

ON THE BOUNDARY BEHAVIOR OF THE DIRICHLET SOLUTIONS AT AN IRREGULAR BOUNDARY POINT

Dedicated to Professor Makoto Ohtsuka on his 60th birthday

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Introduction. In the classical potential theory, O. Frostman [2] investigated the boundary behavior of the Dirichlet solution H_f^y for continuous boundary data f at an irregular boundary point x of a bounded domain U of R^n . And it was revealed that the cluster set of H_f^y at x is a segment with a possible exception. In other words, the cluster set of harmonic measures at x has two extreme points —the Dirac measure ε_x and the balayaged measure $\varepsilon_x^c U$. A generalization of this result was given by Constantinescu-Cornea [1] in an axiomatic setting in a more comprehensive context. Recently, J. Lukeš-J. Malý [6] considered this problem in a relatively compact open subset of a harmonic space. The present paper is a contribution to this problem under a resolutive compactification.

Let X be a \mathcal{P} -harmonic space with countable base in the sense of Constantinescu-Cornea [1] and X^* be a resolutive compactification. Let U be an open set of X . The closure \bar{U} of U in X^* is a resolutive compactification of U . Suppose that $\partial U = (\bar{U} \setminus U) \cap X \neq \emptyset$. For a sequence $\{b_k\}$ converging to $x \in \partial U$ and satisfying $\varepsilon_{b_k}^c U \rightarrow \varepsilon_x^c U$, the harmonic measure of U at b_k converges to a measure λ_x . If x is irregular for \bar{U} , λ_x enjoys remarkable properties stated in Theorem 7, which has a counterpart with the results of Lukeš-Malý [6] and Hyvönen [3], and is connected with a version of maximal sequences considered by Smyrnélis [7]. In view of the work of Lukeš-Malý, we can decide the structure of the cluster set \mathcal{N}_x^y of harmonic measures and reveal that the type of \mathcal{N}_x^y is a local property. We can also conclude the same result for the cluster set of the normalized Dirichlet solutions.

1. Preliminaries

Let X be a \mathcal{P} -harmonic space with countable base in the sense of Constantinescu-Cornea [1] and X^* be a resolutive compactification of X . We assume that there exists a function s_0 which is bounded superharmonic on X and $\inf s_0 > 0$. We write $\Delta = X^* \setminus X$.

For an open subset U of X , we set $\Delta U = \partial U \cup (\Delta \cap \bar{U})$, where $\partial U = (\bar{U} \setminus U) \cap X$ and \bar{U} is the closure of U in X^* .

For a numerical function f on ∂U (resp. ΔU , resp. Δ) we define

$$H_f^{u,x}(a) = \inf \left\{ \begin{array}{l} \text{hyperharmonic on } U, \text{ lower bounded,} \\ v(a); v \geq 0 \text{ outside a compact subset of } X, \\ \underline{\lim} v \geq f \text{ on } \partial U \end{array} \right\}$$

(resp.

$$H_f^u(a) = \inf \left\{ \begin{array}{l} \text{hyperharmonic on } U, \text{ lower bounded,} \\ v(a); \underline{\lim} v \geq f \text{ on } \Delta U \end{array} \right\},$$

resp.

$$H_f(a) = \inf \left\{ \begin{array}{l} \text{hyperharmonic on } X, \text{ lower bounded,} \\ v(a); \underline{\lim} v \geq f \text{ on } \Delta \end{array} \right\},$$

and $H_f^{u,x} = -H_{(-f)}^{u,x}$ (resp. $H_f^u = -H_{(-f)}^u$, resp. $H_f = -H_{(-f)}$).

When $H_f^{u,x} = \underline{H}_f^{u,x}$ and harmonic we write it $H_f^{u,x}$. Similarly we define H_f^u and H_f .

In the following, we denote by $\cdot|A$, the restriction on A .

Proposition 1. *The closure \bar{U} of U in X^* is a resolutive compactification. For $f \in C(\Delta U)$, let f^* be a finite continuous extension of f onto X^* and let $u = H_{f^*|_{\Delta}}$. Then we have*

$$(1.1) \quad H_f^u = H_{f^*-u}^{u,x}.$$

The proposition is proved quite in the same way as in [5], Prop. 1.

In the sequel, we denote by λ_a the harmonic measure of \bar{U} at a , i.e., $\lambda_a(f) = H_f^u(a)$ for every $f \in C(\Delta U)$, and stands ε_x^A for the balayaged measure of the Dirac measure ε_x on A [1].

