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SHARP ESTIMATES FOR THE $\bar{\partial}$-NEUMANN PROBLEM AND THE $\bar{\partial}$-PROBLEM

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0. Introduction

The object of this paper is to establish some estimates for the second order derivatives of the solution of the $\bar{\partial}$-Neumann problem. Similar estimates were obtained by Greiner-Stein [2] when the Levi form is non-degenerate and the metric is a Levi metric. In this article we derive such results merely assuming that the basic estimate (0.2) below holds; the metric may be an arbitrary hermitian metric and we permit some cases where the Levi form is degenerate.

We begin with recalling what the $\bar{\partial}$-Neumann problem is. Let M be a bounded domain in C^n with C^∞-boundary bM. We denote the vector bundle consisting of type (1,0) vectors by S, and the space of smooth (p, q)-forms on $\bar{M}$ by $\mathcal{A}^{p,q}(\bar{M})$. If we write a (p, q)-form ϕ as $\sum'_{I,J} \phi_{I,J}\, dz^I \Lambda \bar{z}^I$, then the $\bar{\partial}$-operator is defined by

$$\bar{\partial}\phi = \textstyle\sum'_{I,J} \sum_{j=1}^n \dfrac{\partial \phi_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \Lambda dz^I \Lambda d\bar{z}^J ,$$

where $\{z_1, \cdots, z_n\} = \{x_1+\sqrt{-1}y_1, \cdots, x_n+\sqrt{-1}y_n\}$ is the canonical coordinate system of C^n, $\partial/\partial z_j = \frac{1}{2}(\partial/\partial x_j - \sqrt{-1}\partial/\partial y_j)$, $j=1, \cdots, n$, and the notation $\sum'$ means that the summation is taken over strictly increasing p-tuples I and q-tuples J of $(1, \cdots, n)$. Let $D^{p,q}$ denote the totality of the smooth (p, q)-forms ϕ on $\bar{M}$ such that $(\psi, \vartheta\phi) = (\bar{\partial}\psi, \phi)$ holds for each $\psi \in \mathcal{A}^{p,q-1}(\bar{M})$, where ϑ is the formal adjoint of $\bar{\partial}$ and $(\, , \,)$ the L^2-inner product on M. We consider the following variational problem: given $\lambda \in C$ and $f \in \mathcal{A}^{p,q}(\bar{M})$ arbitrarily, find $u \in D^{p,q}$ such that

$$(0.1)_\lambda \qquad Q(u, \phi) + \lambda(u, \phi) = (f, \phi) \qquad \text{for any } \phi \in D^{p,q} ,$$

where $Q(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\vartheta\phi, \vartheta\psi) + (\phi, \psi)$. This problem is equivalent to the following boundary value problem:

$$(0.1)'_\lambda \qquad (\square + \lambda + 1)u = f \text{ in } M,\ u \in D^{p,q},\ \bar{\partial}u \in D^{p,q+1} ,$$

where $\square=\bar\partial\vartheta+\vartheta\bar\partial$ stands for the complex Laplacian. Observe that the solution u of $(0.1)_\lambda$, which is required to satisfy the first boundary condition $u\in D^{p,q}$, satisfies automatically the second boundary condition $\bar\partial u\in D^{p,q+1}$. The case $\lambda=-1$ is the $\bar\partial$-Neumann problem. The solution operator N, called the $\bar\partial$-Neumann operator, is defined in such a way that if f is orthogonal to the null space of $\square$ then Nf is the solution of $(0.1)_{-1}$ orthogonal to the null space of $\square$. For precise definition of Nf, see Section 3. The case $\lambda=-1$ is reduced to the case $\lambda=0$ via spectral theory. We thus first consider the case $\lambda=0$.

It is easily seen that there exists a solution u of $(0.1)_0$ in the completion of $D^{p,q}$ with respect to the norm $Q(\phi)=\sqrt{Q(\phi,\phi)}$. The most difficult part of this theory is to prove the smoothness of the solution up to the boundary; that is, to verify that the generalized solution u in the completion of $D^{p,q}$ actually belongs to $D^{p,q}$. This problem has been solved by Kohn under the assumption of the following "basic estimate" due to Morrey [11]:

$$(0.2) \qquad \int_{bM}|\phi|^2dS\leqq CQ(\phi)^2 \qquad \text{for any } \phi\in D^{p,q}\,,$$

where dS stands for the surface element of bM and C is a constant independent of ϕ. In this paper we often use C for different constants without notice. Notice that the estimate (0.2) holds for each $q\geqq 1$ if M is a strongly pseudoconvex domain. Assuming (0.2), Kohn obtained an L^2-estimate with loss of one derivative, see (2.0) below; in particular, the estimate of all the first order derivatives of the solution u of $(0.1)_0$ by f: $\|u\|_1\leqq C\|f\|$. Notice that it is impossible to estimate the L^2-norms of all the second order derivatives of u by that of f under the assumption (0.2) only, see Sweeney [15].

Recently an interest is focused on obtaining sharp estimates without loss of derivatives for the solution. In particular, Greiner–Stein [2] obtained estimates, in various function spaces, of the second order derivatives of the solution u in term of f, where the directions of the derivatives are specialized according to the problem, cf. Sweeney's result cited above. Our main concern is to generalize their result in the case of "Levi metric" to the case of an arbitrary hermitian metric. Though our result will be stated in the case of the standard metric in C^n, it can be generalized easily to the case of an arbitrary hermitian metric on a complex manifold, see Appendix.

In order to specify the directions for which we can estimate the second order derivatives of the solution u, we define the function r by $r(x)=-\mathrm{dis}(x, bM)$ for $x\in M$ and $r(x)=\mathrm{dis}(x, bM)$ for $x\notin M$ where $\mathrm{dis}(x, bM)$ denotes the distance from x to bM; then r is smooth and satisfies $|dr|=1$ in a neighborhood M' of bM. We define the vector field $\partial/\partial n$ on M' by

$$\sum_{j=1}^{n}\left\{\frac{\partial r}{\partial x_j}\frac{\partial}{\partial x_j}+\frac{\partial r}{\partial y_j}\frac{\partial}{\partial y_j}\right\},$$

which is the unit exterior normal vector to M on bM. Then a vector field of type (1,0) is defined by

$$Z'_n=\frac{1}{\sqrt{2}}\{\partial/\partial n-\sqrt{-1}\,J\,\partial/\partial n\}\,,$$

where J is the complex structure tensor. Roughly speaking, Z'_n is the direction along which the estimate is lacking.

Let us state our sharp estimates more precisely. Since the second order derivatives in the interior of M are estimated by using a standard interior regularity theorem for elliptic equations, we need to work only near the boundary. For an arbitrary boundary point, we take a neighborhood U of that point in such a way that there exists an orthonormal frame $\{Z_1, \cdots, Z_n\}$ of S restricted to $\bar U$ with $Z_n=Z'_n$. We set $W_{2j-1}=\mathrm{Re}\,Z_j$ and $W_{2j}=\mathrm{Im}\,Z_j$ for $j=1, \cdots, n-1$. Let ζ_1 and ζ_2 be real smooth functions supported in U such that $\zeta_2\equiv 1$ in a neighborhood of supp ζ_1. Then our estimates for the solution $u=\sum'_{I,J}u_{I,J}\,dz^I\wedge d\bar z^J$ of $(0.1)_0$ are stated as follows:

Estimate A. $\quad \mathrm{Sup}\,\|W_iW_j\zeta_1u_{I,J}\|_k\leqq C_k\{\|\zeta_2f\|_k+\|f\|\}$

for $k=0, 1, 2, \cdots$, where the supremum is taken over $1\leqq i, j\leqq 2n-2$ and all I, J, the norm is the L^2-Sobolev norm and C_k is a constant independent of f.

Estimate B. $\quad \mathrm{Sup}_{I,J}\,\|\bar Z'_n\zeta_1u_{I,J}\|_{k+1}\leqq C_k\{\|\zeta_2f\|_k+\|f\|\}\,.$

Greiner–Stein [2] established Estimates A and B under the conditions that the Levi form is non-degenerate and the metric is a Levi metric, i.e., the metric tensor g on $S_t\otimes\bar S_t$ coincides with the Levi form $\partial\bar\partial r$ where $S_t=S\cap CTbM$. They considered only (0,1)-forms, but proved the results for more general L^t-$(1<t<\infty)$ Sobolev and Lipschitz norms.

In this paper we shall prove Estimates A and B for (p, q)-forms assuming "basic estimate" (0.2) holds for $D^{p,q}$. We shall also prove these estimates for Nf instead of the solution u of $(0.1)_0$ where N is the $\bar\partial$-Neumann operator.

In addition, as an application of estimates for $(0.1)_{-1}$, we give a sharp estimate for the solution of the $\bar\partial$-problem, which is orthogonal to the null space of $\bar\partial$. Such a solution v of the equation $\bar\partial v=\theta$ with $\theta\in\mathcal{A}^{p,q}(\bar M)$ has the following estimate:

(0.3) $\quad \mathrm{Sup}\,\{\|W_j\zeta_1v_{I,J}\|_k+\|\bar Z'_n\zeta_1v_{I,J}\|_k\}\leqq C_k\{\|\zeta_2\theta\|_k+\|\theta\|\}$

for $k=0, 1, 2, \cdots$, where $v=\sum'_{I,J}v_{I,J}\,dz^I\wedge d\bar z^J$, the supremum is taken over $1\leqq j\leqq 2n-2$ and all I, J and C_k is a constant independent of θ.

Some remarks on allowable vector fields and estimates for them will be treated in Section 4.

