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ON THE Λ-MODULE STRUCTURES OF τ-HOMOTOPY GROUPS OF $X \wedge S_+^{1,0}$

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Introduction. Let X be a pointed τ-complex. The stable τ-homotopy group $\pi_{p,q}^S(X \wedge S_+^{1,0})$ is the E^1-term of the forgetful spectral sequence associated with $\pi_{*,*}^S(X)$ [1], and is isomorphic to $\pi_{p+q}^S(X)$ additively since $S_+^{1,0}$ is an equivariant S-dual of itself [3]. Moreover, ρ acts as -1 on $\pi_{p,q}^S(X \wedge S_+^{1,0})$ [2]. (See [2], p. 365 for the definition of ρ). We define Λ to be the ring $\boldsymbol{Z}[\rho]/(1-\rho^2)$. The purpose of this paper is to prove the following theorem on the (unstable) τ-homotopy groups $\pi_{p,q}(X \wedge S_+^{1,0})$.

Theorem. *Let $p \geqq 1$ and $q \geqq 2$. If $\pi_k(X \times X, X \vee X)=0$ for each k, $q+2 \leqq k \leqq p+q+1$, then there exists an isomorphism of abelian groups*

$$\phi_{p,q}\colon \pi_{p,q}(X \wedge S_+^{1,0}) \cong \pi_{p+q}(X) \oplus \pi_{q+1}(X)\,.$$

Furthermore, the ρ-action on $\pi_{p,q}(X \wedge S_+^{1,0})$ is given by

$$\rho\cdot(\alpha, \beta) = (-\alpha, \beta)\,, \qquad p \geqq 2$$
$$\rho\cdot(\alpha, \beta) = (-\alpha, \alpha+\beta)\,, \qquad p = 1$$

where (α, β) is an element of $\pi_{p,q}(X \wedge S_+^{1,0})$ via $\phi_{p,q}$.

See §1 for the definition of $\phi_{p,q}$.

EXAMPLE. If $p+q+2 \leqq 2n$, then we have

$$\pi_{p,q}(S^n X \wedge S_+^{1,0}) \cong \pi_{p+q}(S^n X) \oplus \pi_{q+1}(S^n X)\,,$$

since $\pi_k(S^n X \times S^n X, S^n X \vee S^n X)=0$ for $k \leqq 2n-1$.

Corollary (cf. [2], Proposition 3.6). *Let $(p, q) \in \boldsymbol{Z} \times \boldsymbol{Z}$. Then $\pi_{p,q}^S(X \wedge S_+^{1,0}) \cong \pi_{p+q}^S(X)$ and ρ acts as -1 on $\pi_{p,q}^S(X \wedge S_+^{1,0})$.*

This follows from the above theorem, since $\pi_{q+n+1}(S^{2n} X)=0$ for sufficiently large n.

Notations and elementary results in [2] are used freely.

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1. Let Y be a pointed τ-complex with involution τ. We recall first the forgetful exact sequence ([2], (10.5)),

$$(1)\qquad \cdots\to \pi_{r-1,s+1}(Y)\xrightarrow{\psi_{r-1,s+1}}\pi_{r+s}(Y)\xrightarrow{\delta^*_{r,s}}\pi_{r,s}(Y)\xrightarrow{\chi_{r,s}}\pi_{r-1,s}(Y)\xrightarrow{\psi_{r-1,s}}\pi_{r+s-1}(Y)\to\cdots$$

where $r\geqq 1$. Moreover, by [2] Lemma 12.6, we have

$$(2)\qquad \delta^*_{r,s}\psi_{r,s}=1-\rho \quad \text{(times)}.$$

We denote by $\tau_*\colon \pi_{r+s}(Y)\to\pi_{r+s}(Y)$ the homomorphism induced by τ. Then we have

Proposition 1. $\psi_{r,s}\delta^*_{r,s}=1+(-1)^r\tau_*$.

Proof. Let $\alpha\in\pi_{r+s}(Y)$. Since $\psi_{r,s}\delta^*_{r,s}(\alpha)$ is an element of

$$\pi_{r+s}(Y)=[\Sigma^{r,s}\wedge S^{1,0}_+, Y]^\tau,$$

we obtain

$$\begin{aligned}\psi_{r,s}\delta^*_{r,s}(\alpha)&(s_1,\cdots,s_r,t_1,\cdots,t_s,-1)\\
&=\begin{cases}\alpha(s_1,\cdots,s_{r-1},2s_r+1,t_1,\cdots,t_s,-1), & -1\leqq s_r\leqq 0,\\ \alpha(s_1,\cdots,s_{r-1},1-2s_r,t_1,\cdots,t_s,+1), & 0\leqq s_r\leqq 1,\end{cases}\\
&=\begin{cases}\alpha(s_1,\cdots,s_{r-1},2s_r+1,t_1,\cdots,t_s), & -1\leqq s_r\leqq 0,\\ \tau_*(\alpha)(-s_1,\cdots,-s_{r-1},1-2s_r,t_1,\cdots,t_s), & 0\leqq s_r\leqq 1.\end{cases}\end{aligned}$$

This yields the result.

