Komatsu, G. and Ozawa, S.
Osaka J. Math.
22 (1985), 1–20

VARIATION OF THE BERGMAN KERNEL BY CUTTING A HOLE

In memory of Professor Hitoshi Kumano-go

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(Received January 5, 1984)
(Revised April 13, 1984)

Introduction

The present paper is concerned with the variation of the Bergman kernel of a bounded domain in $\boldsymbol{C}^n$ by cutting a *hole* that is, a relatively closed subset such that the remaining open set is connected and non-empty. In case a hole is small enough, we shall present an explicit formula representing the Bergman kernel of the holed domain in terms of that of the original domain, where the required smallness of the hole will be also determined exactly. A similar problem has been discussed in Schiffer-Spencer [32], §6.15.

Recently, much progress has been made on the study of the dependence of the Bergman kernel on a domain which is assumed to be strictly pseudoconvex, see Fefferman [17], [18] and the references therein; see also Greene-Krantz [20], [21] and Bergman-Schiffer [13], Komatsu [28] for the variation under a smooth perturbation of the boundary. Contrary to these works, our present method is elementary and results are of general character; in particular, no smoothness of the boundary will be required.

The Bergman kernel $K_\Omega(z, w)$ for $z, w \in \Omega$ of a bounded domain Ω in $\boldsymbol{C}^n$ is the reproducing kernel of the space $L^2H(\Omega)$ of square integrable holomorphic functions in Ω, so that it involves, in principle, all information on $L^2H(\Omega)$. Hence, if a hole ω of Ω is so small that the restriction mapping R: $L^2H(\Omega) \to L^2H(\Omega\backslash\omega)$ has a dense range, then one may expect that $K_{\Omega\backslash\omega}(z, w)$ is expressed in terms of $K_\Omega(\cdot, \cdot)$. This is indeed the case; in fact, we shall show that

$$K_{\Omega\backslash\omega}(z, w) = K_\Omega(z, w) + \sum_{m=1}^{\infty} T_{\Omega,\omega}^{(m)}(z, w) \qquad \text{for } z, w \in \Omega\backslash\omega\,,$$

where each $T_{\Omega,\omega}^{(m)}(z, w)$ for $z, w \in \Omega$ is given by

*) The authors were partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

$$\int \cdots \int_{\omega^m} K_\Omega(z, \zeta_1) K_\Omega(\zeta_1, \zeta_2) \cdots K_\Omega(\zeta_m, w) dV(\zeta_1) \cdots dV(\zeta_m) .$$

If a hole ω is too large, then this formula is no more valid. However, its right hand side still makes sense and represents the reproducing kernel of the closure of the range of the mapping R (Theorem 2). In particular, we obtain a refined version of the monotonicity of the Bergman kernel with respect to the domain. We shall also show that if the mapping R has a closed range, then the right hand side of the formula above makes sense for $z, w \in \Omega$ and defines a bounded integral operator in $L^2H(\Omega)$ (Theorem 3).

An application of the formula above will be given. If $n \geqq 2$ and a hole ω is compact, then Hartogs' removable singularity theorem asserts that the restriction mapping R is bijective. In this case, it is easy to see that various regularity properties of the Bergman projector are preserved by cutting ω from Ω. In order to illustrate it, we shall show that the so-called conditions R and Q inherit from Ω to $\Omega \backslash \omega$ (Theorems 4 and 5). These conditions are boundary regularity properties of C^∞ and C^ω classes, respectively, see Subsection 1.2. That the condition R is preserved has been already proved by Bell [8] by a different method assuming the smoothness of the boundary $\partial\Omega$, which we do not require, see Remark 1.2. Our proof reveals that the loss of derivatives is also preserved (Theorem 4′).

The proofs of Theorems 2 and 3 use only elementary Functional Analysis. However, in order to emphasize the simplicity of the proofs of Theorems 4 and 5, we shall present a still more elementary proof of Theorems 2 and 3 in that case by using a so-called doubly orthogonal system (Theorem 1 and its proof). The proof of Theorem 1 will also clarify the situation concerning Theorems 2 and 3, cf. Section 2.

The present paper is organized as follows. The main results (Theorems 1, 2 and 3) are stated in Subsection 1.1 and are proved in Section 2. We present results on the preservation of regularity properties of the Bergman projector (Theorems 4, 5 and 4′) in Subsection 1.2 and prove them in Section 3.

Both authors have had stimulating conversations on the subject of the present paper with Kiyômi Kataoka to whom they express their thanks.

1. Statement of results

1.1. Variation of the Bergman kernel. Suppose given a bounded domain Ω in $\boldsymbol{C}^n$ with $n \geqq 1$. We denote by $L^2H(\Omega)$ the totality of square integrable holomorphic functions in Ω, which is a closed subspace of $L^2(\Omega)$. The orthogonal projector K_Ω: $L^2(\Omega) \to L^2H(\Omega) \subset L^2(\Omega)$ is called the *Bergman projector* of Ω. Recall that

$$K_\Omega u(z) = \int_\Omega K_\Omega(z, w) u(w) dV(w) \qquad \text{for } u \in L^2(\Omega) ,$$

where dV denotes the standard volume element of $\boldsymbol{C}^n$ and the function $K_\Omega(\cdot, \cdot)$ in $\Omega\times\Omega$ stands for the *Bergman kernel* of Ω, which is the reproducing kernel of the space $L^2H(\Omega)$.

Suppose also given a relatively closed subset ω of Ω such that $\Omega\backslash\omega$ is connected and non-empty. We shall call ω a *hole* of Ω. We are concerned with expressing $K_{\Omega\backslash\omega}(\cdot, \cdot)$ in terms of $K_\Omega(\cdot, \cdot)$. Let us begin with the following simplest case.

Theorem 1. *If* $n\geqq 2$ *and* $\omega\Subset\Omega$, *then*

$$K_{\Omega\backslash\omega}(z, w) = K_\Omega(z, w)+\sum_{m=1}^{\infty} T_{\Omega,\omega}^{(m)}(z, w) \qquad \text{for } z, w\in\Omega\,, \tag{1.1}$$

where each $T_{\Omega,\omega}^{(m)}(z, w)$ *is given by*

$$\int\cdots\int_{\omega^m} K_\Omega(z, \zeta_1)K_\Omega(\zeta_1, \zeta_2)\cdots K_\Omega(\zeta_m, w)dV(\zeta_1)\cdots dV(\zeta_m)\,,$$

and the series $\sum_{m=1}^{\infty}T_{\Omega,\omega}^{(m)}(\cdot, \cdot)$ *is absolutely convergent uniformly in every compact subset of* $\Omega\times\Omega$. *Furthermore, the right hand side of* (1.1) *defines a bounded integral operator in* $L^2H(\Omega)$.

Even if a hole ω of Ω is not necessarily compact, we may define $T_{\Omega,\omega}^{(m)}(\cdot, \cdot)$ in $\Omega\times\Omega$ by setting

$$T_{\Omega,\omega}^{(m)}(z, w) = [(K_\Omega\chi_\omega)^m K_\Omega(\cdot, w)]\,(z) \qquad \text{for } m\geqq 0\,, \tag{1.2}$$

so that $T_{\Omega,\omega}^{(0)}(z, w)=K_\Omega(z, w)$, where χ_ω stands for the characteristic function of the set ω. Then, we have in general that:

Theorem 2. *The series* $\sum_{m=0}^{\infty}T_{\Omega,\omega}^{(m)}(\cdot, \cdot)$ *is absolutely convergent uniformly in every compact subset of* $(\Omega\backslash\omega)\times(\Omega\backslash\omega)$ *and represents the integral kernel of the orthogonal projector of* $L^2H(\Omega\backslash\omega)$ *onto the closure* $\widetilde{L^2H(\Omega)}$ *of* $L^2H(\Omega)_{|\Omega\backslash\omega}$. *In other words, it is the reproducing kernel of the space* $\widetilde{L^2H(\Omega)}$.