Corollary 2. *$x \in \partial U$ is regular for \bar{U} if and only if $\varepsilon_x^{CU} = \varepsilon_x$.*

Proof. If g is continuous on ∂U and has a compact support, then g can be extended continuously to be 0 on Δ , thus $H_g^u = H_g^{u,x}$. This proves that if x is regular for \bar{U} then we have $\varepsilon_x^{CU}(g) = g(x)$ for all continuous functions g on ∂U with compact support, since $\varepsilon_x^{CU}(g) = \lim_k \varepsilon_{b_k}^{CU}(g) = \lim_k H_g^{u,x}(b_k) = \lim_k H_g^u(b_k) = g(x)$ for some $\{b_k\}$ converging to x , i.e., $\varepsilon_x^{CU} = \varepsilon_x$. The converse is also true, since $\varepsilon_x^{CU} = \varepsilon_x$ means that $\{\varepsilon_{a_k}^{CU}\}$ converges to ε_x for every $\{a_k\}$ tending to x and the following Corollary 3 deduces the result.

Corollary 3. *Let $x \in \partial U$ and $\{a_k\}$ be a sequence of points of U tending to x . If $\{\varepsilon_{a_k}^{CU}\}$ converges for μ , then $\{\lambda_{a_k}\}$ converges vaguely.*

Proof. Using the same notation as in Proposition 1,

$$H_f^u = H_{f^*-u}^{u,x} + u$$

and $\lim_k \lambda_{a_k}(f) = \lim_k H_{f_i}^U(a_k) = \lim_k H_{f-u}^{U,x}(a_k) + u(x)$. By [1] Cor. 7.2.6, $\lim_k H_{f-u}^{U,x}(a_k) = \lim_k \varepsilon_{a_k}^{CU}(f-u) = \mu(f-u)$, since $|f^*-u| \leq p$ for a potential p on X .

Corollary 4. *The regularity of $x \in \partial U$ is a local property, that is, x is regular for \bar{U} if and only if it is regular for $\overline{U \cap V}$ for every neighborhood V of x .*

Proof. The regularity of x for U is equivalent to $\varepsilon_x^{CU} = \varepsilon_x$ and the latter is equivalent to the fact that $X \setminus U$ is not thin at x . This is also equivalent to the fact that $X \setminus (U \cap V)$ is not thin at x since $X \setminus V$ is thin at x .

2. The definition of λ_x

Lemma 5. *Let $x \in \partial U$, $f \in C(\Delta U)$, f^* be a continuous extension of f on X^* and let $u = H_{f^*|\Delta}$. Then $u(x) - \varepsilon_x^{CU}(u)$ depends only on f .*

Proof. Consider the sequence $\{b_k\}$ such that $b_k \rightarrow x$ and $\varepsilon_{b_k}^{CU} \rightarrow \varepsilon_x^{CU}$. By Corollary 3, the sequence $\{\lambda_{b_k}\}$ of harmonic measures with respect to \bar{U} converges and $\lim_k \lambda_{b_k}(f) = \lim_k H_f^U(b_k) = \lim_k H_{f-u}^{U,x}(b_k) + u(x) = \varepsilon_x^{CU}(f-u) + u(x)$. If we denote this limit by λ , then $\varepsilon_x(u) - \varepsilon_x^{CU}(u) = \lambda(f) - \varepsilon_x^{CU}(f)$, and the last expression shows that $\varepsilon_x(u) - \varepsilon_x^{CU}(u)$ depends only on f and independent of f^* .

We can see also that if $\varepsilon_{a_k}^{CU} \rightarrow \varepsilon_x$ then $\lambda_{a_k} \rightarrow \varepsilon_x$.

We shall denote by λ_x the vague limit of $\{\lambda_{b_k}\}$ corresponding to the sequence $\{b_k\}$ satisfying $\varepsilon_{b_k}^{CU} \rightarrow \varepsilon_x^{CU}$, and call $\{b_k\}$ to be maximal at x in \bar{U} . Thus using the above notation we have

$$(2.1) \quad \lambda_x(f) = \varepsilon_x^{CU}(f) + \varepsilon_x(u) - \varepsilon_x^{CU}(u) \text{ for every } f \in C(\Delta U).$$

If U is a relatively compact open set of X , then λ_x is just ε_x^{CU} . We can see $\lambda_x \neq \varepsilon_x^{CU}$ in general. In fact, we have

Proposition 6. *If $X \setminus U$ is compact, then $\lambda_x|\partial U = \varepsilon_x^{CU}$.*

Proof. For $f \in C(\Delta U)$ with $f=0$ on $\bar{U} \cap \Delta$
 $\lambda_x(f) = \lim_k \lambda_{b_k}(f) = \lim_k H_f^U(b_k) = \lim_k H_{f,x}^{U,x}(b_k) = \lim_k \varepsilon_{b_k}^{CU}(f) = \varepsilon_x^{CU}(f)$.