Let X be a pointed τ-complex. Hereafter, we shall consider the case $Y=X\wedge S^{1,0}_+$. Let $X\vee X$ be a τ-complex with involution defined by $\tau(x,*)=(*,x)$. Since $X\wedge S^{1,0}_+$ is τ-homeomorphic to $X\vee X$ ([2], p. 370), $X\wedge S^{1,0}_+$ may be replaced by $X\vee X$ in the τ-homotopy groups. Thus we may assume that the involution on X is trivial.

As is well known, there exists an isomorphism

$$\pi_k(X\vee X)\cong\pi_k(X)\oplus\pi_k(X)\oplus\pi_{k+1}(X\times X, X\vee X).$$

Here we forget the involution of $X\vee X$, as usual.

If $\pi_{k+1}(X\times X, X\vee X)=0$, then we have

$$\pi_k(X\vee X)\cong\pi_k(X)\oplus\pi_k(X).$$

By this isomorphism, we identify $\pi_k(X\vee X)$ with $\pi_k(X)\oplus\pi_k(X)$.

Let $p\geqq 1$ and $q\geqq 2$. In the following Lemmas 2 and 3, we assume $\pi_{p+q+1}(X\times X, X\vee X)=0$.

Lemma 2. $\psi_{p,q}\delta^{*}_{p,q}(\alpha, \beta)=(\alpha+(-1)^{p}\beta, \beta+(-1)^{p}\alpha)$.

Proof. Let $\tau_{*}: \pi_{p+q}(X\vee X)\rightarrow\pi_{p+q}(X\vee X)$ be the map induced by τ. Then $\tau_{*}(\alpha,\beta)=(\beta,\alpha)$ where $(\alpha,\beta)\in\pi_{p+q}(X\vee X)\cong\pi_{p+q}(X)\oplus\pi_{p+q}(X)$. Thus, Lemma 2 follows from Proposition 1.

Lemma 3. *Let* $u\in\pi_{p,q}(X\vee X)$. *Then* $\psi_{p,q}(u)$ *is of the form* $(\alpha, (-1)^{p}\alpha)$ *with* $\alpha\in\pi_{p+q}(X)$.

Proof. Put $\psi_{p,q}(u)=(\alpha, \beta)$. It is sufficient to prove $\beta=(-1)^{p}\alpha$. Apply $\psi_{p,q}$ to $\delta^{*}_{p,q}\psi_{p,q}(u)=u-\rho\cdot u$ (2). Since $\psi_{p,q}\circ\rho=-\psi_{p,q}$ ([2], (9.9)), Lemma 2 shows that

$$(\alpha+(-1)^{p}\beta, \beta+(-1)^{p}\alpha) = (\alpha, \beta)+(\alpha, \beta)\,,$$

and so $\beta=(-1)^{p}\alpha$ as required.

Since $\pi_{0,k}(X\vee X)=0$, it follows from (1) with $Y=X\vee X$, $r=1$ and $s=q$, that there exists an isomorphism

$$\delta^{*}_{1,q}: \pi_{q+1}(X\vee X)\xrightarrow{\approx}\pi_{1,q}(X\vee X)\,.$$

Suppose that $\pi_{q+2}(X\times X, X\vee X)=0$. We then define the homomorphism $\bar{\chi}_{p,q}: \pi_{p,q}(X\vee X)\rightarrow\pi_{q+1}(X)$ by the composition $p_{2}(\delta^{*}_{1,q})^{-1}\chi_{2,q}\cdots\chi_{p,q}$ if $p\geqq 2$, and define $\bar{\chi}_{1,q}$ by $p_{2}(\delta^{*}_{1,q})^{-1}$, where p_{2} denotes the projection to the second factor.

Moreover, suppose $\pi_{p+q+1}(X\times X, X\vee X)=0$. Then we define the homomorphism

$$\phi_{p,q}: \pi_{p,q}(X\vee X) \rightarrow \pi_{p+q}(X)\oplus\pi_{q+1}(X)$$

by

$$\phi_{p,q}(u) = (p_{1}\psi_{p,q}(u), \bar{\chi}_{p,q}(u))\,,$$

where p_{1} denotes the projection to the first factor.