Let R: $L^2H(\Omega)\rightarrow L^2H(\Omega\backslash\omega)$ denote the restriction mapping. Then, Theorem 2 asserts that the range of R is dense if and only if

$$K_{\Omega\backslash\omega}(z, w) = K_\Omega(z, w)+\sum_{m=1}^{\infty} T_{\Omega,\omega}^{(m)}(z, w) = \sum_{m=0}^{\infty} T_{\Omega,\omega}^{(m)}(z, w)$$

for $z, w\in\Omega\backslash\omega$. In general, the left hand side must be replaced by the reproducing kernel of the space $\widetilde{L^2H(\Omega)}$.

If the range of R is closed, then we can say more, that is:

Theorem 3. *If* $L^2H(\Omega)_{|\Omega\backslash\omega}=\widetilde{L^2H(\Omega)}$, *then the series* $\sum_{m=0}^{\infty}T_{\Omega,\omega}^{(m)}(\cdot, \cdot)$ *is absolutely convergent uniformly in every compact subset of* $\Omega\times\Omega$ *and defines a bounded integral operator in* $L^2H(\Omega)$.

Observe that Theorem 1 is a special case of Theorems 2 and 3 by virtue of Hartogs' removable singularity theorem. Conditions on the range of R will be further discussed in Section 2.

It will be easily seen that $T^{(m)}_{\Omega,\omega}(\cdot,\cdot)\gg 0$ in Ω in the sense that $\sum_{j,k=1}^{N} T^{(m)}_{\Omega,\omega}(z_j, z_k)\xi_j\bar{\xi}_k \geqq 0$ for $z_1, \cdots, z_N \in \Omega$ and $\xi_1, \cdots, \xi_N \in \boldsymbol{C}$, see (2.7) in the proof of Theorem 2. Hence, Theorem 2 provides a refinement of the monotonicity

$$(1.3) \qquad K_{\Omega\backslash\omega}(\cdot,\cdot)-K_{\Omega}(\cdot,\cdot)\gg 0 \qquad \text{in } \Omega\backslash\omega\,,$$

see Aronszajn [2], Theorem II in p. 355. In fact, setting

$$\tilde{K}^{\perp}_{\Omega\backslash\omega}(z, w)=K_{\Omega\backslash\omega}(z, w)-\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(z, w) \qquad \text{for } z, w\in\Omega\backslash\omega\,,$$

we see by Theorem 2 that $\tilde{K}^{\perp}_{\Omega\backslash\omega}(\cdot,\cdot)$ is the reproducing kernel of the orthogonal complement of $\widetilde{L^2H(\Omega)}$ in $L^2H(\Omega\backslash\omega)$, so that $\tilde{K}^{\perp}_{\Omega\backslash\omega}(\cdot,\cdot)\gg 0$ in $\Omega\backslash\omega$, which refines (1.3). Note that (1.3) is a generalization of the well-known monotonicity $K_{\Omega\backslash\omega}(z, z)\geqq K_{\Omega}(z, z)$ for $z\in\Omega\backslash\omega$, see Bergman [12], pp. 44–45.

1.2. Preservation of regularity properties of the Bergman projector. Let us now assume that $n\geqq 2$ and that a hole ω of Ω is compact. Then, by virtue of Theorem 1, one may easily see that various regularity properties of the Bergman projector inherit from Ω to $\Omega\backslash\omega$, for the series in the right hand side of (1.1) will represent a smoothing kernel. In order to illustrate it, we shall show that the conditions R and Q are preserved by cutting ω from Ω. For the sake of simplicity, we shall assume that the boundary of Ω is smooth, see Remark 1.2.

Recall that Ω is said to satisfy the *condition R* if K_{Ω} maps $C^{\infty}(\bar{\Omega})$ into itself (continuously), cf. Bell-Ligocka [11], Bell [5]; and the *condition Q* (or Q1 in [6]) if K_{Ω} maps $C_0^{\infty}(\Omega)$ into the space $C^{\omega}(\bar{\Omega})$ of real analytic functions in Ω which extend real analytically to neighborhoods of $\bar{\Omega}$, where the neighborhood may depend on each function, cf. Bell [6], [7]. (In fact, $K_{\Omega}u$ with $u\in C_0^{\infty}(\Omega)$ is holomorphic near $\bar{\Omega}$ under the condition Q.) Observe that the continuity of K_{Ω} in the condition R follows automatically by virtue of the closed graph theorem. We need not consider any topology of the space $C^{\omega}(\bar{\Omega})$.

We shall prove that

Theorem 4 (Bell [8]). *If Ω satisfies the condition R, so does $\Omega\backslash\omega$.*

Theorem 5. *If Ω satisfies the condition Q, so does $\Omega\backslash\omega$.*

It should be mentioned that Theorem 4 has been already proved by Bell [8]. Our proofs of Theorems 4 and 5 are based on Theorem 1 (and its proof), and are thus very simple. Moreover, our proof of Theorem 4 involves that the loss of derivatives is also preserved by cutting ω from Ω in the following sense.

By definition, the condition R is equivalent to the existence of a mapping M: $\boldsymbol{N}\to\boldsymbol{N}$ such that

$$(R;\, M)\quad K_{\Omega}\colon W^{M(s)}(\Omega) \to W^{s}(\Omega) \ \text{ is bounded for } s\in\boldsymbol{N},$$

where $W^s(\Omega)$ denotes the $L^2(\Omega)$ Sobolev space of order s, and $\boldsymbol{N}=\{1, 2, \cdots\}$. It has been known that the condition R is further equivalent to the existence of a mapping M_0: $\boldsymbol{N}\to\boldsymbol{N}$ such that

$$(R_0;\, M_0)\ K_{\Omega}\colon W_0^{M_0(s)}(\Omega) \to W^{s}(\Omega) \ \text{ is bounded for } s\in\boldsymbol{N},$$

where $W_0^s(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in $W^s(\Omega)$, see Barrett [3], also Bell-Boas [9]. (Again, the boundedness of K_Ω in $(R;\, M)$ and $(R_0;\, M_0)$ is not necessary to assume by virtue of the closed graph theorem, though we shall not use this fact.)

We shall actually show the following:

Theorem 4′. *If Ω satisfies $(R;\, M)$ (resp. $(R_0;\, M_0)$), so does $\Omega\backslash\omega$ with the same mapping M (resp. M_0).*

The condition R has been used successfully in the problem of extending a given biholomorphic or proper holomorphic mapping smoothly up to the boundary, see Ligocka [30], Bell-Ligocka [11], Bell [5], Bell-Catlin [10], Diederich-Fornaess [16], and the references therein. Likewise, the condition Q has appeared in the corresponding holomorphic extension problem, see Bell [6], [7].

The proof of Theorem 4′ is a prototype of that of the preservation of a regularity property of the Bergman projector by cutting ω from Ω. It can be easily modified to be applied to an another regularity property as in Phong-Stein [31]. Also, Theorems 4′ and 5 can be localized.

REMARK 1.1. The conditions R and Q are also related to regularity properties of the $\bar\partial$-Neumann problem of Kohn [24], [25] *via* the formula $K_\Omega = 1-\bar\partial^* N\bar\partial$, where $\bar\partial^*$ denotes the L^2 adjoint of $\bar\partial$ and N stands for the $\bar\partial$-Neumann operator acting on $(0, 1)$-forms, see Kerzman [23]. The property $(R;M)$ with $M(s)=s$ is known to be satisfied if the $\bar\partial$-Neumann problem is subelliptic, a fact which is implicitly involved in the proof of the hypoellipticity, see Kohn-Nirenberg [26], Folland-Kohn [19]. The condition Q follows from the (global) analytic-hypoellipticity of the $\bar\partial$-Neumann problem. For the analytic-hypoellipticity, see Komatsu [27], Derridj-Tartakoff [15], Treves [34], Tartakoff [33].

The works on the hypoellipticity and the analytic-hypoellipticity of the $\bar\partial$-Neumann problem also involve the conclusions of Theorems 4 and 5, respectively, in certain cases, where very special assumptions on both boundaries

$\partial\omega$ and $\partial\Omega$ must be imposed. A typical example of such assumptions is that $n \geqq 3$ and that both Ω and ω are strictly pseudo-convex with C^∞ (resp. C^ω) boundary. See also Bell [8].