Estimate A is obtained from a few devices of the calculus for the commutators and for the integration by parts. For a real tangential differential operator

D of order $k+1$, we shall show

$$Q(D\zeta_1 u)^2 - \mathrm{Re}(D\zeta_1 f, D\zeta_1 u) = O(\|\zeta' u\|_{k+1}^2) \qquad \text{for } f \in \mathfrak{A}^{p,q}(\bar{M}),$$

where ζ' is a real smooth function supported in $\{\zeta_2 \equiv 1\}$ with $\zeta' \equiv 1$ in a neighborhood of supp ζ_1. We use this commutator estimate for D of the form $W_j D'$, where D' is an arbitrary tangential differential operator of order k. Since $Q(W_j D' \zeta_1 u)$ dominates $\sum'_{i,I,J} \|W_i W_j D' \zeta_1 u_{I,J}\|$, the tangential derivatives of $W_i W_j \zeta_1 u_{I,J}$ up to order k are estimated, for the term $O(\|\zeta' u\|_{k+1}^2)$ can be estimated by Kohn's estimate (2.0), while the term $\mathrm{Re}(W_j D' \zeta_1 f, W_j D' \zeta_1 u)$ is treated by integration by parts. The estimates for the normal derivatives are obtained as in the case of standard boundary value problems for elliptic equations.

Estimate B was suggested by Kohn (cf. foot note in pp. 7 of [2]). It is obtained by using the second boundary condition, namely, $\bar{\partial} u \in D^{p,q+1}$. Let $u = \bar{\omega}'_n \wedge u^1 + u^2$ be the decomposition of the solution of $(0.1)_0$, where $\bar{\omega}'_n = \sqrt{2}\,\bar{\partial} r = g(Z'_n, *)$ and u^1, u^2 do not contain $\bar{\omega}'_n$. Then the first boundary condition is written as $u^1 = 0$ on bM, while the second boundary condition implies $\bar{Z}'_n u_{I,J} - B_{I,J} u^2 = 0$ on bM with an appropriate 0-th order operator $B_{I,J}$. The estimate of u^1 is easy. For u^2, we estimate the second order derivatives of u^2 containing $\bar{Z}'_n$ by using the second boundary condition. Hence for the whole u we obtain the desired estimate for the second order derivatives containing $\bar{Z}'_n$.

The outline of this paper is as follows. In Section 1, we review some elementary calculus for differential forms. All the formulae are known, but arranged for the convenience of our use. In particular, expressions for the complex Laplacian and for the second boundary condition are given. We prove Estimates A and B for $(0.1)_0$ in Section 2, and those for the $\bar{\partial}$-Neumann solution in Section 3. In Section 4, we make Estimate A intrisic in terms of allowable vector fields. An application to the $\bar{\partial}$-problem is given in Section 5. In Appendix, we suggest how to extend these results to a complex manifold with an arbitrary hermitian metric.

1. Preliminaries

In this section we give some notations and known facts. We denote $\Lambda^p S^* \otimes \Lambda^q \bar{S}^*$ by $\Lambda^{p,q}$ where S^* and $\bar{S}^*$ are the duals of S and $\bar{S}$ respectively. The canonical metric g in C^n is given by $g(\partial/\partial z_i, \partial/\partial \bar{z}_j) = \delta_{ij}/2$ and $g(\partial/\partial z_i, \partial/\partial z_j) = g(\partial/\partial \bar{z}_i, \partial/\partial \bar{z}_j) = 0$ for $1 \leqq i, j \leqq n$. ∇ stands for the flat connection in C^n, namely, $\nabla_X \phi = \sum'_{I,J} (X\phi_{I,J})\, dz^I \wedge d\bar{z}^J$ for $\phi = \sum'_{I,J} \phi_{I,J} dz^I \wedge d\bar{z}^J$ and for a vector field X.

1.1. The metric on the vector bundle $\Lambda^{p,q}$

Let $\phi^i = \sum'_{I,J} \phi^i_{I,J} dz^I \wedge d\bar{z}^J$, $i=1, 2$ be (p, q)-forms. Then the inner product

$\langle\phi^1, \phi^2\rangle$ on $\Lambda^{p,q}$ is defined by

$$\langle\phi^1, \phi^2\rangle = 2^{p+q}\sum'_{I,J}\phi^1_{I,J}\overline{\phi^2_{I,J}},$$

where the summation $\sum'$ is taken over all ordered p-tuples $I=(i_1, \cdots, i_p)$, $1\leqq i_1 < \cdots < i_p \leqq n$ and q-tuples $J=(j_1, \cdots, j_q)$, $1\leqq j_1 < \cdots < j_q \leqq n$. We set $|\phi| = \sqrt{\langle\phi, \phi\rangle}$. For this inner product we have the equality:

$$X\langle\phi^1, \phi^2\rangle = \langle\nabla_X\phi^1, \phi^2\rangle + \langle\phi^1, \nabla_{\bar X}\phi^2\rangle \qquad \text{for any vector field } X.$$

In C^n, it means the obvious equality

$$X(\sum' \phi^1_{I,J}\overline{\phi^2_{I,J}}) = \sum' (X\phi^1_{I,J})\overline{\phi^2_{I,J}} + \sum' \phi^1_{I,J}(\overline{\bar X\phi^2_{I,J}}).$$

Let ϕ be a (p, q)-form and $X\in\Gamma(\bar S)$. Then the $(p, q-1)$-form $i(X)\phi$ is defined by

$$(i(X)\phi)(X_1, \cdots, X_{q-1}, Y_1, \cdots, Y_p) = \phi(X, X_1, \cdots, X_{q-1}, Y_1, \cdots, Y_p)$$

for $X_1, \cdots, X_{q+1}\in\bar S$ and $Y_1, \cdots, Y_p\in S$. $i(*)$ is called the interior product.

The above norm on $\Lambda^{p,q}$ has the following property. Let $\phi\in\mathfrak{A}^{p,q}(M')$ and $\bar\omega'_n = g(Z'_n, *)$. Then

$$(1.1.1) \qquad |\phi|^2 = |i(\bar Z'_n)\phi|^2 + |\bar\omega'_n\Lambda\phi|^2.$$

This equality will play an important role for Estimate B.

1.2. The boundary conditions

In this paragraph we rewrite the boundary condition $\phi\in D^{p,q}$ as a geometrical condition on bM.

Let $\phi\in\mathfrak{A}^{p,q}(\bar M)$ and $\psi\in\mathfrak{A}^{p,q-1}(\bar M)$. Then the following integral formula holds.

$$(1.2.1) \qquad (\bar\partial\psi, \phi) = (\psi, \vartheta\phi) + \int_{bM}\langle\psi, i(\bar Z'_n)\phi\rangle dS',$$

where dS' stands for $1/\sqrt{2}$ times the volume element of bM.

In Folland–Kohn [1] the second term in the right side was represented as $\int_{bM}\langle\psi\ \sigma(\vartheta, dr)\phi\rangle dS$ where $\sigma(\vartheta, \xi)$ denotes the symbol of ϑ and dS the surface element of bM. Hence we can see that the symbol $\sigma(\vartheta, dr)$ is given by an interior product. Moreover we can see that

$$D^{p,q} = \{\phi\in\mathfrak{A}^{p,q}(\bar M)\,|\,i(\bar Z'_n)\phi = 0 \text{ on } bM\}.$$

It is known that the solution u of $(0.1)_\lambda$ satisfies $\bar\partial u\in D^{p,q+1}$, namely, $i(\bar Z'_n)\bar\partial u = 0$ on bM. This is called the second boundary condition.

1.3. Modified connection $\tilde\nabla$

Let $\phi\in D^{p,q}$ and X be a vector field tangenital to bM. In general $\nabla_X\phi$

does not satisfy the first boundary condition. Hence we modify ∇_X so as to preserve the first boundary condition. Recall that the estimate in the interior of M has been known, so that we may only work in the neighborhood M' of bM. As in Komatsu [8], we define the connection $\tilde{\nabla}$ on M' by

$$\tilde{\nabla}_X Y = P\nabla_X PY + (I-P)\nabla_X (I-P)Y,$$

where P is the orthogonal projection from the tangent bundle to the subbundle spanned by $\partial/\partial n$ and $J\partial/\partial n$, and we extend this connection to p-form ϕ by

$$(\tilde{\nabla}_X\phi)(X_1, \cdots, X_p) = X(\phi(X_1, \cdots, X_p)) - \sum_{j=1}^{p} \phi(X_1, \cdots, \tilde{\nabla}_X X_j, \cdots, X_p).$$

Then, $\tilde{\nabla}_X\phi \in D^{p,q}$ whenever $\phi \in D^{p,q}$ is supported in M'. In fact,

$$i(\bar{Z}_n')\tilde{\nabla}_X\phi = \tilde{\nabla}_X(i(\bar{Z}_n')\phi) - i(\tilde{\nabla}_X\bar{Z}_n')\phi,$$

where both terms in the right side vanish on bM; the first term by the first boundary condition and the second one by

$$\tilde{\nabla}_X\bar{Z}_n' = \mu\bar{Z}_n' \quad \text{with} \quad \mu = \sum_{k=1}^{n} 2g(Z_n', \partial/\partial\bar{z}_k)\, Xg(\partial/\partial z_k, \bar{Z}_n').$$

As in the case of ∇, our connection $\tilde{\nabla}$ satisfies

$$(1.3.1) \qquad X\langle\phi, \psi\rangle = \langle\tilde{\nabla}_X\phi, \psi\rangle + \langle\phi, \tilde{\nabla}_{\bar{X}}\psi\rangle \qquad \text{for } \phi, \psi \in \mathfrak{A}^{p,q}(M').$$

In order to see (1.3.1), we set $S(X) = \nabla_X - \tilde{\nabla}_X$. Then $S(X)$ is an operator of order zero and $\langle S(X)\phi, \psi\rangle = \langle\phi, -S(\bar{X})\psi\rangle$ holds for $\phi, \psi \in \mathfrak{A}^{p,q}(M')$ (see Appendix). Thus $\langle\nabla_X\phi, \psi\rangle + \langle\phi, \nabla_{\bar{X}}\psi\rangle - (\langle\tilde{\nabla}_X\phi, \psi\rangle + \langle\phi, \tilde{\nabla}_{\bar{X}}\psi\rangle) = \langle S(X)\phi, \psi\rangle + \langle\phi, S(\bar{X})\psi\rangle = 0$.