2. We shall prove the theorem by the induction on p. Let $p=1$. Put $(\delta^{*}_{1,q})^{-1}(u)=(\alpha, \beta)$, and we obtain

$$\begin{aligned}\phi_{1,q}(u) &= (p_{1}\psi_{1,q}(u), \bar{\chi}_{1,q}(u)) && \text{by definition}\\ &= (p_{1}\psi_{1,q}\delta^{*}_{1,q}(\alpha, \beta), p_{2}(\alpha, \beta))\\ &= (\alpha-\beta, \beta) && \text{by Lemma 2.}\end{aligned}$$

Therefore, $\phi_{1,q}$ is an isomorphism. When $p\geqq 2$, by the induction hypothesis, we get an isomorphism

$$\phi_{p-1,q}\colon \pi_{p-1,q}(X\vee X)\cong\pi_{p+q-1}(X)\oplus\pi_{q+1}(X)\,.$$

Consider the forgetful exact sequence (1) with $Y=X\vee X$, $r=p$ and $s=q$. Let $u\in\pi_{p,q}(X\vee X)$. Then $\phi_{p-1,q}(\chi_{p,q}(u))=(0, \bar{\chi}_{p,q}(u))$ since $\psi_{p-1,q}\chi_{p,q}=0$.

We assume $\phi_{p,q}(u)=(0, 0)$. Then $p_1\psi_{p,q}(u)=0$ and $\bar{\chi}_{p,q}(u)=0$ by definition. Since $\phi_{p-1,q}$ is an isomorphism, we get $\chi_{p,q}(u)=0$. Therefore, there exists an element $(\alpha, \beta)\in\pi_{p+q}(X\vee X)$ such that $\delta^*_{p,q}(\alpha, \beta)=u$. It follows from Lemma 2 that (α,β) is congruent to $(\alpha+(-1)^p\beta,0)$ mod Im $\psi_{p-1,q+1}$. Thus $\delta^*_{p,q}(\alpha+(-1)^p\beta, 0)=u$. Applying $p_1\psi_{p,q}$ to this, we get an equality

$$\alpha+(-1)^p\beta = p_1\psi_{p,q}(u) = 0$$

which implies $u=0$. Hence $\phi_{p,q}$ is a monomorphism.

We show that $\phi_{p,q}$ is an epimorphism. Let $(\alpha, \beta)\in\pi_{p+q}(X)\oplus\pi_{q+1}(X)$. By Lemma 3, $\psi_{p-1,q}(\phi_{p-1,q})^{-1}(0, \beta)=(0, 0)$. Therefore, there exists an element $v\in\pi_{p,q}(X\vee X)$ such that $\chi_{p,q}(v)=(\phi_{p-1,q})^{-1}(0, \beta)$. This implies $\bar{\chi}_{p,q}(v)=\beta$. Then we have

$$\phi_{p,q}(\delta^*_{p,q}(\alpha, 0)+v-\delta^*_{p,q}(p_1\psi_{p,q}(v), 0)) = (\alpha, \beta)$$

as can be easily checked.

We now turn to the ρ-action on $\pi_{p,q}(X\vee X)$. Let $u\in\pi_{p,q}(X\vee X)$ and $\phi_{p,q}(u)=(\alpha, \beta)$. Then $\psi_{p,q}(u)=(\alpha, (-1)^p\alpha)$ by Lemma 3. From (2), we obtain $\delta^*_{p,q}(\alpha, (-1)^p\alpha)=u-\rho\cdot u$. Recall that $(\alpha,0)$ is congruent to $(0, (-1)^p\alpha)$ mod Im $\psi_{p-1,q+1}$ if $p\geqq 2$. Thus $2\cdot\delta^*_{p,q}(\alpha, 0)=u-\rho\cdot u$. Applying $\phi_{p,q}$, we obtain

$$2\cdot(\alpha, 0) = \phi_{p,q}(u)-\phi_{p,q}(\rho\cdot u)\,.$$

This shows that $\phi_{p,q}(\rho\cdot u)=(-\alpha, \beta)$ for $p\geqq 2$. Let $u\in\pi_{1,q}(X\vee X)$ and $\phi_{1,q}(u)=(\alpha, \beta)$. Then $\psi_{1,q}(u)=(\alpha, -\alpha)$ by Lemma 3. The same method gives rise to

$$\phi_{1,q}\delta^*_{1,q}(\alpha, -\alpha) = (\alpha, \beta)-\phi_{1,q}(\rho\cdot u)\,.$$

By the definition of $\phi_{1,q}$, the left hand side coincides with $(2\alpha, -\alpha)$. Hence $\phi_{1,q}(\rho\cdot u)=(-\alpha, \alpha+\beta)$ as required. This completes the proof of the theorem stated in the introduction.

Remark. Let $X\times X$ be a τ-space with involution defined by $\tau(x_1, x_2)=(x_2, x_1)$. Then we have an isomorphism of abelian groups

$$\xi_{p,q}\colon \pi_{p,q}(X\times X)\cong\pi_{p+q}(X) \qquad \text{for } p\geqq 1,\ q\geqq 2.$$

The correspondence is given by $\xi_{p,q}(u)=p_1\psi_{p,q}(u)$. Moreover, ρ acts as -1 on $\pi_{p,q}(X\times X)$.

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