Bouleau, N.
Osaka J. Math.
22 (1985), 31–34

# ABOUT STOCHASTIC INTEGRALS WITH RESPECT TO PROCESSES WHICH ARE NOT SEMI-MARTINGALES

NICOLAS BOULEAU

(Received April 26, 1984)

## 1. Introduction

Let $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ be a probability space with an increasing right continuous family of $(\mathcal{F}_\infty, \boldsymbol{P})$-complete $\sigma$-algebras $(\mathcal{F}_t)$, and let $\mathcal{P}$ be the predictable $\sigma$-algebra induced on $\Omega\times\boldsymbol{R}_+$ by the family $(\mathcal{F}_t)$.

For $H\in\mathcal{P}$, we write $H_s$ for the random variable $\omega\rightarrow 1_H(s, \omega)$. If $Z=N+B$ is a semi-martingale such that $N$ is a square integrable martingale and $B$ an adapted process with square integrable variation, the mapping

$$(1) \qquad H\rightarrow\int_0^\infty H_s\,dZ_s$$

defines a $\sigma$-additive vector measure on $(\Omega\times\boldsymbol{R}_+, \mathcal{P})$ with values in $L^2(\Omega, \mathcal{F}_\infty, \boldsymbol{P})$. It has been shown by several authors that conversely if $\mu$ is a $\sigma$-additive measure from $\mathcal{P}$ to $L^2(\Omega, \mathcal{F}_\infty, \boldsymbol{P})$ given on the elementary predictable sets $H$ of the form

$$H=h\times]s, t] \qquad 0<s<t\,, \quad h\in\mathcal{F}_s$$

by

$$(2) \qquad \mu(H)=1_h(Z_t-Z_s)$$

for a mean square right-continuous adapted process $Z$, then there is a modification of $Z$ which is a semi-martingale [2].

Nevertheless, if we consider an other probability space $(W, \mathcal{W}, \boldsymbol{Q})$, an adapted process $(\omega, t)\rightarrow Z_t(\omega, w)$ depending on $w\in W$, and a measure $\mu$ which satisfies (2) for elementary predictable sets, and if we replace $\sigma$-additivity in $L^2(\boldsymbol{P})$ for each $w\in W$ by $\sigma$-additivity in $L^2(\boldsymbol{P}\times\boldsymbol{Q})$, it becomes possible that $Z_t$ fails to be a semi-martingale for fixed $w$.

In the example that we give, $Z_t$ is, for fixed $w$, the sum of a martingale and a process of zero energy similar to those considered by Fukushima [3] in order to give a probabilistic interpretation of functions in a Dirichlet space.

## 2. Random mixing of semi-martingales

Let $(U_\alpha(w))_{\alpha\in\boldsymbol{R}}$ be a second order process on $(W, \mathcal{W}, Q)$ which is right continuous in $L^2$, with orthogonal increments and $\mathcal{B}(\boldsymbol{R})\times\mathcal{W}$ measurable and let $m$ be the positive Radon measure on $\boldsymbol{R}$ associated to $U_\alpha$ by

$$m(]\alpha, \beta]) = \boldsymbol{E}_Q(U_\beta - U_\alpha)^2\,, \quad \alpha<\beta\,.$$

Let $(M_t^\alpha(\omega))_{\alpha\in\boldsymbol{R}}$ be a family of right continuous and left limited martingales, and $(A_t^\alpha(\omega))_{\alpha\in\boldsymbol{R}}$ a family of continuous increasing adapted processes on $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ such that the maps $(\alpha, \omega, s)\rightarrow M_s^\alpha(\omega)$ and $(\alpha, \omega, s)\rightarrow A_s^\alpha(\omega)$ are $\mathcal{B}(\boldsymbol{R})\times\mathcal{F}_t\times\mathcal{B}(\boldsymbol{R}_+)$ measurable on $\boldsymbol{R}\times\Omega\times[0, t]$ and such that

$$(3)\qquad \int_{\alpha\in\boldsymbol{R}} \boldsymbol{E}_P[(M_\infty^\alpha)^2+(A_\infty^\alpha)^2]\, dm(\alpha)<+\infty\,.$$

Then we set $Z_t^\alpha(\omega) = M_t^\alpha(\omega)+A_t^\alpha(\omega)$ and

$$(4)\qquad Z_t(\omega, w) = \int_{\alpha\in\boldsymbol{R}} Z_t^\alpha(\omega)\, dU_\alpha(w)$$

where the stochastic integral is of Wiener's type and exists for $\boldsymbol{P}$ almost all $\omega$ since by (3) $Z_t^\alpha(\omega)$ belongs to $L^2(\boldsymbol{R}, \mathcal{B}(\boldsymbol{R}), dm(x))$ for $\boldsymbol{P}$-almost all $\omega$.