When we prove Estimates A and B, we first establish the estimates of the tangential derivatives of the solution of $(0.1)_0$ or $(0.1)_{-1}$, and then those of its derivatives containing the normal one. Let U and $\{Z_1, \cdots, Z_n\}$ be as in the introduction. Then the complex Laplacian has the local expression:

$$(1.3.2) \qquad \square \equiv \bar{\partial}\vartheta + \vartheta\bar{\partial} = -\sum_{j=1}^{n} \tilde{\nabla}_{Z_j}\tilde{\nabla}_{\bar{Z}_j} + \cdots,$$

where $\cdots$ is at most first order terms. We often use this expression in the following form:

$$(1.3.3) \qquad \tilde{\nabla}_{Z_n'}\tilde{\nabla}_{\bar{Z}_n'}u = -f - \sum_{j=1}^{2n-2} \tilde{\nabla}_{W_j}\tilde{\nabla}_{W_j} + \cdots,$$

where $\cdots$ contains at most first order derivatives of u.

We also need the second boundary condition $\bar{\partial}u \in D^{p,q+1}$ in order to prove Estimate B. We set

$$B_0 = -S(\bar{Z}_n') + \sum_{j=1}^{n-1} \bar{\omega}_j \wedge i(\bar{Z}_n')S(\bar{Z}_j),$$

where $\{\bar{\omega}_1, \cdots, \bar{\omega}_n\}$ is the dual frame of $\{\bar{Z}_1, \cdots, \bar{Z}_n\}$, hence $\bar{\omega}_n = \bar{\omega}_n' = g(Z_n', *)$.

Then for $u\in D^{p,q}$ satisfying $\bar\partial u\in D^{p,q+1}$, we have

(1.3.4) $\quad \bar\omega'_n\Lambda(\tilde\nabla_{\bar Z'_n}u-B_0u)=0$ on bM.

2. Estimates

In this section we fix U, $\{Z_1, \cdots, Z_n\}$, $\{W_1, \cdots, W_{2n-2}\}$, ζ_1 and ζ_2 stated in the introduction. Assuming that the basic estimate (0.2) holds for $D^{p,q}$, we shall prove Estimates A and B for $(0.1)_0$. Firstly we show them in the case that the derivatives are restricted to tangential directions, namely, only for $J\partial/\partial n$ and $\{W_1, \cdots, W_{2n-2}\}$. Secondly we show them for general case with the aid of a usual method for elliptic boundary value problems.

We use the following estimate without not ce.

Kohn's estimate. Let ζ and ζ_0 be real smooth functions supported in U with $\zeta_0\equiv 1$ in a neighborhood of supp ζ. Then for each non-negative integer m, there exists a constant C_m such that

(2.0) $\quad ||\zeta u||_{m+1}\leqq C_m\{||\zeta_0 f||_m+||f||\}\qquad$ for any $f\in\mathfrak{A}^{p,q}(\bar M)$,

where u is the solution of $(0.1)_0$ for f and $||f||=||f||_0$.

We shall prove the following theorem, which is slightly stronger than Estimate A, and will be used in the proof of Estimate B.

Theorem 1. *Let ζ_1 and ζ_2 be as in Estimate A. Then*

$$\operatorname*{Sup}_{i,j}||\tilde\nabla_{W_i}\tilde\nabla_{W_j}\zeta_1 u||_k+\operatorname*{Sup}_{j}||\tilde\nabla_{W_j}\tilde\nabla_{\bar Z'_n}\zeta_1 u||_k\leqq C_k\{||\zeta_2 f||_k+||f||\}$$

for $f\in\mathfrak{A}^{p,q}(\bar M)$, where u is the solution of $(0.1)_0$ for f.

2.1. The proofs of Theorem 1 and Estimate A

The main tool for the proof of Theorem 1 is the following proposition, whose proof is given in 2.2, and it is a part of Theorem 1 that the derivatives are restricted to tangential directions. To state such cases we introduce a few notations.

We set $e_j=W_j$ for $j=1, \cdots, 2n-2$ and $e_{2n-2}=J\partial/\partial n$. $\{e_1, \cdots, e_{2n-1}\}$ is a local basis of tangential vectors on $\bar U$. We denote $\tilde\nabla_{e_{\alpha(1)}}\cdots\tilde\nabla_{e_{\alpha(m)}}$ by $\tilde\nabla_e^\alpha$ and set $|\alpha|=m$ where $1\leqq\alpha(1), \cdots, \alpha(m)\leqq 2n-1$.

Proposition 2.1. *Let $D=\tilde\nabla_{W_j}\tilde\nabla_e^\alpha$ with $|\alpha|=k$. Then*

$$Q(D\zeta_1 u)\leqq C_k\{||\zeta_2 f||_k+||f||\}\qquad \textit{for any } f\in\mathfrak{A}^{p,q}(\bar M),$$

where u is the solution of $(0.1)_0$ for f.

Proof of Theorem 1. We denote $\frac{1}{\sqrt 2}\partial/\partial n$ by N' and JN' by T'. We shall prove the following inequality by induction on l.

$$(\sharp 1;\, l)\qquad \operatorname{Sup} \|\tilde{\nabla}_{W_i}\tilde{\nabla}_{W_j}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l}\zeta_1 u\| + \operatorname{Sup} \|\tilde{\nabla}_{W_j}\tilde{\nabla}_{\bar{Z}'_n}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l}\zeta_1 u\| \leqq C_k\{\|\zeta_2 f\|_k + \|f\|\} \qquad 0 \leqq l \leqq k,$$

where the supremums are taken over $1 \leqq i, j \leqq 2n-2$ and $|\alpha| = k-l$.

Let us prove $(\sharp 1;\, l)$. $(\sharp 1;\, 0)$ follows from Proposition 2.1 and the fact

$$\operatorname{Sup}_j \|\tilde{\nabla}_{W_j}\phi\| + \|\tilde{\nabla}_{\bar{Z}'_n}\phi\| \leqq KQ(\phi) \qquad \text{for } \phi \in D^{p,q} \cap \mathfrak{A}^{p,q}(\bar{M} \cap U).$$

Suppose that $(\sharp 1;\, i)$ is known to hold for $i \leqq l-1$. Noticing $N' = \bar{Z}'_n - \sqrt{-1}\, T'$, we get

$$\begin{aligned}
&\|\tilde{\nabla}_{W_i}\tilde{\nabla}_{W_j}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l}\zeta_1 u\| \leqq \\
&\|\tilde{\nabla}_{W_i}\tilde{\nabla}_{W_j}(\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{T'})\tilde{\nabla}_{N'}^{l-1}\zeta_1 u\| + \|\tilde{\nabla}_{W_i}\tilde{\nabla}_{\bar{Z}'_n}(\tilde{\nabla}_{W_j}\tilde{\nabla}_e^{\alpha})\tilde{\nabla}_{N'}^{l-1}\zeta_1 u\| + C\|\zeta_1 u\|_{k+1} \\
&\leqq C\{\|\zeta_2 f\|_k + \|f\|\} \qquad \text{by (2.0) and } (\sharp 1;\, l-1).
\end{aligned}$$

On the other hand, noting $N' = Z'_n + \sqrt{-1}\, T'$, we obtain

$$\begin{aligned}
&\|\tilde{\nabla}_{W_j}\tilde{\nabla}_{\bar{Z}'_n}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l}\zeta_1 u\| \leqq \\
&\|\tilde{\nabla}_{W_j}\tilde{\nabla}_{\bar{Z}'_n}\tilde{\nabla}_{Z'_n}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l-1}\zeta_1 u\| + \|\tilde{\nabla}_{W_j}\tilde{\nabla}_{\bar{Z}'_n}(\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{T'})\tilde{\nabla}_{N'}^{l-1}\zeta_1 u\| + C\|\zeta_1 u\|_{k+1} \\
&\leqq \|\tilde{\nabla}_{W_j}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l-1}\tilde{\nabla}_{Z'_n}\tilde{\nabla}_{\bar{Z}'_n}\zeta_1 u\| + C\{\|\zeta_2 f\|_k + \|f\|\}.
\end{aligned}$$

Now in view of (1.3.3), $\|\tilde{\nabla}_{W_j}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l-1}\tilde{\nabla}_{Z'_n}\tilde{\nabla}_{\bar{Z}'_n}\zeta_1 u\| \leqq$

$$\begin{aligned}
&\textstyle\sum_{p=1}^{2n-2} \|(\tilde{\nabla}_{W_p})^2\tilde{\nabla}_{W_j}\tilde{\nabla}_e^{\alpha}\tilde{\nabla}_{N'}^{l-1}\zeta_1 u\| + \|\zeta_1 f\|_k + C\|\zeta' u\|_{k+1} \leqq \\
&C\{\|\zeta_2 f\|_k + \|f\|\}.
\end{aligned}$$

Thus the proof of the induction is complete, and hence so is that of Theorem 1.

Now we can show Estimate A. Since $\nabla_X\phi = \sum'_{I,J}(X\phi_{I,J})\,dz^I \wedge d\bar{z}^J$ for $\phi = \sum'_{I,J}\phi_{I,J}\,dz^I \wedge d\bar{z}^J$, we have only to show

$$\|\nabla_{W_i}\nabla_{W_j}\zeta_1 u\|_k \leqq C\{\|\zeta_2 f\|_k + \|f\|\}.$$

The difference of the connections ∇_X and $\tilde{\nabla}_X$, namely, $S(X)$, contains no derivatives. Therefore

$$\begin{aligned}
&\|\nabla_{W_i}\nabla_{W_j}\zeta_1 u\|_k \leqq \|\tilde{\nabla}_{W_i}\tilde{\nabla}_{W_j}\zeta_1 u\|_k + C\|\zeta_1 u\|_{k+1} \leqq \\
&C\{\|\zeta_2 f\|_k + \|f\|\}, \quad \text{by Theorem 1 and (2.0).} \qquad \text{q.e.d.}
\end{aligned}$$

2.2. Proof of Proposition 2.1

The proof of Proposition 2.1 is a consequence of the following estimate: for $D = \tilde{\nabla}_e^{\alpha}$ with $|\alpha| = k+1$,

$$(2.2)\qquad Q(D\zeta_1 u,\, D\zeta_1 u) - \operatorname{Re}(D\zeta_1 f,\, D\zeta_1 u) = O(\|\zeta' u\|_{k+1}^2),$$

where ζ' is a real smooth function supported in $\{\zeta_2\equiv 1\}$ with $\zeta'\equiv 1$ in a neighborhood of supp ζ_1. In fact, suppose that (2.2) has been proved. Take $D=\tilde\nabla_{W_j}\tilde\nabla_e^{\alpha}$ with $|\alpha|=k$. Then the term $O(\|\zeta' u\|_{k+1}^2)$ is treated by using Kohn's estimate and the term $\mathrm{Re}(D\zeta_1 f, D\zeta_1 u)$ is dominated by $\|D_1\zeta_1 f\|Q(D\zeta_1 u)$ where $D_1=\tilde\nabla_e^{\alpha}$. Hence it remains only to prove (2.2).

