}}$ be the quaternion Hermitian inner product of $\boldsymbol{H}^{n+1}$. Define a right complex scalar product by $v(a+b\sqrt{-1})=(a+b\tilde{I})v$ for $v\in\boldsymbol{H}^{n+1}$ and a complex Hermitian inner product $(\ ,\ )_{\boldsymbol{C}}$ by $(u,\ v)_{\boldsymbol{C}}=$ the complex conjugate of the complex part of $(u,\ v)_{\boldsymbol{H}}$ for $u,\ v\in\boldsymbol{H}^{n+1}$. Let $\{e_0,\ \cdots,\ e_n\}$ be the canonical basis of $\boldsymbol{H}^{n+1}$. Then $\{e_0, \tilde{J}e_0, \cdots, e_n, \tilde{J}e_n\}$ is a unitary basis with respect to this complex Hermitian inner product. Thus we may view $\boldsymbol{H}^{n+1}$ as $\boldsymbol{C}^{2(n+1)}$. Define a complex skew-symmetric bilinear form Ω on $\boldsymbol{C}^{2(n+1)}$ by $\Omega(u,\ v)=(\tilde{J}u,\ v)_{\boldsymbol{C}}$ $u,\ v\in\boldsymbol{C}^{2(n+1)}$. Then, for $T\in\mathfrak{sp}(n+1)$, T is a complex linear endomorphism of $\boldsymbol{C}^{2(n+1)}$ and T leaves the complex Hermitian inner product $(\ ,\ )_{\boldsymbol{C}}$ and the bilinear form Ω invariant. That is, we have $(Tu,\ v)_{\boldsymbol{C}}+(u,\ Tv)_{\boldsymbol{C}}=0$ and $\Omega(Tu,\ v)+\Omega(u,\ Tv)=0$. Therefore the Lie algebra homomorphism ρ: $\mathfrak{g}\rightarrow\mathfrak{sp}(n+1)$ induces the Lie algebra homomorphism ρ: $\mathfrak{g}\rightarrow\mathfrak{su}(2(n+1))$ and the complexification ρ: $\mathfrak{g}^{C}\rightarrow\mathfrak{gl}(2(n+1),\ \boldsymbol{C})$ is a symplectic representation.

Let $\mathfrak{su}(2(n+1))=\hat{\mathfrak{k}}+\hat{\mathfrak{p}}$ be the canonical decomposition of $\mathfrak{su}(2(n+1))$ which corresponds to $P_{2n+1}(\boldsymbol{C})$. Since $\rho(T)=\tau(T)$ for $T\in\mathfrak{k}$ has the form:

$$\begin{bmatrix} i\lambda & o \\ 0 & Y \end{bmatrix}, \qquad \lambda\in\boldsymbol{R},\ \ Y\in\mathfrak{sp}(n),$$

$\rho(\mathfrak{k})$ is contained in $\hat{\mathfrak{k}}$. Extend the Lie algebra homomorphism ρ: $\mathfrak{g}\rightarrow\mathfrak{su}(2(n+1))$ to a Lie group homomorphism ρ: $G\rightarrow SU(2(n+1))$. Then $\rho(K)$ is contained in $S(U(1)\times U(2n+1))$ and we obtain a G-equivariant C^{∞}-mapping $\hat{f}$ of M into $P_{2n+1}(\boldsymbol{C})$ defined by $\hat{f}(xK)=\pi_{\boldsymbol{C}}(\rho(x)e_0)$, where $\pi_{\boldsymbol{C}}$ denotes the Hopf fibration $\pi_{\boldsymbol{C}}$: $S\boldsymbol{C}^{2(n+1)}\rightarrow P_{2n+1}(\boldsymbol{C})$.

Proposition 7.8. *The mapping $\hat{f}$ is a full Kaehler immersion of M into $P_{2n+1}(\boldsymbol{C})$. Moreover we have $f=\pi\circ\hat{f}$, where π denotes the Riemannian submersion π: $P_{2n+1}(\boldsymbol{C})\rightarrow P_n(\boldsymbol{H})$.*

Proof. Let $j_{\boldsymbol{C}}$ and $j_{\boldsymbol{H}}$ be the projections of $\mathfrak{su}(2(n+1))$ and $\mathfrak{sp}(n+1)$ onto $\hat{\mathfrak{p}}$ and $\tilde{\mathfrak{p}}$ with respect to the decompositions $\mathfrak{su}(2(n+1))=\hat{\mathfrak{k}}+\hat{\mathfrak{p}}$ and $\mathfrak{sp}(n+1)$