For $\boldsymbol{P}$-almost $\omega$ the process $Z_t(\omega, w)$ is right continuous and left limited in $L^2(W, \mathcal{W}, \boldsymbol{Q})$.

If $G$ is an elementary predictable process on $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ given by:

$$G_s(\omega) = G_0(\omega)\, 1_{]0,t_1]}(s)+\cdots+G_n(\omega)\, 1_{]t_n,t_{n+1}]}(s)$$

for $0<t_1<\cdots<t_{n+1}$, where $G_i$ is a $\mathcal{F}_{t_i}$-measurable bounded random variable, it follows immediately

$$G_0(Z_{t_1}-Z_0)+\cdots+G_n(Z_{t_{n+1}}-Z_{t_n}) = \int_{\alpha\in\boldsymbol{R}} \left(\int_0^\infty G_s\, dZ_s^\alpha\right) dU_\alpha\,.$$

And we have:

**Proposition 1.** *The map $H\in\mathcal{P}\rightarrow\int_0^t H_s\, dZ_s$ defined by*

$$\int_0^t H_s\, dZ_s = \int_{\alpha\in\boldsymbol{R}} \left(\int_0^t H_s\, dZ_s^\alpha\right) dU_\alpha$$

*is a $\sigma$-additive $L^2(\boldsymbol{P}\times\boldsymbol{Q})$ valued measure on $(\Omega\times\boldsymbol{R}_+, \mathcal{P})$.*

Proof. Let $H^{(n)}$ be a sequence of disjoint predictable subsets of $\Omega\times\boldsymbol{R}_+$, we have

$$\boldsymbol{E}_P\,\boldsymbol{E}_Q\left[\int_{\alpha\in\boldsymbol{R}}\left(\int_0^t \sum_{n=N}^\infty H_s^{(n)}\, dZ_s^\alpha\right) dU_\alpha\right]^2$$

$$= \int_{\alpha\in\boldsymbol{R}} \boldsymbol{E}_P\left(\int_0^t \sum_{n=N}^\infty H_s^{(n)}\, dZ_s^\alpha\right)^2 dm(\alpha)$$

which can be made arbitrarily small for $N$ large enough because

$$\boldsymbol{E}_{\boldsymbol{P}}\left(\int_0^t \sum_{n=N}^{\infty} H_s^{(n)}\, dZ_s^{\alpha}\right)^2$$

tends to zero and remains bounded by

$$2\, \boldsymbol{E}_{\boldsymbol{P}}\, [(M_{\infty}^{\alpha})^2+(A_{\infty}^{\alpha})^2]<+\infty. \ \square$$

Set

$$Z_t^{(1)} = \int_{\alpha\in\boldsymbol{R}} M_t^{\alpha}\, dU_{\alpha} \quad \text{and} \quad Z_t^{(2)} = \int_{\alpha\in\boldsymbol{R}} A_t^{\alpha}\, dU_{\alpha}\,.$$

**Lemma 2.** *There is a $\boldsymbol{P}\times\boldsymbol{Q}$-modification $\tilde{Z}_t^{(1)}$ of $Z_t^{(1)}$ which is a $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ right continuous and left limited martingale for $\boldsymbol{Q}$-almost all $w$.*

Proof. Let $G\in\mathcal{F}_s$, the following equalities hold in $L^2(W, \mathcal{W}, \boldsymbol{Q})$ for $s<t$:

$$\begin{aligned}\boldsymbol{E}_{\boldsymbol{P}}[1_G\, Z_t^{(1)}] &= \int_{\alpha\in\boldsymbol{R}} \boldsymbol{E}_{\boldsymbol{P}}[1_G\, M_t^{\alpha}]\, dU_{\alpha} = \int_{\alpha\in\boldsymbol{R}} \boldsymbol{E}_{\boldsymbol{P}}[1_G\, M_s^{\alpha}]\, dU_{\alpha}\\ &= \boldsymbol{E}_{\boldsymbol{P}}[1_G\, Z_s^{(1)}]\,,\end{aligned}$$

therefore, if we choose a $\mathcal{F}_t\times\mathcal{W}$-measurable element $z_t^{(1)}(\omega, w)$ in the $L^2(\boldsymbol{P}\times\boldsymbol{Q})$ equivalence class of $Z_t^{(1)}$, for $w$ outside a $\boldsymbol{Q}$-negligible set $\mathcal{N}$, $z_s^{(1)}$ is a $(\mathcal{F}_s, \boldsymbol{P})$-martingale for rational $s$.
Then, if we put $\tilde{Z}_t^{(1)}= \lim_{\substack{s \text{ rational} \\ s\downarrow t}} z_s^{(1)}$, for $w\notin\mathcal{N}$, $\tilde{Z}_t^{(1)}$ is $\boldsymbol{P}$-almost surely a right continuous and left limited $(\mathcal{F}_t)$-martingale and