In order to prove (2.2), we recall that

$$(D\zeta_1 f,\ D\zeta_1 u)=\underset{\sim}{Q}(u,\ \zeta_1 D^* D\zeta_1 u)\,,$$

where D^* is the formal adjoint of D. Setting $\phi=\zeta_1 v$ for $v\in\mathfrak{A}^{p,q}(\bar M)$ and $B=\bar\partial$ or ϑ, we have

$$\|BD\phi\|^2-\mathrm{Re}(Bv,\ B\zeta_1 D^* D\phi)=\mathrm{Re}(I_1+I_2+II_1+II_2)\,,$$

where $I_1=([B,\ D]\phi,\ BD\phi)$, $I_2=(B\phi,\ [D^*,\ B]D\phi)$,

$$II_1=([B,\ \zeta_1]v,\ BD^*D\phi)\ \text{ and }\ II_2=(Bv,\ [\zeta_1,\ B]D^*D\phi)\,.$$

Hence, in order to prove (2.2), it suffices to show that

(2.3) $\mathrm{Re}(I_1+I_2)=O(\|\phi\|_{k+1}^2)$, and

(2.4) $\mathrm{Re}(II_1+II_2)=O(\|\zeta' v\|_{k+1}^2)$.

Proof of (2.3). We prove (2.3) by showing

(2.3)′ $(B\phi,\ [D^*,\ B]D\phi)=(BD\phi,\ [D,\ B]\phi)+O(\|\phi\|_{k+1}^2)$.

Decompose the proof of (2.3)′ into two steps. The first and the second steps are to show

(2.3.1) $(BD^*\phi,\ [D^*,\ B]\phi)=(BD\phi,\ [D,\ B]\phi)+O(\|\phi\|_{k+1}^2)$, and

(2.3.2) $(B\phi,\ [[D^*,\ B],\ D]\phi)=O(\|\phi\|_{k+1}^2)$.

Under (2.3.1) and (2.3.2) one can see easily (2.3)′ and hence (2.3).

Proof of (2.3.1): It suffices to show by induction on k,

$(*;\,k)$ $D-(-1)^{k+1}D^*$ is a sum of differentials of order $\leqq k$.

For the case $k=0$, it is trivial, since $(\tilde\nabla_e)^*=-\tilde\nabla_e-\mathrm{div}\,e$. We assume that $(*;\,i)$ is known to hold for $i\leqq k-1$. Let $D=D_1D_2$, where the order of D_1 is k and that of D_2 is one. Then $D^*=D_2^*D_1^*$. Hence

$$D-(-1)^{k+1}D^*=D_1(D_2+D_2^*)-D_2^*(D_1-(-1)^kD_1^*)-[D_1,\ D_2^*]\,.$$

Thus $(*;\,k)$ follows.

Proof of (2.3.2). It is easily seen that $[D^*,\ B]$ is a sum of terms of the form $\tilde{\nabla}_e^{*\beta} P_1 \tilde{\nabla}_e^{*\gamma}$ with $|\beta|+|\gamma|\leqq k$ and an appropriate first order differential operator P_1. Hence $[[D^*, B], D]$ is a sum of terms of the form $D_1P_2D_2$ where P_2 is a first order differential operator, D_1 and D_2 are hybrid products of $\tilde{\nabla}_e$, $\tilde{\nabla}_e^*$ and their commutators with the sum of the orders of D_1 and D_2 smaller than $2k+1$. Therefore the conclusion follows from the following lemma, which is used in also showing (2.4).

Lemma 2.2. *Let $D=\tilde{\nabla}_e^{\alpha}$ with $|\alpha|=m$ and L be a first order differential operator. Then*

$$|(\phi, LD\psi)|\leqq C_i\|\phi\|_{m-i}\|\psi\|_{i+1} \qquad 0\leqq i\leqq m\,,$$

for ϕ, $\psi\in\mathfrak{A}^{p,q}(\bar{M}\cap U)$.

Proof. We shall prove the following by induction on l.

$(C; l)$ for any first order differential operator P_1,

$$|(\phi, P_1\tilde{\nabla}_e^{\beta}\psi)|\leqq C(P_1, l, i)\|\phi\|_{l-i}\|\psi\|_{i+1}\,, \qquad 0\leqq i\leqq l$$

for ϕ, $\psi\in\mathfrak{A}^{p,q}(\bar{M}\cap U)$, where $|\beta|=l$.
Case $l=0$ is trivial. We assume that $(C; i)$ is known to hold for $i\leqq l-1$. Let $D=\tilde{\nabla}_e^{\alpha}$ with $|\alpha|=l$. Then

$$(\phi, P_1D\psi) = (D_1^*\phi, P_1D_2\psi)+(\phi, [P_1, D_1]D_2\psi)$$

where $D=D_1D_2$ and the order of D_1 is one. Therefore

$$|(\phi, P_1D\psi)|\leqq C_i\|\psi\|_{i+1}\{\|\phi\|_{l-1-i}+\|D_1^*\phi\|_{l-1-i}\}$$

for $0\leqq i\leqq l-1$ by $(C; l-1)$.
Hence $(C; l)$ follows from this and the following trivial estimate.

$$|(\phi, P_1D\psi)|\leqq C\|\phi\|\,\|\psi\|_{l+1}\,.$$

Proof of (2.4). We decompose the proof of (2.4) into two steps. The first and the second steps are to show

(2.4.1) $([B, \zeta_1]v, BD^*D\phi) = (D[B, \zeta_1]v, BD\phi)+O(\|\zeta'v\|_{k+1}^2)$, and

(2.4.2) $(Bv, [\zeta_1, B]D^*D\phi) = (BD\phi, [\zeta_1, B]Dv)+O(\|\zeta'v\|_{k+1}^2)$.

Under (2.4.1) and (2.4.2), one can see that $\mathrm{Re}(II_1+II_2)=\mathrm{Re}([D, [B, \zeta_1]]v, BD\phi)+O(\|\zeta'v\|_{k+1}^2)$, and in view of Lemma 2.2, $([D, [B, \zeta_1]]v, BD\phi)=O(\|\zeta'v\|_{k+1}^2)$. Therefore (2.4) holds.

Proof of (2.4.1). Following the proof of (2.3.2), we can see $([B, \zeta_1]v, BD^*D\phi)$ $=(D[B, \zeta_1]v, BD\phi)+([B, \zeta_1]v, [B, D^*]D\phi)=(D[B, \zeta_1]v, BD\phi)+O(\|\zeta'v\|_{k+1}^2)$.

q.e.d.

Proof of (2.4.2). As the proof above, we get $(Bv, [\zeta_1, B]D^*D\phi)=(BDv, [\zeta_1, B]D\zeta_1 v)+O(\|\zeta' v\|_{k+1}^2)$, and in a similar way, $(BDv, [\zeta_1, B]D\zeta_1 v)=(BD\phi, [\zeta_1, B]Dv)+O(\|\zeta' v\|_{k+1}^2)$. The proof is complete.

2.3. Estimates for $\tilde\nabla_{Z'_n}\tilde\nabla_{\bar Z'_n}u$ and $\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}u$ (Proof of Estimate B)

In Theorem 1, we have already shown a part of Estimate B:

(*; A) $\quad \sup_j \|\tilde\nabla_{\bar Z'_n}\tilde\nabla_{W_j}\zeta_1 u\|_k \leqq C_k\{\|\zeta_2 f\|_k+\|f\|\}$.

Since the tangent space on $\bar U$ is spanned by $\{W_1, \cdots, W_{2n-2}\}$ and $\{Z'_n, \bar Z'_n\}$, in order to prove Estimate B, we need to show

(2.5) $\quad \|\tilde\nabla_{Z'_n}\tilde\nabla_{\bar Z'_n}\zeta_1 u\|_k \leqq C_k\{\|\zeta_2 f\|_k+\|f\|\}$, and

(2.6) $\quad \|\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}\zeta_1 u\|_k \leqq C_k\{\|\zeta_2 f\|_k+\|f\|\}$.

In what follows, we shall prove (2.5) and (2.6). The proof of (2.5) is easy in view of (1.3.3), while that of (2.6) is slightly difficult. In view of (*; A) and (2.5), the crucial part of the proof of (2.6) is to show

(2.6.1) $\quad \|\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}\tilde\nabla_{T'}^k\zeta_1 u\| \leqq C_k\{\|\zeta_2 f\|_k+\|f\|\}$.

We decompose the proof of (2.6.1) into two parts: one part is treated by using the first boundary condition and the other one by using the second boundary condition in view of (1.1.1).

Proof of (2.5). From (1.3.3),

$$\|\tilde\nabla_{Z'_n}\tilde\nabla_{\bar Z'_n}\zeta_1 u\|_k \leqq \textstyle\sum_{l=1}^{2n-2}\|(\tilde\nabla_{W_l})^2\zeta_1 u\|_k+C\|\zeta' u\|_{k+1}+\|\zeta_1 f\|_k .$$

Hence (2.5) follows from Theorem 1 and (2.0).