The property $(R_0;\ M_0)$ with $M_0(s)=s$ is known to be equivalent to the validity of a certain duality between spaces of holomorphic functions in L^2 Sobolev spaces of positive and negative order, see Komatsu [29].

REMARK 1.2. No smoothness of the boundary $\partial\Omega$ will be required in the proof of Theorem 4 if we regard the space $C^\infty(\bar{\Omega})$ as the intersection $W^\infty(\Omega) = \cap\, W^s(\Omega)$, cf. Theorem 4′. In fact, the smoothness of the boundary $\partial\Omega$ is only used in order to guarantee that $(R_0;\ M_0)$ with an arbitrary M_0 fixed implies $(R;\ M)$ with some M. By virtue of Sobolev's lemma, the relation $W^\infty(\Omega)=C^\infty(\bar{\Omega})$ holds in the usual sense if Ω has, for instance, a Lipschitz boundary.

On the other hand, Theorem 5 requires some regularity condition on the boundary $\partial\Omega$. It is enough to assume that Ω has a Lipschitz boundary, or, more generally, that Ω has the so-called cone property if the definition of the space $C^\omega(\bar{\Omega})$ is modified appropriately. For the detail, see Subsection 3.3.

It should be noticed that we need not assume any regularity condition on $\partial\omega$, for $K_{\Omega\backslash\omega}u$ with $u\in L^2(\Omega\backslash\omega)$ extends holomorphically to Ω by virtue of Hartogs' removable singularity theorem.

2. Proofs of Theorems 1, 2 and 3

Proof of Theorem 1. By Montel's theorem, the restriction mapping $L^2H(\Omega)\to L^2(\omega)$ is compact. It then follows from F. Riesz' representation theorem that there exists a compact Hermitian operator T in $L^2H(\Omega)$ such that

$$(2.1)\qquad (Tf,g)^\Omega=(f,g)^\omega \qquad \text{for } f,g\in L^2H(\Omega)\,,$$

where $(\cdot,\cdot)^\Omega$ denotes the $L^2(\Omega)$ scalar product with the corresponding norm $\|\cdot\|^\Omega$, and similarly for $(\cdot,\cdot)^\omega$. (In fact, $T=K_\Omega\chi_\omega$, cf. (1.2).) Hence, T admits a discrete spectral decomposition

$$T=\sum_{j=0}^{\infty}\lambda_j\phi_j\otimes\bar{\phi}_j \qquad \text{with } 1>\lambda_j\geqq\lambda_{j+1}\geqq 0\,,$$

where $\{\phi_j\}_j$ is a complete orthonormal system of $L^2H(\Omega)$ and satisfies

$$(\phi_j,\phi_k)^\omega=\lambda_j\delta_{jk}\,,\qquad \lambda_j=(\|\phi_j\|^\omega)^2\,.$$

That is, $\{\phi_j\}_j$ is a so-called *doubly orthogonal system*, cf. Bergman [12].

Since $n\geqq 2$ and ω is a compact hole of Ω, it follows from Hartogs' removable singularity theorem that the restriction mapping R: $L^2H(\Omega)\to L^2H(\Omega\backslash\omega)$ is bijective. Hence, $\{(1-\lambda_j)^{-1/2}\phi_{j|\Omega\backslash\omega}\}_j$ is a complete orthonormal system of

$L^2H(\Omega\backslash\omega)$, so that

$$K_{\Omega\backslash\omega}(z, w) = \sum_{j=0}^{\infty} (1-\lambda_j)^{-1} \phi_j(z)\overline{\phi_j(w)} \qquad \text{for } z, w\in\Omega ,$$

where the series in the right hand side is absolutely convergent uniformly in every compact subset of $\Omega\times\Omega$ and defines a bounded integral operator in $L^2H(\Omega)$. Noticing that

$$T^{(m)}_{\Omega,\omega}(z, w) = \sum_{j=0}^{\infty} (\lambda_j)^m \phi_j(z)\overline{\phi_j(w)} ,$$

we obtain the desired conclusion. q.e.d.

REMARK 2.1. The largest eigenvalue λ_0 of T provides the maximum of $(||f||^{\omega}/||f||^{\Omega})^2$ for all non-zero $f\in L^2H(\Omega)$, and the maximum is achieved by the corresponding eigenfunction $f=\phi_0$, see (2.1). Compare it with a variational problem characterizing the Bergman kernel, that is, for $z\in\Omega$ fixed,

$$K_\Omega(z, z) = \max\{(|g(z)|/||g||^{\Omega})^2;\ 0\neq g\in L^2H(\Omega)\} ,$$

and the maximum is achieved by $g=K_\Omega(\cdot, z)/K_\Omega(z, z)$, see Bergman [12], pp. 21–22.

REMARK 2.2. Observe that $||T||=\lambda_0<1$, where $||T||$ denotes the operator norm of T. By Remark 2.1, we have

$$(1-||T||)(||f||^{\Omega})^2 \leqq (||f||^{\Omega\backslash\omega})^2 \qquad \text{for } f\in L^2H(\Omega) .$$

Compare it with a fact that the restriction mapping R is bijective, see also Remark 2.5.

Since $||T||<1$, we have a Neumann series expansion

$$(1-T)^{-1} = 1+T+T^2+\cdots \qquad \text{in } L^2H(\Omega) ,$$

which is an operator version of (1.1).

REMARK 2.3. In Remarks 2.1 and 2.2 above, we have used a fact that T is a compact operator, which is a consequence of the assumption of Theorem 1. More can be said from a weaker assumption

$$\int_\omega K_\Omega(\zeta, \zeta)dV(\zeta)<+\infty . \tag{2.2}$$

Namely, (2.2) holds if and only if $T=K_\Omega\chi_\omega$ is of trace class. In fact, T has the integral kernel $T^{(1)}_{\Omega,\omega}(\cdot, \cdot)$, while

$$\int_\Omega T^{(1)}_{\Omega,\omega}(z, z)dV(z) = \iint_{\Omega\times\omega} |K_\Omega(z, \zeta)|^2 dV(z)dV(\zeta)$$
$$= \int_\omega K_\Omega(\zeta, \zeta)dV(\zeta) .$$

Moreover, $|T^{(1)}_{\Omega,\omega}(z, w)|^2 \leqq T^{(1)}_{\Omega,\omega}(z, z) T^{(1)}_{\Omega,\omega}(w, w)$, implying that

$$\int_{\Omega\times\Omega} |T^{(1)}_{\Omega,\omega}(z, w)|^2 dV(z) dV(w) \leqq \Big(\int_\omega K_\Omega(\zeta, \zeta) dV(\zeta)\Big)^2 .$$

Hence, if (2.2) is satisfied, then T is compact and the trace $\sum_{j=0}^\infty \lambda_j$ of T is given by the left hand side of (2.2). (Recall that $T\geqq 0$ and $T^{(1)}_{\Omega,\omega}(\cdot, \cdot) \gg 0$ in Ω.) Conversely, (2.2) holds if T is of trace class.

Suppose that (2.2) is satisfied. Then $T^{(1)}_{\Omega,\omega}(\cdot, \cdot) \in L^2(\Omega\times\Omega)$. Furthermore,

$$\iint_{\Omega\times\Omega} |T^{(1)}_{\Omega,\omega}(z, w)|^2 dV(z) dV(w) = \sum_{j=0}^{\infty} (\lambda_j)^2 < +\infty .$$

Recalling that

$$\sum_{m=1}^{\infty} T^{(m)}_{\Omega,\omega}(z, w) = \sum_{j=0}^{\infty} \frac{\lambda_j}{1-\lambda_j} \phi_j(z) \overline{\phi_j(w)} ,$$

we also see that the operator $\sum_{m=1}^\infty T^m$ is of trace class, and that its integral kernel $\sum_{m=1}^\infty T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$ belongs to $L^2(\Omega\times\Omega)$. More precisely,

$$\int_\Omega \sum_{m=1}^{\infty} T^{(m)}_{\Omega,\omega}(z, z) dV(z) = \sum_{j=0}^{\infty} \frac{\lambda_j}{1-\lambda_j} < +\infty ,$$

$$\iint_{\Omega\times\Omega} \Big|\sum_{m=1}^{\infty} T^{(m)}_{\Omega,\omega}(z, w)\Big|^2 dV(z) dV(w) = \sum_{j=0}^{\infty} \Big(\frac{\lambda_j}{1-\lambda_j}\Big)^2 < +\infty .$$

Compare these inequalities with Lemmas 3.1 and 3.2.