$$\tilde{Z}_t^{(1)} = Z_t^{(1)} \qquad \boldsymbol{P}\times\boldsymbol{Q}\text{-a.e.}$$

because $Z_t^{(1)}$ is right continuous in $L^2(\boldsymbol{P}\times\boldsymbol{Q})$. $\square$

As concerns $Z_t^{(2)}$, it is a zero energy process:

**Lemma 3.** *Let $\tau_n$ be a sequence of partitions of* $[0, t]$ *with diameter tending to zero, then*

$$\boldsymbol{E}_{\boldsymbol{Q}}\, \boldsymbol{E}_{\boldsymbol{P}}[\sum_{t_i\in\tau_n} (Z_{t_{i+1}}^{(2)}-Z_{t_i}^{(2)})^2] \underset{n\uparrow\infty}{\rightarrow} 0\,.$$

Proof. The expression is equal to

$$\boldsymbol{E}_{\boldsymbol{P}} \int_{\alpha\in\boldsymbol{R}} \sum_{\tau_n} (A_{t_{i+1}}^{\alpha}-A_{t_i}^{\alpha})^2\, dm(\alpha)\,,$$

and $\sum_{\tau_n} (A_{t_{i+1}}^{\alpha}-A_{t_i}^{\alpha})^2$ tends to zero, because $A_t^{\alpha}$ is continuous, and remains majorized by $(A_{\infty}^{\alpha})^2$, which gives the result by (3).

Nevertheless, in general $Z_t^{(2)}$ has no modification with finite variation, as shown by the following example:

Let $X$ be a continuous martingale on $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ such that

$$\boldsymbol{E}_{\boldsymbol{P}} X_\infty^2 < +\infty .$$

Let

$$M_t^\alpha = \int_0^t 1_{\{X_s > \alpha\}} \, dX_s$$

and $A_t^\alpha = \dfrac{1}{2} L_t^\alpha$

where $L_t^\alpha$ is the local time of $X$ at $\alpha$. Condition (3) is satisfied as soon as the measure $m$ is finite. If we put

$$Z_t = \int_{\alpha \in \boldsymbol{R}} M_t^\alpha \, dU_\alpha + \int_{\alpha \in \boldsymbol{R}} A_t^\alpha \, dU_\alpha$$

we have, from Meyer-Tanaka's formula:

$$Z_t = \int_{\alpha \in \boldsymbol{R}} [(X_t - \alpha)^+ - (X_0 - \alpha)^+] \, dU_\alpha = \int_{X_0}^{X_t} U_\lambda \, d\lambda \quad \boldsymbol{P} \times \boldsymbol{Q} \text{ a.e.}$$

If $Z_t$ had a $\boldsymbol{P}\times\boldsymbol{Q}$-modification such that, for fixed $w \in W$, $\tilde{Z}_t$ were a $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ semi-martingale, then, since $\tilde{Z}_t$ and $\int_{X_0}^{X_t} U_\lambda d\lambda$ are both right continuous, $\int_{X_0}^{X_t} U_\lambda d\lambda$ would be a semi-martingale. So, from ([1], theorem 5, 6), if we took for $X$ a real stopped brownian motion starting at 0, the map

$$x \overset{\psi}{\rightarrow} \int_0^x U_\lambda(w) \, d\lambda$$

would be the difference of two convex functions. But, if for example, $U$ itself is a stopped brownian motion, that can be true only on a $\boldsymbol{Q}$-negligible set because almost all brownian sample paths have not finite variation. So, in this case, the $\boldsymbol{P}\times\boldsymbol{Q}$-modifications of $Z_t$ are $\boldsymbol{Q}$ a.e. not semi-martingales.

---

## References

[1] E. Cinlar, J. Jacod, P. Protter and M.J. Sharpe: *Semimartingales and Markov processes*, Z. Wahrsch. Verw. Gebiete **54** (1980), 161–219.

[2] C. Dellacherie and P.A. Meyer: Probabilités et potentiel, théorie des martingales, Hermann, 1980.

[3] M. Fukushima: Dirichlet forms and Markov processes, North-Holland, 1980.

Ecole Nationale des Ponts et
Chaussées
28, rue des Saints-Pères
75007, Paris
France