3. The properties of λ_x

We denote by $\Gamma(U)$ the harmonic boundary of \bar{U} , i.e.,

$$\Gamma(U) = \overline{\bigcup_{a \in U} \text{Supp} \cdot \lambda_a}$$

We define, for $x \in \partial U$

$$\mathcal{N}_x^U = \{\lambda; \exists \{a_k\} \subset U, a_k \rightarrow x, \lambda_{a_k} \rightarrow \lambda \text{ vaguely}\}.$$

Theorem 7. *Let $x \in \partial U$ be irregular for \bar{U} . Then we have:*

$$(1) \quad \lambda_x \neq \varepsilon_x \text{ and } \mathcal{N}_x^U \subset \{t \varepsilon_x + (1-t)\lambda_x; 0 \leq t \leq 1\},$$

- (2) $\lambda_x(s) = \underline{\lim}_x H_s^U$ for every s continuous on \bar{U} and superharmonic on U ,
- (3) if $x \notin \Gamma(U)$ then $\mathcal{N}_x^U = \{\lambda_x\}$, thus for every $f \in C(\Delta U)$ H_f^U is extendable continuously at x to the value $\lambda_x(f)$.
- (4) for $f \in C(\Delta U)$, non-negative and $f=0$ in a neighborhood of x we have $\lambda_x(f) = \overline{\lim}_x H_f^U$,
- (5) $\varepsilon_x \in \mathcal{N}_x^U$ if and only if $x \in \Gamma(U)$,
- (6) let $x \in \Gamma(U)$ and $f \in C(\Delta U)$ such that $f(x) \neq \lambda_x(f)$, then only one of the following cases occurs:
 - (i) $\underline{\lim}_x H_f^U = f(x) < \lambda_x(f) = \overline{\lim}_x H_f^U$,
 - (ii) $\underline{\lim}_x H_f^U = \lambda_x(f) < f(x) = \overline{\lim}_x H_f^U$.

Proof. To prove (1), we note that the convergence of $\{\lambda_{a_k}\}$ is equivalent to the convergence of $\{\varepsilon_{a_k}^U\}$. Suppose that there exists $\{a_k\}$ such that $a_k \rightarrow x$ and $\lambda_{a_k} \rightarrow \lambda$, then we may find $t \in [0, 1]$ so that $\varepsilon_{a_k}^U \rightarrow t\varepsilon_x + (1-t)\varepsilon_x^U$. For $f \in C(\Delta U)$ we have $\lambda(f) = \lim_k \lambda_{a_k}(f) = \lim_k H_f^U(a_k) = \lim_k H_f^U(a_k) + u(x) = \lim_k \varepsilon_{a_k}^U(f-u) + u(x) = t\varepsilon_x(f-u) + (1-t)\varepsilon_x^U(f-u) + u(x) = t\varepsilon_x(f) + (1-t)[\varepsilon_x(u) - \varepsilon_x^U(u) + \varepsilon_x^U(f)] = t\varepsilon_x(f) + (1-t)\lambda_x(f)$.

(2): let $\{b_k\}$ be maximal at x in \bar{U} and fix a function s .

$$\underline{\lim}_x H_s^U \leq \lim_k H_s^U(b_k) = \lambda_x(s) \leq s(x).$$

Then there exists $\lambda \in \mathcal{N}_x^U$ so that $\lambda(s) = \underline{\lim}_x H_s^U$, for there is a sequence $\{a_k\}$ satisfying $a_k \rightarrow x$ and $\lim_k \lambda_{a_k}(s) = \underline{\lim}_x H_s^U$, and therefore there is a subsequence of $\{\lambda_{a_k}\}$ converging to λ vaguely. We can not conclude $\lambda_x(s) > \underline{\lim}_x H_s^U$ since, by (1), $\lambda = t\varepsilon_x + (1-t)\lambda_x$ for some $t \in [0, 1]$.