Proof of Theorem 2. Recall the definition (1.2) of $T^{(m)}_{\Omega,\omega}(z, w)$; that is, for $m\geqq 0$ and $z, w\in\Omega$,

$$T^{(m)}_{\Omega,\omega}(z, w) = [T^m K_\Omega(\cdot, w)](z) = (T^m K_\Omega(\cdot, w), K_\Omega(\cdot, z))^\Omega , \tag{2.3}$$

where $T=K_\Omega \chi_\omega$. In order to define an integral operator $T^{(m)}_{\Omega\backslash\omega}$ in $L^2H(\Omega\backslash\omega)$ with the kernel $T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$, we begin with observing that the adjoint operator R^*: $L^2H(\Omega\backslash\omega) \to L^2H(\Omega)$ of the restriction mapping R: $L^2H(\Omega) \to L^2H(\Omega\backslash\omega)$ is given by $R^*=K_\Omega E$, where E: $L^2(\Omega\backslash\omega) \to L^2(\Omega)$ is an extension mapping defined by setting $Eu(z)=0$ for $z\in\omega$. We set $T^{(m)}_{\Omega\backslash\omega}=RT^m R^*$ for $m\geqq 0$. Since $R^* K_{\Omega\backslash\omega}(\cdot, z) = K_\Omega(\cdot, z)$, it follows from (2.3) that

$$T^{(m)}_{\Omega\backslash\omega} K_{\Omega\backslash\omega}(\cdot, z) = R T^{(m)}_{\Omega,\omega}(\cdot, z) \qquad \text{for } z\in\Omega\backslash\omega .$$

Hence, for $f\in L^2H(\Omega\backslash\omega)$ and $z\in\Omega\backslash\omega$,

$$\begin{aligned} T^{(m)}_{\Omega\backslash\omega} f(z) &= (T^{(m)}_{\Omega\backslash\omega} f, K_{\Omega\backslash\omega}(\cdot, z))^{\Omega\backslash\omega} \\ &= (f, T^{(m)}_{\Omega\backslash\omega} K_{\Omega\backslash\omega}(\cdot, z))^{\Omega\backslash\omega} = (f, T^{(m)}_{\Omega,\omega}(\cdot, z))^{\Omega\backslash\omega} . \end{aligned} \tag{2.4}$$

Therefore, $T^{(m)}_{\Omega\backslash\omega}$ has the integral kernel $T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$.

We next show that the series $\sum_{m=0}^{\infty} T^{(m)}_{\Omega\backslash\omega}$ converges strongly to the orthogonal projector

$$\tilde{K}_{\Omega\backslash\omega}\colon L^2H(\Omega\backslash\omega) \to \widetilde{L^2H(\Omega)} \subset L^2H(\Omega\backslash\omega)\,.$$

It is immediately seen from the definition that $T^{(m)}_{\Omega\backslash\omega}$ is a bounded non-negative Hermitian operator and that

$$\text{(2.5)}\qquad \mathrm{Range}(T^{(m)}_{\Omega\backslash\omega}) \subset L^2H(\Omega)_{|\Omega\backslash\omega},\ \mathrm{Ker}(T^{(m)}_{\Omega\backslash\omega}) \supset (\widetilde{L^2H(\Omega)})^{\perp}\,,$$

where the second relation is obtained by taking the orthogonal complements of the both hand sides of the first one with respect to $L^2H(\Omega\backslash\omega)$. By virtue of (2.5), we may work substantially in $L^2H(\Omega)_{|\Omega\backslash\omega}$. Observe that

$$R^*R = K_\Omega \chi_{\Omega\backslash\omega} = 1 - T\,.$$

Then, given $\tilde{f},\ \tilde{g} \in L^2H(\Omega)$, we have

$$(T^{(m)}_{\Omega\backslash\omega} R\tilde{f},\ R\tilde{g})^{\Omega\backslash\omega} = (T^m R^* R\tilde{f},\ \tilde{g})^{\Omega\backslash\omega} = (T^m \tilde{f} - T^{m+1}\tilde{f},\ \tilde{g})^{\Omega\backslash\omega}\,,$$

so that, for $N \geqq 1$,

$$\begin{aligned}(\tilde{f},\ \tilde{g})^{\Omega\backslash\omega} - \sum_{m=0}^{N-1} (T^{(m)}_{\Omega\backslash\omega} R\tilde{f},\ R\tilde{g})^{\Omega\backslash\omega} \\ = (T^N \tilde{f},\ \tilde{g})^{\Omega\backslash\omega} = ((T^N - T^{N+1})\tilde{f},\ \tilde{g})^{\Omega}\,.\end{aligned}$$

Since $0 \leqq T \leqq 1$ as a Hermitian operator in $L^2H(\Omega)$, it follows that the sequence $\{T^N\}_N$ is non-increasing and bounded, so that it converges strongly. Hence,

$$\text{(2.6)}\qquad \sum_{m=0}^{\infty} (T^{(m)}_{\Omega\backslash\omega} R\tilde{f},\ R\tilde{g})^{\Omega\backslash\omega} = (\tilde{f},\ \tilde{g})^{\Omega\backslash\omega} = (\tilde{K}_{\Omega\backslash\omega} R\tilde{f},\ R\tilde{g})^{\Omega\backslash\omega}\,.$$

Taking (2.5) into account, we obtain by using (2.6) that, for $N \geqq 0$,

$$\sum_{m=0}^{N} (T^{(m)}_{\Omega\backslash\omega} f,\ f)^{\Omega\backslash\omega} \leqq (\tilde{K}_{\Omega\backslash\omega} f,\ f)^{\Omega\backslash\omega} \qquad \text{for } f \in L^2H(\Omega\backslash\omega)\,.$$

Thus, the sequence $\{\sum_{m=0}^{N} T^{(m)}_{\Omega\backslash\omega}\}_N$ is non-decreasing and bounded, so that it converges strongly. Therefore, by using (2.6) again,

$$\sum_{m=0}^{\infty} (T^{(m)}_{\Omega\backslash\omega} f,\ g)^{\Omega\backslash\omega} = (\tilde{K}_{\Omega\backslash\omega} f,\ g)^{\Omega\backslash\omega} \qquad \text{for } f,\ g \in L^2H(\Omega\backslash\omega)\,,$$

that is, $\sum_{m=0}^{\infty} T^{(m)}_{\Omega\backslash\omega} = \tilde{K}_{\Omega\backslash\omega}$.