Proof of (2.6). We want to establish the following inequality:

(*; B) $\quad \|\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}D\zeta_1 u\| \leqq C\{\|\zeta_2 f\|_k+\|f\|\}$,

where $D=\tilde\nabla_{N'}^l\tilde\nabla_{T'}^m\tilde\nabla_W^\alpha$ with $l+m+|\alpha|=k$.
But this inequality is known to hold if $\alpha\neq 0$ by (*; A). Hence we can assume $\alpha=0$. Changing N' to $Z'_n+\sqrt{-1}\,T'$, we obtain

$$\|\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}\tilde\nabla_{N'}^l\tilde\nabla_{T'}^m\zeta_1 u\| \leqq \|\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}\tilde\nabla_T^{m+l}\zeta_1 u\|+C\{\|\zeta_2 f\|_k+\|f\|\} .$$

Therefore in order to prove (2.6), we have only to show (2.6.1). In view of (1.3.3), we need to show the following

(2.6.2) $\quad \|i(\bar Z'_n)\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}\tilde\nabla_{T'}^k\zeta_1 u\| \leqq C\{\|\zeta_2 f\|_k+\|f\|\}$, and

(2.6.3) $\quad \|\bar\omega'_n\Lambda\tilde\nabla_{\bar Z'_n}\tilde\nabla_{\bar Z'_n}\tilde\nabla_{T'}^k\zeta_1 u\| \leqq C\{\|\zeta_2 f\|_k+\|f\|\}$.

In order to show (2.6.2), we prove the following, which is used later in the proof of Theorem 3.

(2.6.2)′ $\|i(\bar{Z}_n')\zeta_1 u\|_{k+2} \leqq C\{\|\zeta_2 f\|_k + \|f\|\}$.

Compute $\sum_{j=1}^n \tilde{\nabla}_{Z_j}\tilde{\nabla}_{\bar{Z}_j} i(\bar{Z}_n')\zeta_1 u$. Then in view of (1.3.2), we get

$$\begin{aligned}&\|\textstyle\sum_{j=1}^n \tilde{\nabla}_{Z_j}\tilde{\nabla}_{\bar{Z}_j} i(\bar{Z}_n')\zeta_1 u\|_k \leqq C\|i(\bar{Z}_n')\zeta_1 f\|_k + C\|\zeta' u\|_{k+1}\\ &\leqq C\{\|\zeta_2 f\|_k + \|f\|\}\ .\end{aligned}$$

Since $i(\bar{Z}_n')\zeta_1 u = 0$ on bM, (2.6.2)′ follows from a well-known result on elliptic boundary value problems (cf. Nirenberg [12]).

We prove (2.6.3). To do so, we first see that for any $\psi \in \mathfrak{A}^{p,q+1}(\bar{M}\cap U)$ with $\psi = 0$ on bM,

$$\|\tilde{\nabla}_{\bar{Z}_n'}\psi\|^2 = \|\tilde{\nabla}^*_{\bar{Z}_n'}\psi\|^2 + 2\,\mathrm{Re}\,(\tilde{\nabla}_W\psi, \psi) + O(\|\psi\|^2)\,,$$

by integration by parts, where $W = \nabla_{\bar{Z}_n'} Z_n'$.
In view of (1.3.4), we can apply this calculus to

$$\tilde{\nabla}^k_{T'}\zeta_1\{\bar{\omega}_n'\Lambda(\tilde{\nabla}_{\bar{Z}_n'}u - B_0 u)\}\ ,$$

where B_0 is the operator which appeared there.
Therefore we obtain

$$\|\tilde{\nabla}_{\bar{Z}_n'}\psi\| \leq \|\tilde{\nabla}_{Z_n'}\psi\| + C\{\|\zeta' u\|_{k+1} + \|\tilde{\nabla}_W\psi\|\}\ .$$

Since W belongs to the subspace spanned by $\{W_1, \cdots, W_{2n-2}\}$ and Z_n', in view of Theorem 1 and (2.5), we obtain

$$\|\tilde{\nabla}_W\psi\| \leq C\{\|\zeta_2 f\|_k + \|f\|\}\ .$$

(2.6.3) follows from these inequalities. Collecting these results, we get the following theorem.

Theorem 2.

$$\|\tilde{\nabla}_{\bar{Z}_n'}\zeta_1 u\|_{k+1} \leqq C\{\|\zeta_2 f\|_k + \|f\|\} \qquad \textit{for any } f \in \mathfrak{A}^{p,q}(\bar{M})\,,$$

where u is the solution of $(0.1)_0$.

Estimate B readily follows from Theorem 2 as Estimate A follows from Theorem 1.

3. Estimates for the $\bar{\partial}$-Neumann operator

In this section we establish sharp estimates for the $\bar{\partial}$-Neumann operator. Such estimates are easily derived from Theorems 1 and 2, for the equations

$(0.1)_0$ and $(0.1)_{-1}$ differ only up to a lower order term. In order to state it more precisely, we need to recall the precise definition of the $\bar\partial$-Neumann operator.

3.1. Review of the $\bar\partial$-Neumann problem

The $\bar\partial$-Neumann operator N: $L^2_{p,q}(M)\to L^2_{p,q}(M)$ is defined as follows.
If $f\in L^2_{p,q}(M)$ belongs the the harmonic space,

$$H^{p,q}=\{h\in D^{p,q}\,|\,\bar\partial h=0,\ \vartheta h=0\}\ ,$$

then $Nf=0$.
If $f\in L^2_{p,q}(M)$ is orthogonal to $H^{p,q}$, then Nf is the unique solution of $(0.1)_{-1}$, which is orthogonal to $H^{p,q}$.

Since the basic estimate (0.2) is assumed, it has been known that $H^{p,q}$ is finite dimensional, in particular, closed in $L^2_{p,q}(M)$. Hence the operator N is well-defined.
Notice that, setting $u=Nf$,

$$(\Box+1)u=f+u,\ \ u\in D^{p,q},\ \bar\partial u\in D^{p,q+1}\ ,$$

that is, $u=Nf$ is the solution of $(0.1)_0$ with $f+u$ in place of f in the right side.

The following lemma corresponds to Kohn's estimate (2.0). Though it might be well-known, we can not find any references, so we shall give the proof for the sake of completeness.

Lemma 3.1 (Kohn's estimate for the $\bar\partial$-Neumann operator). *Let ζ and ζ_0 be the functions in* (2.0). *Then*

$$(3.1)\qquad \|\zeta Nf\|_{k+1}\leqq C_k\{\|\zeta_0 f\|_k+\|f\|\}\ .$$

Proof. Let $\{\zeta'_l\}_{l=1}^{\infty}$ be a family of smooth real functions supported in $\{\zeta_0\equiv 1\}$ with $\zeta'_1=\zeta$ and supp $\zeta'_l\subset\{\zeta'_{l+1}\equiv 1\}$ for $l=1, 2, \cdots$.
Let $v=Nf$. Then v is the solution of $(0.1)_0$ for $v+f$ instead of f. We show by induction on k:

$$(I;\,k)\qquad \|\zeta'_l v\|_{k+1}\leqq C(k,\,l)\{\|\zeta_0 f\|_k+\|f\|\}\qquad \text{for } l=1,\,2,\,\cdots\ ,$$

where $C(k,\,l)$ is a constant independant of f.
Firstly in view of (2.0),

$$\|\zeta'_l v\|_1\leqq C\{\|\zeta'_{l+1}(v+f)\|+\|v+f\|\}\leqq C\|f\|\ .$$

Assume that $(I;\,i)$ is known to hold for $i\leqq k-1$. Then

$$\begin{aligned}&\|\zeta'_l v\|_{k+1}\leqq C\{\|\zeta'_{l+1}(v+f)\|_k+\|v+f\|\}\leqq\\ &C\|\zeta'_{l+1}v\|_k+C\{\|\zeta_0 f\|_k+\|f\|\}\leqq C'\{\|\zeta_0 f\|_k+\|f\|\}\ .\quad \text{q.e.d.}\end{aligned}$$

Proofs of Estimates A and B for the $\bar\partial$-Neumann operator.

Now we are in a position to prove

(3.2) (Estimate A) $\|\tilde{\nabla}_{W_i}\tilde{\nabla}_{W_j}\zeta_1 Nf\|_k \leqq C\{\|\zeta_2 f\|_k + \|f\|\}$,

for any $f \in \mathfrak{A}^{p,q}(\bar{M})$ and $1 \leqq i, j \leqq 2n-2$, and

(3.3) (Estimate B) $\|\tilde{\nabla}_{\bar{Z}'_n}\zeta_1 Nf\|_{k+1} \leqq C\{\|\zeta_2 f\|_k + \|f\|\}$

for any $f \in \mathfrak{A}^{p,q}(\bar{M})$.

As in the case of the problem $(0.1)_0$, (3.2) and (3.3) will follow from Theorems 1 and 2, respectively. In order to estimate error terms in the present case, we need (3.1) in place of (2.0).

Proof of (3.2). In view of Theorem 1,

$$\|\tilde{\nabla}_{W_i}\tilde{\nabla}_{W_j}\zeta_1 Nf\|_k \leqq C\{\|\zeta'(v+f)\|_k + \|v+f\|\} \leqq$$
$$C\|\zeta' v\|_k + C\{\|\zeta_2 f\|_k + \|f\|\} ,$$

where ζ' is as in the proof of (2.2) and $v = Nf$.
Therefore the conclusion follows from (3.1).

Proof of (3.3). In view of Theorem 2,

$$\|\tilde{\nabla}_{\bar{Z}'_n}\zeta_1 Nf\|_{k+1} \leqq C\{\|\zeta'(v+f)\|_k + \|v+f\|\} .$$

Hence as the proof of (3.2), the conclusion follows.

4. Remarks on estimates for allowable vector fields

In this section we consider estimates for allowable vector fields. The notation "allowable" appeared in a paper of E.M. Stein at first, and depends only on the complex structure and the boundary bM. A real vector field X on $\bar{M}$ is called allowable if both X and JX are tangential to bM.

Any allowable vector fields are spanned by $\{W_1, \cdots, W_{2n-2}\}$ and $\{rN', rT'\}$ on $\bar{U}$, where r is the defining function of bM stated in the introduction.

We shall prove the following theorem.

Theorem 1′. *Let X and Y be allowable vector fields. Then*

$$\|\tilde{\nabla}_X\tilde{\nabla}_Y\zeta_1 v\|_k \leqq C\{\|\zeta_2 f\|_k + \|f\|\} \qquad \text{for any } f \in \mathfrak{A}^{p,q}(\bar{M}) ,$$

where v is Nf or the solution of $(0.1)_0$.