$=\tilde{\mathfrak{k}}+\tilde{\mathfrak{p}}$ respectively. Let $H_{e_0}SC^{2(n+1)}$ and $H_{e_0}SH^{n+1}$ be the horizontal subspaces with respect to the fibrations π_C: $SC^{2(n+1)}\to P_{2n+1}(C)$ and π_H: $SH^{n+1}\to P_n(H)$ respectively (see § 4). We note that $H_{e_0}SC^{2(n+1)}=\{\tilde{J}e_0\}_C+H_{e_0}SH^{n+1}$, where $\{\tilde{J}e_0\}_C$ denotes a complex subspace of $C^{2(n+1)}$ spanned by $\tilde{J}e_0$. Let q_C and q_H be the identification mappings of $\hat{\mathfrak{p}}$ and $\tilde{\mathfrak{p}}$ with $H_{e_0}SC^{2(n+1)}$ and $H_{e_0}SH^{n+1}$ respectively (also see § 4). By the form of $\mu(X)$, we have $\rho(X)e_0=(\mu(X)+s(X))e_0=s(X)e_0\in H_{e_0}SH^{n+1}$ for $X\in\mathfrak{p}$ and hence $q_Cj_C\rho(X)=q_Hj_H\rho(X)=\rho(X)e_0$ for $X\in\mathfrak{p}$. The real linear mapping $q_H\circ j_H\circ\rho$: $\mathfrak{p}\to H_{e_0}SH^{n+1}$ is complex linear, i.e., $q_Hj_H\rho(IX)=\tilde{I}q_Hj_H\rho(X)$ and leaves the real scalar product. So the real linear mapping $q_C\circ j_C\circ\rho$: $\mathfrak{p}\to H_{e_0}SC^{2(n+1)}$ has the same properties. Therefore the mapping $j_C\circ\rho$: $\mathfrak{p}\to\hat{\mathfrak{p}}$ is a complex linear and preserves the Hermitian inner product. This, together with the G-equivariance of $\hat{f}$, implies that $\hat{f}$ is a Kaehler immersion of M into $P_{2n+1}(C)$. By construction, it is obvious that $f=\pi\circ\hat{f}$.

Next we shall prove that $\hat{f}$ is full. Let $TP_{2n+1}(C)=\mathcal{H}+\mathcal{V}$ be the decomposition with respect to the Riemannian submersion π: $P_{2n+1}(C)\to P_n(H)$ (cf. the proof of Theorem 6.8). At the point $\pi_C(e_0)\in P_{2n+1}(C)$, $\mathcal{H}$ and $\mathcal{V}$ are given by $\mathcal{H}=H_{e_0}SH^{n+1}$ and $\mathcal{V}=\{\tilde{J}e_0\}_C$ under the identification of $T_{\pi_C(e_0)}P_{2n+1}(C)$ with $H_{e_0}SC^{2(n+1)}$. Let X and Y be local horizontal vector fields of $P_{2n+1}(C)$ around $\pi_C(e_0)$. By a computation similar to the example in [15], we get

$$\mathcal{V}(\hat{\nabla}_XY)=-(\tilde{J}e_0)\langle\tilde{J}X,\ Y\rangle-(\tilde{K}e_0)\langle\tilde{K}X,\ Y\rangle, \tag{7.7}$$

where $\mathcal{V}(\hat{\nabla}_XY)$ denotes the vertical component of $\hat{\nabla}_XY$ at $\pi_C(e_0)$ and $\langle X, Y\rangle$ denotes the real part of the complex Hermitian inner product $(X, Y)_C$. Let $\hat{h}$ and h be the second fundamental forms of $\hat{f}$ and f respectively. Since f is a totally complex immersion, by (7.7), we have $\hat{h}(X, Y)$ is horizontal for $X, Y\in T_oM$ and $\pi\hat{h}(X, Y)=h(X, Y)$. Therefore we have $\mathcal{H}_{\pi_C(e_0)}=\hat{f}_*T_oM+\hat{h}(\otimes^2T_oM)$. Calculating as in the proof of Corollary 6.11, we get $\langle\bar{\nabla}\hat{h}(X, Y, Z), \hat{h}(U, V)\rangle=0$ for $X, Y, Z, U, V\in T_oM$ and by (7.7) we have $\mathcal{V}_{\pi_C(e_0)}=\bar{\nabla}\hat{h}(\otimes^3T_oM)$. Consequently $\hat{f}$ is full.

Proof of Theorem 7.3. By Theorem 2.1 and Theorem 2.2 in Takeuchi [17] we see that the Kaehler immersion $\hat{f}$ in Proposition 7.8 is obtained by the way described in § 5. Especially the representation ρ: $\mathfrak{g}\to\mathfrak{su}(2(n+1))$ is irreducible. We showed that the complexification ρ: $\mathfrak{g}^C\to\mathfrak{gl}(2(n+1), C)$ is a symplectic representation. Therefore the totally complex immersion f is obtained by the same way as in the proof of Theorem 6.5. Then by Theorem 6.8 and Table 2, Theorem 7.3 follows.

By the arguments in the proof of Theorem 7.3, the following is easily seen.

Corollary 7.9. *The local version of Theorem* 7.3 *holds. Namely if* f: $M\to P_n(H)$ *is a totally complex parallel immersion of a connected Riemannian* *manifold M with* $\dim_{\boldsymbol{R}} M \geqq 4$ *into* $P_n(\boldsymbol{H})$ *such that* $O_p^1(M)=T_{f(p)}P_n(\boldsymbol{H})$ *for some point* $p\in M$, *then* M *is locally isometric to one of the Hermitian symmetric spaces given in Corollary* 6.11 *and* f *is locally equivalent to its totally complex immersion given in Corollary* 6.11.

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