Since $\sum_{m=0}^{\infty} T^{(m)}_{\Omega\backslash\omega}$ converges strongly to $\tilde{K}_{\Omega\backslash\omega}$, it follows from (2.4) that

$$\tilde{K}_{\Omega\backslash\omega} f(z) = (f,\ \sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot,\ z))^{\Omega\backslash\omega} \qquad \text{for } f \in L^2H(\Omega\backslash\omega)\,,$$

where the series in the right hand side converges in $L^2H(\Omega\backslash\omega)$ for $z \in \Omega\backslash\omega$ arbitrarily fixed. That is, $\tilde{K}_{\Omega\backslash\omega}$ has the integral kernel $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot,\ \cdot)$. Hence, it remains only to show that $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot,\ \cdot)$ is absolutely convergent uniformly in every compact subset of $(\Omega\backslash\omega)\times(\Omega\backslash\omega)$. In order to prove it, let us observe that (2.3) implies

$$
(2.7)\qquad \begin{aligned} T^{(2m)}_{\Omega,\omega}(z, w) &= (T^{(m)}_{\Omega,\omega}(\cdot, w), T^{(m)}_{\Omega,\omega}(\cdot, z))^{\Omega}, \\ T^{(2m+1)}_{\Omega,\omega}(z, w) &= (T^{(m)}_{\Omega,\omega}(\cdot, w), T^{(m)}_{\Omega,\omega}(\cdot, z))^{\omega}, \end{aligned}
$$

for $m \geqq 0$ and $z, w \in \Omega$. In particular,

$$
(2.8)\qquad T^{(m)}_{\Omega,\omega}(z, z) \geqq 0 \qquad \text{for } z \in \Omega .
$$

Furthermore, by Schwarz' inequality,

$$
(2.9)\qquad |T^{(m)}_{\Omega,\omega}(z, w)|^2 \leqq T^{(m)}_{\Omega,\omega}(z, z) T^{(m)}_{\Omega,\omega}(w, w) \qquad \text{for } z, w \in \Omega .
$$

Hence, it suffices to dominate $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(z, z)$, which is carried out as follows:

$$
\begin{aligned} \sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(z, z) &= (\tilde{K}_{\Omega\setminus\omega} K_{\Omega\setminus\omega}(\cdot, z), K_{\Omega\setminus\omega}(\cdot, z))^{\Omega\setminus\omega} \\ &\leqq (\|K_{\Omega\setminus\omega}(\cdot, z)\|^{\Omega\setminus\omega})^2 = K_{\Omega\setminus\omega}(z, z) \end{aligned}
$$

for $z \in \Omega\setminus\omega$. Noting that the sum $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$ together with each term $T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$ is sesqui-holomorphic, that is, holomorphic and conjugate holomorphic in the first and the second variables, respectively, so that continuous in $(\Omega\setminus\omega)\times(\Omega\setminus\omega)$, we see that the absolute convergence is uniform in every compact subset. Therefore, the proof is complete. q.e.d.

REMARK 2.4. The strong limit of the sequence $\{T^m\}_m$ is zero. In fact, since the series $\sum_{m=0}^{\infty} T^{(m)}_{\Omega\setminus\omega}$ converges strongly, it follows that

$$
(\|T^m R^* f\|^{\Omega})^2 = (T^{(2m)}_{\Omega\setminus\omega} f, f)^{\Omega\setminus\omega} \to 0 \text{ as } m \to +\infty
$$

for $f \in L^2H(\Omega\setminus\omega)$. Recalling that the restriction mapping R: $L^2H(\Omega) \to L^2H(\Omega\setminus\omega)$ is injective and bounded, we see that its adjoint operator R^* has a dense range. Therefore, $T^m \to 0$ strongly as $m \to +\infty$.

Consequently, given $\tilde{f} \in L^2H(\Omega)$, we have

$$
(1-T)\sum_{m=0}^{N} T^m \tilde{f} = \sum_{m=0}^{N} T^m (1-T)\tilde{f} = \tilde{f} - T^{N+1}\tilde{f} \to \tilde{f}
$$

as $N \to +\infty$. However, the series $\sum_{m=0}^{\infty} T^m$ may not converge in general to a bounded operator in $L^2H(\Omega)$ even weakly, for if it does then the operator $1-T$ is boundedly invertible in $L^2H(\Omega)$ so that $\|T\| < 1$. (In fact, if $\|T\| = 1$, then 0 is a spectrum of $1-T$.) On the other hand, it is possible that $\|T\| = 1$, see Remark 2.7, also Examples 2.1 and 2.3 below.

Proof of Theorem 3. By Banach's theorem, the restriction mapping R: $L^2H(\Omega) \to L^2H(\Omega\setminus\omega)$ is an injective Banach space isomorphism. Hence, there exists a constant $C > 1$ such that

$$
(2.10)\qquad \|f\|^{\Omega} \leqq C \|f\|^{\Omega\setminus\omega} \qquad \text{for } f \in L^2H(\Omega) .
$$

That is, $(\|f\|^{\Omega}/C)^2 \leqq ((1-T)f, f)^{\Omega}$, so that $\|T\| \leqq 1-C^{-2} < 1$.

Recalling the definition (1.2) or (2.3) of $T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$, we see that the series $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot, z)$ converges to $(1-T)^{-1}K_{\Omega}(\cdot, z)$ in $L^2H(\Omega)$ for every $z \in \Omega$. Moreover,

$$(f, \sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot, z))^{\Omega} = (1-T)^{-1}f(z)$$

for $f \in L^2H(\Omega)$ and $z \in \Omega$. That is, $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$ is the integral kernel of the bounded operator $(1-T)^{-1}$ in $L^2H(\Omega)$.

It remains to show that the series $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$ is absolutely convergent uniformly in every compact subset of $\Omega \times \Omega$. As in the proof of Theorem 2, it suffices to dominate $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(z, z)$ by virtue of (2.8) and (2.9). Recalling by (2.3) that $T^{(m)}_{\Omega,\omega}(z, z) = (T^m K_{\Omega}(\cdot, z), K_{\Omega}(\cdot, z))^{\Omega}$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(z, z) &\leqq \sum_{m=0}^{\infty} \|T\|^m (\|K_{\Omega}(\cdot, z)\|^{\Omega})^2 \\ &= K_{\Omega}(z, z)/(1-\|T\|) \qquad \text{for } z \in \Omega .\end{aligned}$$

Therefore, the desired conclusion is obtained as in the proof of Theorem 2. q.e.d.

REMARK 2.5. In view of the proof of Theorem 3, we see that the condition $L^2H(\Omega)|_{\Omega\setminus\omega} = \widetilde{L^2H(\Omega)}$ is indeed equivalent to $\|T\| < 1$ *via* the inequality (2.10). Hence, the following conditions are equivalent:

(i) $\|T\| < 1$,
(ii) R: $L^2H(\Omega) \to L^2H(\Omega\setminus\omega)$ has a closed range,
(iii) R is an injective Banach space isomorphism.

Also, it is elementary in Functional Analysis that (iii) holds if and only if

(iv) $R^* = K_{\Omega}E$: $L^2H(\Omega\setminus\omega) \to L^2H(\Omega)$ is surjective.

In Remark 2.4, we have observed that if (i) is violated, then the series $\sum_{m=0}^{\infty} T^m$ does not converge even weakly, though $\sum_{m=0}^{\infty} T^{(m)}_{\Omega\setminus\omega}$ converges strongly to $\tilde{K}_{\Omega\setminus\omega}$. Namely, the convergence of $\sum_{m=0}^{\infty} T^m$ depends on the validity of (iv), a fact which is reasonable in view of the relation

$$(T^{(m)}_{\Omega\setminus\omega} f, g)^{\Omega\setminus\omega} = (T^m R^* f, R^* g)^{\Omega} \qquad \text{for } f, g \in L^2H(\Omega\setminus\omega) .$$

REMARK 2.6. In Remark 2.3, we have observed that a sufficient condition for $\|T\| < 1$ is given by

$$\int_{\omega} K_{\Omega}(\zeta, \zeta) dV(\zeta) < +\infty . \tag{2.2}$$

Let us add some comments on (2.2) in case $\partial\Omega$ is sufficiently smooth and $n \geqq 2$.

It is well known that $K_{\Omega}(z, z)$ is regular around a boundary point at which some eigenvalue of the Levi form is negative. Hence, if $\bar{\omega}\cap\partial\Omega$ consists only of such points, then (2.2) is valid.

On the other hand, if $\bar{\omega}\cap\partial\Omega$ contains a pseudo-convex point, then $K_\Omega(z, z)$ may grow up to infinity as z tends to that point, cf. Hörmander [22], Theorem 3.5.1. In this case, ω must be sufficiently small around that point in order that (2.2) is satisfied.

REMARK 2.7. Contrary to (2.2), if a hole ω is so large that

$$\int_{\Omega\setminus\omega} K_\Omega(\zeta, \zeta)dV(\zeta)<+\infty\,, \tag{2.11}$$

then the operator $1-T$ is of trace class, cf. Remark 2.3. In particular, $1-T$ is compact and admits a discrete spectral decomposition. Then, as in the proof of Theorem 1, we see that $||T||=1$. We also have $||1-T||<1$, for (2.11) implies that the measure of ω is positive.