Proof. It suffices to show

(I') $\|r\zeta_1 v\|_{k+2} \leqq C\{\|\zeta_2 f\|_k + \|f\|\}$.

In order to show (I'), we compute $\square(r\zeta_1 v)$, then

$$\|\square(r\zeta_1 v)\|_k \leqq \|r\zeta_1 f\|_k + C\|\zeta' v\|_{k+1} \leqq C\{\|\zeta_2 f\|_k + \|f\|\}$$

Hence (I') follows from a standard theorems for the boundary value problems of elliptic equations (cf. Nirenberg [12]).

5. Estimate for the $\bar\partial$-problem

In this section we show the sharp estimate for the solution of the equation $\bar\partial v = \theta$, orthogonal to the null space of $\bar\partial$, announced in the introduction. In Greiner–Stein [2], they showed the sharp estimate as that in Theorem 3, for L^t-Sobolev norms with $1 < t < \infty$, however their solution is not orthogonal to the null space of $\bar\partial$ by the standard metric in C^n.

Theorem 3.

$$\sum_{j=1}^{2n-2} \|\nabla_{W_j}\zeta_1 V\|_k + \|\nabla_{\bar Z'_n}\zeta_1 V\|_k \leqq C\{\|\zeta_2 f\|_k + \|f\|\}$$

for any $f \in \mathfrak{A}^{p,q}(\bar M)$, where $V = \vartheta N f$.

Proof. In view of Proposition A.5.4, $\vartheta = -\sum_{j=1}^{n} i(\bar Z_j)\nabla_{Z_j}$. Hence applying Estimate A to $i(\bar Z_j)\nabla_{Z_j} Nf$ for $1 \leqq j \leqq n-1$, Estimate B to $\nabla_{\bar Z'_n} V$ and $(2.6.2)'$ to $i(\bar Z'_n)\nabla_{Z'_n} Nf$, we get

$$\sum_{j=1}^{n} \|\nabla_{\bar Z_j}\zeta_1 V\|_k + \sum_{j=1}^{n-1} \|\nabla_{Z_j}\zeta_1 V\|_k \leqq C\{\|\zeta_2 f\|_k + \|f\|\} .$$

The conclusion follows from this estimate.

REMARK. Several types of Hölder estimates for the equation $\bar\partial u = f$ with $f \in \mathfrak{A}^{0,1}(\bar M)$ appeared when M is a strongly pseudoconvex domain. Kerzman [5] showed the Hölder estimate for any exponent smaller than 1/2, and then Henkin–Romanov [4] proved the exact 1/2-Hölder estimate. These papers dealt with the Hölder norm of the solution u itself. Siu [14] showed the 1/2-Hölder estimate for the derivatives of the solution. The sharp estimate for "good directions", that is, Hölder estimate with any exponent smaller than 1 for derivatives of u with at least one allowable direction, was showed in [12] and Krantz [9]. These solutions are not orthogonal to the null space of $\bar\partial$, i.e., holomorphic functions. The solution operator used in Kerzman [5] consists of integrals only on M and that in the other papers, which was constracted in Henkin [3], contains an integral terms on bM.

Appendix

The results of this paper can be extended to the case when M is a relatively compact subdomain with C^∞-boundary bM of a complex manifold M_1 with an arbitrary hermitian metric g. We collect here necessary materials for such extension and suggest the outline of it.

A.0. Notations. We denote the subbundle of the complexified tangent bundle consisting of the type (1,0) vectors by S. $\{Z_1, \cdots, Z_n\}$ denotes a local orthogonal basis of S ($\{Z_1, \cdots, Z_n\}$ is defined on some open set U of M_1, however we do not specify such U unless necessary.)

A.1. Complex connection

DEFINITION A.1.1. The canonical complex connection ∇ is defined as follows.

I. $\qquad \nabla_X\bar{Y} = [X, \bar{Y}]_{\bar{S}} \qquad$ for $X, Y \in \Gamma(S)$,

where $[X, \bar{Y}]_{\bar{S}}$ is the (0, 1)-part of $[X, \bar{Y}]$.

II. Let $X, Y \in \Gamma(S)$. Then $\nabla_X Y \in \Gamma(S)$ is defined by

$$Xg(Y, \bar{Z}) = g(\nabla_X Y, \bar{Z}) + g(Y, \nabla_X \bar{Z}) \qquad \text{for any } Z \in \Gamma(S).$$

III. $\qquad \nabla_{\bar{X}}\bar{Y} = \overline{\nabla_X Y}$ and $\nabla_{\bar{X}} Y = \overline{\nabla_X \bar{Y}} \qquad$ for any $X, Y \in \Gamma(S)$.

This connection coincides with what is defined in Wells [16] Chap. III, 2 and Kobayashi-Nomizu [6] II, Chap. IX, Prop. 10.2 and it is called hermitian connection there.

Proposition A.1.2. i) $\nabla g=0$. ii) $\nabla J=0$. iii) $T(X, \bar{Y})=0$ *for* $X, Y \in S$, *where* T *is the torsion tensor.*

Proof. Since g vanishes on $S\otimes S$ and $\bar{S}\otimes\bar{S}$, i) follows from II of Definition A.1.1.

To show ii), let $X, Y \in \Gamma(S)$. Then $J\bar{X} = -\sqrt{-1}\,\bar{X}$. Hence $\nabla_X(J\bar{Y}) = [X, -\sqrt{-1}\,\bar{Y}]_{\bar{S}} = -\sqrt{-1}\,[X, \bar{Y}]_{\bar{S}} = -\sqrt{-1}\,\nabla_X\bar{Y} = J(\nabla_X\bar{Y})$. $g(\nabla_X(JY) - J(\nabla_X Y), \bar{Z}) = Xg(JY, \bar{Z}) - g(JY, \nabla_X\bar{Z}) - Xg(Y, J\bar{Z}) + g(Y, \nabla_X(J\bar{Z})) = g(Y, \nabla_X(J\bar{Z}) - J\nabla_X\bar{Z}) = 0$ for any $Z \in \Gamma(S)$. Thus $\nabla_X J = 0$ for $X \in \Gamma(S)$, and $\nabla_{\bar{X}} J = 0$ can be proved in a similar way.

$T(X, \bar{Y}) = 0$ for $X, Y \in \Gamma(S)$ is a direct consequence from I of Definition A.1.1.

We extend the connection ∇ to p-form ϕ by

$$(\nabla_X\phi)(X_1, \cdots, X_p) = X(\phi(X_1, \cdots, X_p)) - \sum_{j=1}^{p} \phi(X_1, \cdots, \nabla_X X_j, \cdots, X_p).$$

A.2. The canonical metric on the vector bundle $\Lambda^{p,q}$

Let ϕ and ψ be (p, q)-forms. We define the inner product $\langle\phi, \psi\rangle$ on $\Lambda^{p,q}$ by

$$\sum\nolimits'_{I,J} \phi(Z^I, \bar{Z}^J)\overline{\psi(Z^I, \bar{Z}^J)},$$

where $I=(i_1, \cdots, i_p)$ (resp. $J=(j_1, \cdots, j_q)$) runs over $1 \leqq i_1 < \cdots < i_p \leqq n$ (resp. $1 \leqq$

$j_1<\cdots<j_q\leqq n$) and $\phi(Z^I, \bar{Z}^J)$ means

$$\phi(Z_{i_1}, \cdots, Z_{i_p}, \bar{Z}_{j_1}, \cdots, \bar{Z}_{j_q})$$

This inner product does not depend on the choice of $\{Z_1, \cdots, Z_n\}$ and we set $|\phi|=\sqrt{\langle\phi, \phi\rangle}$.

Proposition A.2.1. *Let ϕ and ψ be (p, q)-forms and X a vector field. Then*

$$X\langle\phi, \psi\rangle = \langle\nabla_X \phi, \psi\rangle+\langle\phi, \nabla_{\bar{X}}\psi\rangle .$$

Proof. Let P be any point of M_1. We take a system of coordinate $\{z_1, \cdots, z_n\}$ with the origin at P. Then there exist holomorphic vector fields $\{w_1, \cdots, w_n\}$ such that $g(w_i, \bar{w}_j)=\delta_{ij}+O(|z|^2)$ (Wells [16] Chap. III, Sec. 2, Lemma 2.3). Applying Gram–Schmidt's orthonormalization to $\{w_1, \cdots, w_n\}$, we get an o.n.s. $\{W_1, \cdots, W_n\}$ of $\Gamma(S)$ such that $\nabla W_j=0$ at P for $j=1, \cdots, n$. Then at P,

$$X(\phi(W^I, \bar{W}^J)\overline{\psi(W^I, \bar{W}^J)}) = (\nabla_X\phi)(W^I, \bar{W}^J)\overline{\psi(W^I, \bar{W}^J)}$$
$$+\phi(W^I, \bar{W}^J)\overline{(\nabla_{\bar{X}}\psi)(W^I, \bar{W}^J)} .$$

Hence $X\langle\phi, \psi\rangle=\langle\nabla_X \phi, \psi\rangle+\langle\phi, \nabla_{\bar{X}}\psi\rangle$ at P. q.e.d.