EXAMPLE 2.1. It is possible that the restriction mapping R: $L^2H(\Omega)\to L^2H(\Omega\setminus\omega)$ has a dense range, but is not bijective. One of the simplest example is given by the case where $\Omega\setminus\omega$ is a ball with $\Omega\setminus\omega\Subset\Omega$. In this case, polynomials are dense in $L^2H(\Omega\setminus\omega)$, so that the range of R is dense. However, $K_{\Omega\setminus\omega}(\cdot, w)\notin L^2H(\Omega)$ if $w\in\Omega\setminus\omega$ is close to the boundary, so that R is not surjective. Namely, the range of R is not closed, cf. Remark 2.7.

More examples will be constructed by a similar idea in view of Hörmander [22], Theorem 2.3.5 and the subsequent Remark (1).

EXAMPLE 2.2. It is also possible that the range of R is closed but not dense. A one dimensional example is given simply by setting

$$\Omega=\Delta_1,\ \omega=\bar{\Delta}_r,\ \Omega\setminus\omega=A_r \qquad \text{with } 0<r<1\,,$$

where $\Delta_r=\{z\in\boldsymbol{C}^1;\ |z|<r\}$, so that A_r is an annulus. In this case, every element of $L^2H(\Omega\setminus\omega)$ admits a Laurent series expansion, while any term of negative power cannot be approximated in $L^2H(\Omega\setminus\omega)$ by elements of $L^2H(\Omega)$. Hence, the range of R is not dense. Observe also that $||T||<1$, for the hole ω is compact.

A higher dimensional example can be obtained by setting

$$\Omega=\Delta_1\times\Omega',\ \omega=\bar{\Delta}_r\times\Omega',\ \Omega\setminus\omega=A_r\times\Omega'\,,$$

where Ω' is an arbitrary bounded domain in $\boldsymbol{C}^{n-1}$. In this example, T is not compact, though $||T||<1$.

EXAMPLE 2.3. Let us finally present an example such that the range of R is not dense nor closed. Suppose that Ω is a unit ball in $\boldsymbol{C}^n$ with $n\geq 2$ centered at the origin and that $\Omega\setminus\omega$ is relatively compact in Ω. Then, by virtue of

Remark 2.7, the range of R is not closed. Now if $\Omega\backslash\omega$ is of the form

$$\Omega\backslash\omega = \{(z', z_n)\in \boldsymbol{C}^{n-1}\times \boldsymbol{C}^1;\ a<|z_n|<b,\ |z'|<c\}$$

with some positive constants a, b and c, then polynomials are not dense in $L^2H(\Omega\backslash\omega)$, while they are in $L^2H(\Omega)$. Therefore, the range of R is not dense.

3. Proofs of Theorems 4 and 5

3.1. A smoothing kernel. By virtue of Theorem 1, we may consider the function

$$\mathcal{T}_{\Omega,\omega}(z, w) = K_{\Omega\backslash\omega}(z, w) - K_{\Omega}(z, w) = \sum_{m=1}^{\infty} T^{(m)}_{\Omega,\omega}(z, w)$$

for $z, w\in\Omega$. Then,

Lemma 3.1. *If Ω satisfies the condition R, then $\mathcal{T}_{\Omega,\omega}(\cdot, \cdot)\in C^{\infty}(\bar{\Omega}\times\bar{\Omega})$.*

Lemma 3.2. *If Ω satisfies the condition Q, then $\mathcal{T}_{\Omega,\omega}(\cdot, \cdot)\in C^{\omega}(\bar{\Omega}\times\bar{\Omega})$.*

If no smoothness of the boundary $\partial\Omega$ is assumed, then the statement of Lemma 3.1 should be interpreted as follows: *If Ω satisfies $(R_0;\ M_0)$ with some M_0, then $\mathcal{T}_{\Omega,\omega}(\cdot, \cdot)$ belongs to $W^s(\Omega\times\Omega)$ for all $s\in \boldsymbol{N}$*, see Remark 1.2. We shall actually prove this statement.

By definition, the conclusion of Lemma 3.2 states that $\mathcal{T}_{\Omega,\omega}(\cdot, \cdot)$ extends real analytically to a neighborhood of $\bar{\Omega}\times\bar{\Omega}$, where some regularity condition on the boundary $\partial\Omega$ is required implicitly. (Namely, Ω must lie on only one side of its boundary $\partial\Omega$, see Subsection 3.3.) Note that the extension is indeed sesqui-holomorphic, that is, holomorphic and conjugate holomorphic in the first and the second variables, respectively.

We specify a norm $\|\cdot\|^{\Omega}_s$ on $W^s(\Omega)$ by setting

$$\|u\|^{\Omega}_s = \Big(\int_{\Omega} \sum_{|\alpha|+|\beta|\leq s} |\partial^{\alpha}_z\partial^{\beta}_{\bar{z}}u(z)|^2 dV(z)\Big)^{1/2},$$

where $\partial_z=\partial/\partial z$ and $\partial_{\bar{z}}=\partial/\partial\bar{z}$.

Proof of Lemma 3.1. Recall that

$$\begin{aligned}\mathcal{T}_{\Omega,\omega}(z, w) &= \sum_{m=1}^{\infty} T^{(m)}_{\Omega,\omega}(z, w)\\ &= \int_{\omega} K_{\Omega}(z, \zeta)K_{\Omega}(\zeta, w)dV(\zeta)\\ &\quad +\iint_{\omega\times\omega} K_{\Omega}(z, \zeta_1)\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\zeta_1, \zeta_2)K_{\Omega}(\zeta_2, w)dV(\zeta_1)dV(\zeta_2)\end{aligned}$$

for $z, w\in\Omega$. Since $\sum_{m=0}^{\infty} T^{(m)}_{\Omega,\omega}(\cdot, \cdot)$ is bounded in $\omega\times\omega$, it suffices to show that

(3.1) $\sup_{w\in\omega}\|K_\Omega(\cdot, w)\|_s^\Omega<+\infty$ for $s\in\boldsymbol{N}$.

Let us prove (3.1). By the mean value property for harmonic functions, we have

$$\partial_z^\alpha K_\Omega(z, w)=\int_\Omega \partial_z^\alpha K_\Omega(z, \zeta)\phi_w(\zeta)dV(\zeta)=\partial_z^\alpha K_\Omega\phi_w(z)$$

for $\alpha\in\boldsymbol{Z}_+^n$, where $\phi_w\in C_0^\infty(\Omega)$ is radially symmetric around w and satisfies $\int\phi_w dV=1$. (This expression is due to Bell [4], cf. also Kerzman [23].) By virtue of $(R_0; M_0)$ for Ω, we get

(3.2) $\|\partial_z^\alpha K_\Omega(\cdot, w)\|_0^\Omega\leqq C_\alpha\|\phi_w\|_{M_0(|\alpha|)}^\Omega$

with some constant $C_\alpha>0$. Fixing $\alpha\in\boldsymbol{Z}_+^n$ arbitrarily, we want to estimate the right hand side of (3.2) uniformly in $w\in\omega$. This can be done by choosing ϕ_w to be of the form $\phi_w(z)=\phi(|z-w|)$ with $\phi\in C_0^\infty([0, \infty))$ satisfying $\phi(r)=0$ whenever $2r\geqq$distance $(\omega, \partial\Omega)$. q.e.d.

Proof of Lemma 3.2. As in the proof of Lemma 3.1, it suffices to show that

(3.3) $K_\Omega(\cdot, \cdot)\in C^\omega(\bar{\Omega}\times\omega)$.

In order to prove (3.3), we recall that $K_\Omega(z, w)=K_\Omega\phi_w(z)$ for z, $w\in\Omega$ with the same $\phi_w\in C_0^\infty(\Omega)$ as in the proof of Lemma 3.1. By virtue of the condition Q for Ω, we then get

(3.3)′ $K_\Omega(\cdot, w)\in C^\omega(\bar{\Omega})$ for $w\in\Omega$.