Proposition A.2.2. *Let $Z\in\Gamma(S)$ have the unit length and ω the $(0, 1)$-form defined by $\omega(X)=g(Z, X)$ for $X\in S\oplus\bar{S}$. Then*
i) *for any (p, q)-form ϕ and $(p, q+1)$-form ψ*

$$\langle\omega\Lambda\phi, \psi\rangle = \langle\phi, i(\bar{Z})\psi\rangle ,$$

where Λ is the wedge product.
ii) *For any (p, q)-form ϕ, $|\phi|^2=|\omega\Lambda\phi|^2+|i(\bar{Z})\phi|^2$.*

Proof. Let $\{Z_1, \cdots, Z_n\}$ be a local orthonormal basis of S with $Z_1=Z$.
i) Let $I=(i_1, \cdots, i_p)$ (resp. $J=(j_1, \cdots, j_{q+1})$) be any ordered p-tuple (resp. $q+1$-tuple). Then

$$(\omega\Lambda\phi)(Z^I, \bar{Z}^J) = \textstyle\sum_{l=1}^{q+1}(-1)^{p+l+1}\omega(\bar{Z}_{j_l})\phi(Z^I, \bar{Z}^{J(l)}) ,$$

where $J(l)=(j_1, \cdots, \hat{l}, \cdots, j_{q+1})$.
Hence it vanishes unless $j_1=1$. Therefore

$$\langle\omega\Lambda\phi, \psi\rangle = (-1)^p \textstyle\sum_{|I|=p,|J'|=q} \phi(Z^I, \bar{Z}^{J'})\overline{\psi(Z^I, \bar{Z}^{J'})} ,$$

where J' runs over all ordered q-tuples $(j_1, \cdots, j_q)$, $2\leqq j_1<\cdots<j_q\leqq n$.
On the other hand,

$$\langle\phi, i(\bar{Z})\psi\rangle = \textstyle\sum'\phi(Z^I, \bar{Z}^J)\overline{\psi(\bar{Z}_1, Z^I, \bar{Z}^J)} = (-1)^p \textstyle\sum'\phi(Z^I, \bar{Z}^J)\overline{\psi(Z^I, \bar{Z}_1, \bar{Z}^J)} .$$

In this sum we can consider that J runs over all ordered q-tuples $(j_1, \cdots, j_q)$, $2 \leqq j_1 < \cdots < j_q \leqq n$, since $\psi(Z^I, \bar{Z}_1, \bar{Z}^J)=0$ if $j_1=1$.
Hence $\langle \omega\Lambda\phi, \psi \rangle = \langle \phi, i(\bar{Z})\psi \rangle$.

ii) By the calculus of i), $|\omega\Lambda\phi|^2 = \sum' |\phi(Z^I, \bar{Z}^J)|^2$, where J runs over $(j_1, \cdots, j_q)$, $2 \leqq j_1 < \cdots < j_q \leqq n$. On the other hand, $|i(\bar{Z})\phi|^2 = \sum' |\phi(Z^I, \bar{Z}_1, \bar{Z}^K)|^2$, where K runs over $(k_1, \cdots, k_{q-1})$, $2 \leqq k_1 < \cdots < k_{q-1} \leqq n$. Hence $|\phi|^2 = |\omega\Lambda\phi|^2 + |i(\bar{Z})\phi|^2$.

A.3. Notations about boundary

Let M be a relatively compact subdomain of M_1 with C^∞-boundary bM. The function r is defined as in the introduction. We take a neighborhood M' of bM as there and define $\partial/\partial n$ by $g(\partial/\partial n, X)=(dr)(X)$ for $X \in TM'$ and set Z'_n as there. For an arbitrary point of the boundary bM, we take a neighborhood U of that point and the functions ζ_1, ζ_2 as in the introduction. $\{Z_1, \cdots, Z_n\}$ and $\{W_1, \cdots, W_{2n-1}\}$ are stated as there.

A.4. Divergence and integral formula

Let X be a vector field. Then the following formula holds (Kobayashi-Nomizu [6] I, Appendix).

(A.4.1) $$\operatorname{div} X = \sum_{j=1}^{2n} g(e_j, \nabla_{e_j} X + T(X, e_j)),$$

where $\{e_1, \cdots, e_{2n}\}$ is an o.n.s. of the tangent space. Therefore if $X \in \Gamma(S)$, then

(A.4.2) $$\operatorname{div} X = \sum_{j=1}^{n} g(\bar{Z}_j, \nabla_{Z_j} X + T(X, Z_j)).$$

The following integral formula holds by Stoke's theorem (Matsushima [10] pp. 292).

(A.4.3) $$\int_{bM} f\, i(X) dV = \int_M (f \operatorname{div} X + Xf)\, dV.$$

DEFINITION A.4.1. The vector field $\alpha \in \Gamma(\bar{S})$ on M_1 is defined by

$$\sum_{j,k=1}^{n} g(\bar{Z}_j, T(Z_k, Z_j))\bar{Z}_k.$$

The value of the above sum does not depend on the choice of $\{Z_1, \cdots, Z_n\}$. Therefore α is defined on the whole M_1.

Proposition A.4.2. *Let $\omega \in \mathfrak{A}^{0,1}(\bar{M})$. Then*

$$\int_M \{\sum_{j=1}^{n} (\nabla_{Z_j}\omega)(\bar{Z}_j)\} dV = \int_{bM} \omega(\bar{Z}'_n) dS' - \int_M \omega(\alpha) dV,$$

where dS' is $1/\sqrt{2}$ times the surface element of bM.

REMARK. The value of $\sum_{j=1}^{n} (\nabla_{Z_j}\omega)(\bar{Z}_j)$ does not depend on the choice of $\{Z_1, \cdots, Z_n\}$. Hence it is defined on the whole M.

Proof. Let $X=\sum_{j=1}^n \omega(\bar{Z}_j)Z_j$. Then X is defined on the whole $\bar{M}$ and

$$\operatorname{div} X = \sum_{j=1}^n \{g(\nabla_{Z_j}X, \bar{Z}_j)+g(T(X, Z_j), \bar{Z}_j)\} .$$

Now $\sum_{j=1}^n g(T(X, Z_j), \bar{Z}_j)=\sum_{j,k=1}^n \omega(\bar{Z}_k)g(T(Z_k, Z_j), \bar{Z}_j)=\omega(\alpha)$. On the other hand, $\sum_{j=1}^n g(\nabla_{Z_j}X, \bar{Z}_j)=$

$$\sum_{j=1}^n \{Z_j g(X, \bar{Z}_j)-g(X, \nabla_{Z_j}\bar{Z}_j)\} = \sum_{j=1}^n (\nabla_{Z_j}\omega)(\bar{Z}_j) .$$

Hence in view of (A.4.3),

$$\int_M \{\sum_{j=1}^n (\nabla_{Z_j}\omega)(\bar{Z}_j)+\omega(\alpha)\}\, dV = \int_{bM} i(X)dV .$$

Now if we take $\{Z_1, \cdots, Z_n\}$ with $Z_n=Z'_n$, then $i(\bar{Z}_j)dV$ on bM vanishes as a volume form on bM for $j=1, \cdots, n-1$ and $i(\bar{Z}'_n)dV$ is equal to dS' as the volume form on bM. Therefore

$$\int_{bM} i(X)dV = \int_{bM} \omega(\bar{Z}'_n)dS' . \quad \text{q.e.d.}$$

A.5. $\bar{\partial}$ and ϑ

Lemma A.5.1. *Let ϕ be a p-form. Then*

$$(d\phi)(X_1, \cdots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1}(\nabla_{X_j}\phi)(X_1, \cdots, \hat{j}, \cdots, X_{p+1})$$
$$-\sum_{i<j}(-1)^{i+j}\phi(T(X_i, X_j), X_1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, X_{q+1}) .$$

Proof. $(d\phi)(X_1, \cdots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1}X_j(\phi(X_1, \cdots, \hat{j}, \cdots, X_{p+1}))$
$+\sum_{i<j}(-1)^{i+j}\phi([X_i, X_j], X_1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, X_{p+1})$,

(Matsushima [10] pp. 140).

Replacing $X_j(\phi(X_1, \cdots, \hat{j}, \cdots, X_{p+1}))$ by $(\nabla_{X_j}\phi)(X_1, \cdots, \hat{j}, \cdots, X_{p+1})+$
$\sum_{i<j} \phi(X_1, \cdots, \nabla_{X_j}X_i, \cdots, \hat{j}, \cdots, X_{p+1})+\sum_{j<i}\phi(X_1, \cdots, \hat{j}, \cdots,$
$\nabla_{X_j}X_i, \cdots, X_{p+1})$,

we get

$$(d\phi)(X_1, \cdots, X_{p+1}) = \sum_{j=1}^{p+1}(-1)^{j+1}(\nabla_{X_j}\phi)(X_1, \cdots, \hat{j}, \cdots, X_{p+1})-$$
$$\sum_{i<j}(-1)^{i+j}\phi(\nabla_{X_i}X_j-\nabla_{X_j}X_i-[X_i, X_j], X_1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, X_{p+1}) .$$

This is equal to the conclusion.

Proposition A.5.2. *Let ϕ be a (p, q)-form. Then*

$$(\bar{\partial}\phi)(X_1, \cdots, X_{q+1}, Y_1, \cdots, Y_p) =$$
$$\sum_{j=1}^{q+1} (-1)^{j+1}(\nabla_{X_j}\phi)(X_1, \cdots, \hat{j}, \cdots, X_{q+1}, Y_1, \cdots, Y_p)-$$

$$\sum_{i<j}(-1)^{i+j}\phi(T(X_i, X_j), X_1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, X_{q+1}, Y_1, \cdots, Y_p)$$

for $X_1, \cdots, X_{q+1} \in \bar{S}$ *and* $Y_1, \cdots, Y_p \in S$.

Proof. $\bar{\partial}\phi$ is the restriction of $d\phi$ on $S^p \otimes \bar{S}^{q+1}$. Hence the conclusion follows from iii) of Proposition A.1.2. and the fact ϕ vanishes on $S^{p-1} \otimes \bar{S}^{q+1}$.