More precisely, for any $w\in\Omega$, there exists a domain $\Omega(w)$ in $\boldsymbol{C}^n$ such that $\bar{\Omega}\subset\Omega(w)$ and that $K_\Omega(\cdot, w)$ extends holomorphically to $\Omega(w)$. It has been known that (3.3)′ implies (3.3) in case the boundary $\partial\Omega$ is of C^2-class, see, e.g., Zorn [35]. (In Subsection 3.3, we shall prove this fact for a domain Ω with Lipschitz boundary.) Since we assume here the smoothness of the boundary $\partial\Omega$, the proof is finished. q.e.d.

3.2. Proofs of Theorems 5 and 4′. We have almost finished the proofs of Theorems 5 and 4′. In fact,

Proof of Theorem 5. By virtue of Lemma 3.2, the kernel $\mathcal{I}_{\Omega,\omega}(\cdot, \cdot)$ defines an integral operator $\mathcal{I}_{\Omega\backslash\omega}$ in $L^2(\Omega\backslash\omega)$ satisfying $\mathcal{I}_{\Omega\backslash\omega}u\in C^\omega(\bar{\Omega})$ for $u\in L^2(\Omega\backslash\omega)$. Observe that

(3.4) $K_{\Omega\backslash\omega}u=K_\Omega u+\mathcal{I}_{\Omega\backslash\omega}u$ for $u\in L^2(\Omega\backslash\omega)$,

where u in $K_\Omega u$ is regarded as an element of $L^2(\Omega)$ by setting $u_{|\omega}=0$. If $u\in C_0^\infty(\Omega\backslash\omega)\subset C_0^\infty(\Omega)$, then $K_\Omega u\in C^\omega(\bar{\Omega})$ so that

$$K_{\Omega\backslash\omega}u\in C^{\infty}(\bar{\Omega})\subset C^{\infty}(\overline{\Omega\backslash\omega}),$$

obtaining the desired conclusion. q.e.d.

Proof of Theorem 4′. Suppose first that Ω satisfies $(R_0;\, M_0)$. By virtue of Lemma 3.1, we still have (3.4) and

$$(3.5)\qquad \|\mathcal{G}_{\Omega\backslash\omega}u\|_s^{\Omega\backslash\omega}\leqq C_s\|u\|_0^{\Omega\backslash\omega}\qquad \text{for } u\in L^2(\Omega\backslash\omega)$$

with some constant $C_s>0$. By $(R_0;\, M_0)$ for Ω, there exists $C_s'>0$ such that, for $u\in C_0^{\infty}(\Omega\backslash\omega)\subset C_0^{\infty}(\Omega)$,

$$\|K_{\Omega}u\|_s^{\Omega\backslash\omega}\leqq\|K_{\Omega}u\|_s^{\Omega}\leqq C_s'\|u\|_{M_0(s)}^{\Omega}=C_s'\|u\|_{M_0(s)}^{\Omega\backslash\omega}.$$

Therefore, $\Omega\backslash\omega$ also satisfies $(R_0;\, M_0)$.

Suppose next that Ω satisfies $(R;\, M)$. We again have (3.4) and (3.5), so that it suffices to show that

$$(3.6)\qquad \|K_{\Omega}u\|_s^{\Omega\backslash\omega}\leqq C_s''\|u\|_{M(s)}^{\Omega\backslash\omega}\qquad \text{for } u\in W^{M(s)}(\Omega\backslash\omega)$$

with some constant $C_s''>0$. In order to prove (3.6), we choose and fix $\xi_1\in C_0^{\infty}(\Omega)$ satisfying $\xi_1=1$ near ω and set $\xi_2=1-\xi_1$. Then, $\xi_2u\in W^{M(s)}(\Omega)$ and $\xi_2u=0$ near ω. Hence, by $(R;\, M)$ for Ω, there exists $C_s'''>0$ such that

$$\|K_{\Omega}(\xi_2u)\|_s^{\Omega\backslash\omega}\leqq\|K_{\Omega}(\xi_2u)\|_s^{\Omega}\leqq C_s'''\|\xi_2u\|_{M(s)}^{\Omega},$$

while $\|\xi_2u\|_{M(s)}^{\Omega}=\|\xi_2u\|_{M(s)}^{\Omega\backslash\omega}\leqq C_s(\xi_2)\|u\|_{M(s)}^{\Omega\backslash\omega}$ with some constant $C_s(\xi_2)>0$. In order to estimate $K_{\Omega}(\xi_1u)$, we observe that (3.1) remains valid with $\mathrm{supp}(\xi_1)$ in place of ω without changing the proof. Then,

$$\|K_{\Omega}(\xi_1u)\|_s^{\Omega\backslash\omega}\leqq C_s(\xi_1)\|\xi_1u\|_0^{\Omega}\leqq C_s'(\xi_1)\|u\|_0^{\Omega\backslash\omega}$$

with some positive constants $C_s(\xi_1)$ and $C_s'(\xi_1)$, where $C_s(\xi_1)$ depends on the support of ξ_1. Since ξ_1 and ξ_2 are independent of u, we obtain (3.6). Therefore, $\Omega\backslash\omega$ also satisfies $(R;\, M)$. q.e.d.

3.3. A remark on Theorem 5. We have proved Theorem 5 assuming that the boundary of Ω is of C^2-class, cf. the proof of Lemma 3.2. In the present subsection, we shall generalize this result to Ω having the so-called cone property.

Recall that a domain Ω in $\boldsymbol{R}^N$ is said to have the *cone property* if there exists a finite cone C in $\boldsymbol{R}^N$ such that each point $x\in\Omega$ is the vertex of a finite cone C_x in Ω congruent to C under a Euclidean motion, see Adams [1]. Here, a *finite cone* C (in $\boldsymbol{R}^N$ with vertex at $x_0\in\boldsymbol{R}^N$) is a set of the form

$$C=\{x_0+\lambda(y-x_0);\, y\in B_1,\, \lambda>0\}\cap B_2,$$

where B_1 and B_2 are open balls in $\boldsymbol{R}^N$ such that $x_0\notin B_1$ and that B_2 is centered at x_0. It is easy to see that a bounded domain Ω in $\boldsymbol{R}^N$ has the cone property if it has a Lipschitz boundary, namely, if the boundary $\partial\Omega$ is locally expressed as a graph of a Lipschitz continuous function from $\boldsymbol{R}^{N-1}$ to $\boldsymbol{R}$. Conversely, a theorem of Gagliardo asserts that if a bounded domain Ω in $\boldsymbol{R}^N$ has the cone property then it is a union of a finite number of domains with Lipschitz boundary, see [1].

Let Ω be a bounded domain in $\boldsymbol{C}^n$ having the cone property. We shall show that Theorem 5 remains valid. Since the cone property does not require Ω to lie on only one side of its boundary, the definition of the space $C^\omega(\bar{\Omega})$ must be modified appropriately. We define $C^\omega(\bar{\Omega})$ in such a way that $f\in C^\omega(\bar{\Omega})$ if and only if $f\in C^\omega(\Omega)$ and there exists $r>0$, possibly depending on f, such that f admits the (real) power series expansion with radius of convergence $\geqq r$ at every point of Ω. (There may be no confusion of the notation $C^\omega(\bar{\Omega})$, though the space $C^\omega(\bar{\Omega})$ depends on Ω and not on the set $\bar{\Omega}$.) This definition is certainly motivated by the notion of analytic continuation, as the following example illustrates: if $n=1$ and $\Omega=\{z\in\boldsymbol{C};\ 1<|z|<2,\ z\notin(1, 2)\}$, then (a branch of) the logarithm $f(z)=\log z$ belongs to $C^\omega(\bar{\Omega})$.

Under the definition of the space $C^\omega(\bar{\Omega})$ as above, we shall show that:

Lemma 3.2′. *If a bounded domain Ω in $\boldsymbol{C}^n$ has the cone property, then* (3.3)′ *implies that*

$$K_\Omega(\cdot,\ \cdot)\in C^\omega(\overline{\Omega\times\omega_0})$$

for any relatively compact open subset ω_0 of Ω.

Note that Lemma 3.2′ implies the conclusion of Theorem 5 under the assumption that Ω has the cone property, for the previous proof remains valid except for Lemma 3.2.

Proof of Lemma 3.2′. The following argument is inspired by the paper of Zorn [35]. We begin with observing that:

Claim 1. *Given a non-empty open subset U of Ω, there exists a non-empty open subset V of U such that $K_\Omega(\cdot,\ \cdot)\in C^\omega(\overline{\Omega\times V})$.*

In order to prove Claim 1, we set

$$S_{l,m}=\bigcap_{s=1}^{\infty}\{w\in U;\ \|K_\Omega(\cdot,\ w)\|_s^\Omega\leqq lm^s s!\}\qquad\text{for } l,\ m\in\boldsymbol{N}.$$

Then, (3.3)′ implies that $\bigcup_{l,m\in\boldsymbol{N}} S_{l,m}=U$. Let us observe that each $S_{l,m}$ is a relatively closed subset of U. Since Ω is bounded and has the cone property, it follows from Rellich's lemma that the inclusion mapping $W^sH(\Omega)\to L^2H(\Omega)$ is compact for $s\in\boldsymbol{N}$, where $W^sH(\Omega)=W^s(\Omega)\cap L^2H(\Omega)$ regarded as a closed subspace of $W^s(\Omega)$. Hence, as in the proof of Theorem 1, we see that there exists a complete orthogonal system $\{\psi_j^s\}_j$ of $W^sH(\Omega)$ which is orthonormal in $L^2H(\Omega)$, cf. Komatsu [29]. We then get a Fourier series expansion

$$K_\Omega(\cdot, w) = \sum_{j=0}^{\infty} a_j^s(w)\psi_j^s \text{ in } L^2H(\Omega) \qquad \text{for } w \in U \subset \Omega ,$$

and that

$$(\|K_\Omega(\cdot, w)\|_s^\Omega)^2 = \sum_{j=0}^{\infty} |a_j^s(w)|^2(\|\psi_j^s\|_s^\Omega)^2 .$$

By the reproducing property for the Bergman kernel, we have

$$a_j^s(w) = (K_\Omega(\cdot, w), \psi_j^s)^\Omega = \overline{\psi_j^s(w)} ,$$

which is a continuous function of w. Therefore,

$$S_{l,m} = \bigcap_{s=1}^{\infty} \bigcap_{N=1}^{\infty} \{w \in U;\ \sum_{j=0}^{N} |\psi_j^s(w)|^2(\|\psi_j^s\|_s^\Omega)^2 \leqq (lm^s s!)^2\} ,$$

which is a relatively closed subset of U.

It then follows from Baire's category theorem that some $S_{l,m}$ includes a non-empty open subset V' of U. Namely,

$$\|K_\Omega(\cdot, w)\|_s^\Omega \leqq lm^s s! \qquad \text{for } w \in V' .$$

Assume for a while that Ω has a Lipschitz boundary. Then, Sobolev's lemma implies that

$$\max_{|\alpha| \leqq s} |\partial_z^\alpha K_\Omega(z, w)| \leqq C_0 C_1^s s! \qquad \text{for } (z, w) \in \Omega \times V' , \tag{3.7}$$

with some positive constants C_0 and C_1 independent of s and (z, w). Hence, there exists $r>0$ such that $K_\Omega(\cdot, w)$ for $w \in V'$ admits the power series expansion with radius of convergence $\geqq r$ at every point of Ω. With V being an arbitrary non-empty relatively compact open subset of V', the desired conclusion of Claim 1 follows from Generalized Hartogs' Lemma in Bochner-Martin [14], pp. 141–142.

In case of a general domain Ω, we choose a subdomain Ω_0 of Ω with smooth boundary. Then, (3.7) holds with Ω_0 in place of Ω, where C_1 may be chosen to be independent of Ω_0; we may set $C_1 = m+1$. Since Ω_0 is arbitrary, the previous argument is still valid, and the desired conclusion of Claim 1 follows from Hartogs' lemma as in the proof of Generalized Hartogs' Lemma. So far, the cone property has been used only to guarantee the validity of Rellich's lemma.

We next observe that:

Claim 2. *If $K_\Omega(\cdot, \cdot) \in C^\infty(\overline{\Omega \times \Delta_1})$ with some polydisc Δ_1 in Ω, then $K_\Omega(\cdot, \cdot) \in C^\infty(\overline{\Omega \times \Delta_2})$ for any relatively compact polydisc Δ_2 in Ω with the same center as that of Δ_1.*

Note that the conclusion of Lemma 3.2′ follows from Claims 1 and 2 together with a simple compactness argument. In fact, it suffices to cover ω_0 by a finite number of open sets U in such a way that every point of U is the center of a polydisc Δ_2 satisfying $U\subset\Delta_2\Subset\Omega$.

In order to prove Claim 2, we reformulate it as follows:

Claim 2′. *Let F be a holomorphic function in $\Omega\times\Delta_0$, where Δ_0 is the unit polydisc in $\boldsymbol{C}^n$ centered at the origin* 0. *If $F\in C^\omega(\overline{\Omega\times\Delta_1})$ for some polydisc Δ_1 in $\boldsymbol{C}^n$ of polyradius <1 centered at* 0, *then $F\in C^\omega(\overline{\Omega\times\Delta_2})$ for any polydisc Δ_2 of the same type as Δ_1.*

Let us prove Claim 2′. By virtue of Gagliardo's theorem, we may assume that Ω has a Lipschitz boundary. This assumption will be used at the final stage of the proof. By definition, we have to show that if F has the polyradius of convergence $\geqq r_1$ at every point of $\Omega\times\Delta_1$ for some $r_1>0$, then there exists $r_2>0$ such that F has the polyradius of convergence $\geqq r_2$ at every point of $\Omega\times\Delta_2$. We reduce the problem to that of polydiscs as follows. For each $z\in\Omega$, we denote by $\Delta(z; r)$ the polydisc centered at z of polyradius $r>0$. We also set $r(z)=d(z, \partial\Omega)$, where $d(\cdot, \cdot)$ stands for the distance measured with respect to $d(z, z')=\max_{1\leq j\leq n}|z_j-z'_j|$ for $z=(z_1, \cdots, z_n)$ and $z'=(z'_1, \cdots, z'_n)$. Then, F extends holomorphically to

$$(\Delta(z; r_1)\times\Delta_1)\cup(\Delta(z; r(z))\times\Delta_0) \qquad \text{for each } z\in\Omega\,.$$

Given Δ_2, we choose a polydisc Δ'_2 centered at 0 in such a way that $\Delta_2\Subset\Delta'_2\Subset\Delta_0$. It then follows from the logarithmic convexity for the polyradii of convergence of power series in a product space that F extends holomorphically to

$$\Delta(z; r_2(z))\times\Delta'_2, \quad \text{where} \quad r_2(z)=r_1^\theta r(z)^{1-\theta}\,,$$

with some θ, $0<\theta<1$, independent of $z\in\Omega$. We now recall the assumption that Ω is a bounded domain with Lipschitz boundary. Then, the desired conclusion of Claim 2 will be obtained if the family $\{\Delta(z; r_2(z))\}_{z\in\Omega}$ covers $\bar{\Omega}$. Obviously, this family covers Ω. Hence, it suffices to show that for each $z'\in\partial\Omega$ there exists $z''\in\Omega$ such that $z'\in\Delta(z''; r_2(z''))$. This is possible by virtue of the Lipschitz regularity assumption on $\partial\Omega$. In fact, z' is the vertex of a finite cone C included in Ω, while

$$r_2(z)/r(z)=(r_1/r(z))^\theta\to+\infty \quad \text{as} \quad z\to z'\,.$$

Thus, the desired conclusion is obtained by approaching z to z' along the axis of rotation of the cone C. Therefore, we get Claim 2′ and the proof of Lemma 3.2′ is complete. q.e.d.

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