DEFINITION A.5.3. i) Let ϕ be a (p, q)-form. Then the $(p, q+1)$-form $T\phi$ is defined by

$$(T\phi)(X_1, \cdots, X_{q+1}, Y_1, \cdots, Y_p) =$$
$$-\sum_{i<j}(-1)^{i+j}\phi(T(X_i, X_j), X_1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, X_{q+1}, Y_1, \cdots, Y_p)$$

for $X_1, \cdots, X_{q+1} \in \bar{S}$ and $Y_1, \cdots, Y_p \in S$.
ii) T^* is the adjoint of T with respect to the canonical metric on $\Lambda^{p,q}$, i.e., for each $(p, q+1)$-form ϕ, the (p, q)-form $T^*\phi$ is defined by

$$\langle T\psi, \phi\rangle = \langle \psi, T^*\phi\rangle \qquad \text{for any } \psi \in \Lambda^{p,q}.$$

Proposition A.5.4. *Let* ϕ *be a* $(p, q+1)$*-form. Then*

$$\phi = -\sum_{j=1}^{n} i(\bar{Z}_j)\nabla_{Z_j}\phi + T^*\phi - i(\alpha)\phi.$$

Proof. Let ψ be a (p, q)-form. We define the $(0, 1)$-form ω by $\omega(X) = \langle i(X)\phi, \psi\rangle$ for $X \in S \oplus \bar{S}$ and apply Proposition A.4.2 to it. In view of Proposition A.2.1. and A.2.2,

$$(\nabla_{Z_j}\omega)(\bar{Z}_j) = \langle \nabla_{Z_j}\{i(\bar{Z}_j)\phi\}, \psi\rangle + \langle i(\bar{Z}_j)\phi, \nabla_{\bar{Z}_j}\psi\rangle - \langle i(\nabla_{Z_j}\bar{Z}_j)\phi, \psi\rangle$$
$$= \langle i(\bar{Z}_j)\nabla_{Z_j}\phi, \psi\rangle + \langle \phi, \bar{\omega}_j \Lambda_{\bar{Z}_j}\psi\rangle,$$

where $\{\bar{\omega}_1, \cdots, \bar{\omega}_n\}$ is the dual of $\{\bar{Z}_1, \cdots, \bar{Z}_n\}$.

Hence $(\phi, \sum_{j=1}^{n} \bar{\omega}_j \Lambda \nabla_{\bar{Z}_j}\psi) + (\sum_{j=1}^{n} i(\bar{Z}_j)\nabla_{Z_j}\phi, \psi) =$

$$(-i(\alpha)\phi, \psi) + \int_{bM} \langle i(\bar{Z}_n')\phi, \psi\rangle dS'.$$

Therefore

$$(\phi, \bar{\partial}\psi) = (-\sum_{j=1}^{n} i(\bar{Z}_j)\nabla_{Z_j}\phi + T^*\phi - i(\alpha)\phi, \psi) + \int_{bM} \langle i(\bar{Z}_n')\phi, \psi\rangle dS'.$$

Hence the conclusion follows.

Proposition A.5.5. *Let* U *and* $\{Z_1, \cdots, Z_n\}$ *be as in* A.3. *Then*

$$\square = -\sum_{j=1}^{n} \nabla_{Z_j}\nabla_{Z_j} + \textit{lower order derivatives.}$$

Proof. Let $\{\omega_1, \cdots, \omega_n\}$ be the dual basis of $\{Z_1, \cdots, Z_n\}$. In view of Proposition A.5.2 and A.5.4, the principal parts of $\bar{\partial}\vartheta$ and $\vartheta\bar{\partial}$ are equal to those of

$$-\sum_{j,k=1}^{n} \overline{\omega}_j\Lambda i(\bar{Z}_k)\nabla_{\bar{Z}_j}\nabla_{Z_k} \text{ and } -\sum_{j,k=1}^{n} i(\bar{Z}_k)\{\overline{\omega}_k\Lambda\nabla_{Z_k}\nabla_{\bar{Z}_j}\} .$$

Therefore using the relations

$$i(\bar{Z}_j)\{\overline{\omega}_k\Lambda\phi\}+\overline{\omega}_k\Lambda i(\bar{Z}_j)\phi = \delta_{jk}\phi\ ,\ \ 1\leqq j,\ k\leqq n\ ,$$

we get the conclusion.

A.6. Boundary conditions

In the proof of Proposition A.5.4, we get

$$\text{(A.6.1)}\quad (\phi,\ \bar\partial\psi) = (\vartheta\phi,\ \psi)+\int_{bM}\langle i(\bar{Z}_n')\phi,\ \psi\rangle dS'$$

for $\phi\in\mathfrak{A}^{p,q+1}(\bar{M})$ and $\psi\in\mathfrak{A}^{p,q}(\bar{M})$.
Hence we can see that the first and second boundary conditions are the same in $\boldsymbol{C}^n$.

A.7. Modified connection $\tilde\nabla$

As Paragraph 1.3, we can define the modified connection $\tilde\nabla$ and extend it to differential forms and hence, we set $S(X)=\nabla_X-\tilde\nabla_X$.

Lemma A.7.1. $g(S(X)Y,\ Z) = -g(Y,\ S(X)Z)$.

Proof. $S(X)Y = P\nabla_X(I-P)Y+(I-P)\nabla_X PY$ from the definition.

Hence $g(S(X)Y,\ Z) = g(\nabla_X(I-P)Y,\ PZ)+g(\nabla_X PY,\ (I-P)Z)=$
$X\{g((I-P)Y, PZ)+g(PY, (I-P)Z)\}-g((I-P)Y, \nabla_X PZ)-g(PY, \nabla_X(I-PZ)=$
$-g(Y, (I-P)\nabla_X PZ+P\nabla_X(I-P)Z) = -g(Y,\ S(X)Z)$. q.e.d.

Proposition A.7.2.

$$\langle S(X)\phi,\ \psi\rangle = \langle\phi,\ -S(\bar{X})\psi\rangle \textit{ holds for } \phi,\ \psi\in\mathfrak{A}^{p,q}(M')\ .$$

Proof. Let $\{e_1, \cdots, e_{2n}\}$ be an o.n.s. of the tangent space. Then we can see

$$\langle\phi,\ \psi\rangle = {\sum}'_{|K|=m}\phi(e^K)\overline{\psi(e^K)} = \frac{1}{m!}\sum_{k_1,\cdots,k_m=1}^{2n}\phi(e^K)\overline{\psi(e^K)}\ ,$$

where $m = p+q$ and $K = (k_1,\ \cdots,\ k_m)$.
Hence $m!\langle S(X)\phi,\ \psi\rangle = \sum_K (S(X)\phi)\ (e^K)\overline{\psi(e^K)} =$

$$-\sum_K\sum_{j=1}^{m}\phi(e_{k_1},\ \cdots,\ S(X)e_{k_j},\ \cdots,\ e_{k_m})\overline{\psi(e_{k_1},\ \cdots,\ e_{k_m})}$$
$$= -\sum_{j=1}^{m}\sum_K\sum_{l=1}^{2n} g(S(X)e_{k_j},\ e_l)\phi(e_{k_1},\ \cdots,\ e_l,\ \cdots,\ e_{k_m})\overline{\psi(e^K)}$$
$$= \sum_{j=1}^{m}\sum_{K,l}\phi(e_{k_1},\ \cdots,\ e_l,\ \cdots,\ e_{k_m})\overline{g(e_{k_j},\ S(\bar{X})e_l)\psi(e^K)}$$
$$= -m!\langle\phi,\ S(\bar{X})\psi\rangle .\quad \text{q.e.d.}$$

Proposition A.7.3. *There is a linear transformation* B_0 *of* $\Lambda^{p,q}$ *such that*

$u\in D^{p,q}$ and $\bar\partial u\in D^{p,q+1}$ imply

$$\varpi_n'\Lambda(\tilde\nabla_{\bar Z_n'}u-B_0u)=0 \quad on \quad bM,$$

where $\varpi_n'=\sqrt{2}\,\bar\partial r=g(Z_n', *)$.

Proof. Let $\phi\in\mathfrak{A}^{p,q}(U)$. Then $\bar\partial\phi=\sum_{j=1}^{n}\varpi_j\Lambda\nabla_{\bar Z_j}\phi+T\phi$. Hence $i(\bar Z_n')\bar\partial\phi=\nabla_{\bar Z_n'}\phi-\sum_{j=1}^{n}\varpi_j\Lambda i(\bar Z_n')\nabla_{\bar Z_j}\phi+i(\bar Z_n')T\phi$. Therefore

$$\varpi_n'\Lambda i(\bar Z_n')\bar\partial u=\varpi_n'\Lambda\{\nabla_{\bar Z_n'}u-\textstyle\sum_{j=1}^{n-1}\varpi_j\Lambda i(\bar Z_n')\nabla\bar Z_ju+i(\bar Z_n')Tu\}=0 \text{ on } bM.$$

Thus changing ∇_X to $\tilde\nabla_X+S(X)$ and noticing $i(\bar Z_n')\tilde\nabla_{\bar Z_j}u=0$ on bM for $j=1,\cdots,n-1$, we get

$$0=\varpi_n'\Lambda[\tilde\nabla_{\bar Z_n'}u+\{S(\bar Z_n')+i(\bar Z_n')T-\textstyle\sum_{j=1}^{n-1}\varpi_j\Lambda i(\bar Z_n')S(\bar Z_j)\}u] \text{ on } bM.$$

Hence the conclusion holds taking B_0 as

$$-S(\bar Z_n')-i(\bar Z_n')T+\textstyle\sum_{j=1}^{n-1}\varpi_j\Lambda i(\bar Z_n')S(\bar Z_j)\,.$$

A.8. The outline of the extension

Let U and $\{Z_1,\cdots,Z_n\}$ be as before. In general there is not an appropriate basis of $\mathfrak{A}^{1,0}(U)$ as in C^n. So we give Estimates A and B in the following forms:

Estimate A. $\sup_{i,j}\|\nabla_{W_i}\nabla_{W_j}\zeta_1u\|_k\leqq C_k\{\|\zeta_2f\|_k+\|f\|\}$,

Estimate B. $\|\nabla_{\bar Z_n'}\zeta_1u\|_{k+1}\leqq C_k\{\|\zeta_2f\|_k+\|f\|\}$.

In order to show these estimates it suffices to prove Theorems 1 and 2. We can easily see that Theorem 1 hold in this case, since the proofs of Theorem 1 and Proposition 2.1 do not require the speciality of C^n.

In order to show Theorem 2, we need some formulas in Section 1. We showed (1.1.1) in Proposition A.2.2, (1.3.1) by Propositions A.2.1 and A.7.2, (1.3.2) in Proposition A.5.5, and (1.3.4) in Proposition A.7.3. Hence we can show (2.5) and (2.6.2) as before. We need (1.3.1) to integrate by parts when we prove (2.6.3). Therefore we can show (2.6), hence Theorem 2.

(3.2) and (3.3) are obtained by the same method as there and the conclusion in Section 4 holds in this case.

The difference of ϑ in Proposition A.5.4 and in Theorem 3 consists of terms without derivatives, hence Theorem 3 holds in this case.

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