(3): suppose that there is a function $f \in C(\Delta U)$ such that $\underline{\lim}_x H_f^U < \overline{\lim}_x H_f^U$. Then as in the proof (2) there exist $\lambda', \lambda'' \in \mathcal{N}_x^U$ satisfying

$$(2.2) \quad \begin{aligned} \underline{\lim}_x H_f^U &= \lambda'(f), \lambda' = t'\varepsilon_x + (1-t')\lambda_x, 0 \leq t' \leq 1 \\ \overline{\lim}_x H_f^U &= \lambda''(f), \lambda'' = t''\varepsilon_x + (1-t'')\lambda_x, 0 \leq t'' \leq 1. \end{aligned}$$

Hence, $(t'' - t')[f(x) - \lambda_x(f)] > 0$. This is impossible, since the support of λ_x is contained in $\Gamma(U)$ and $x \notin \Gamma(U)$.

(4): let $f \in C(\Delta U)$, $f \geq 0$, $f=0$ on a neighborhood of x and let $\{b_k\}$ be maximal at x in \bar{U} . As above, we have $\lambda \in \mathcal{N}_x^U$ such that $\lambda(f) = \overline{\lim}_x H_f^U$ and $\lambda = t\varepsilon_x + (1-t)\lambda_x$ for $t \in [0, 1]$. We claim that $t=0$; in fact, $\overline{\lim}_x H_f^U = \lambda(f) = tf(x) + (1-t)\lambda_x(f) = (1-t)\lambda_x(f) = (1-t)\lim_k H_f^U(b_k) \leq \lim_k H_f^U(b_k) \leq \overline{\lim}_x H_f^U$, which implies that $t=0$.

(5), (6): if $x \notin \Gamma(U)$ then by (2), $\mathcal{N}_x^U = \{\lambda_x\}$ and $\lambda_x \neq \varepsilon_x$ thus $\varepsilon_x \notin \mathcal{N}_x^U$, i.e., $\varepsilon_x \in \mathcal{N}_x^U$ implies that $x \in \Gamma(U)$. To prove the converse suppose that $x \in \Gamma(U)$. Then, by (1), we have $\lambda_x \in \mathcal{N}_x^U$, $\lambda_x \neq \varepsilon_x$. Letting $f \in C(\Delta U)$ with $f(x) \neq \lambda_x(f)$, by virtue of the definition of the harmonic boundary $\Gamma(U)$ and the fact that for a continuous extension f^* of f on \bar{U} there is a potential q on U such that $|H_f^U - f^*| \leq q$, we obtain

$$\underline{\lim}_x H_f^U \leq f(x) \leq \overline{\lim}_x H_f^U.$$

However, the only possible cases are (i) $\underline{\lim}_x H_f^U = f(x)$ or (ii) $\overline{\lim}_x H_f^U = f(x)$; for if $\underline{\lim}_x H_f^U < f(x) < \overline{\lim}_x H_f^U$ then as in the proof of (3), there are $t', t'' \in [0, 1]$ so that

$t'f(x) + (1-t')\lambda_x(f) < f(x) < t''f(x) + (1-t'')\lambda_x(f)$,
 i.e., $(1-t')[f(x) - \lambda_x(f)] > 0$ and $(1-t'')[f(x) - \lambda_x(f)] < 0$, which is absurd. We shall consider the case (i). We may find also $\lambda', \lambda'' \in \mathcal{N}_x^U$ such that

$$\lambda'(f) = \underline{\lim}_x H_f^U, \lambda' = t' \varepsilon_x + (1-t')\lambda_x, 0 \leq t' \leq 1,$$

$$\lambda''(f) = \overline{\lim}_x H_f^U, \lambda'' = t'' \varepsilon_x + (1-t'')\lambda_x, 0 \leq t'' \leq 1.$$

The equality $t'f(x) + (1-t')\lambda_x(f) = f(x)$ means that $t' = 1$ and $\varepsilon_x = \lambda' \in \mathcal{N}_x^U$. On the other hand, the inequality $f(x) < t''f(x) + (1-t'')\lambda_x(f)$ means that $t'' < 1$ and $\lambda_x(f) > f(x)$. This implies $t'' = 0$, since if $t'' > 0$ then we are led to the contradiction that $\lambda''(f) = \overline{\lim}_x H_f^U = t''f(x) + (1-t'')\lambda_x(f) < \lambda_x(f) \leq \overline{\lim}_x H_f^U$. Similarly in the case (ii), we have $\varepsilon_x \in \mathcal{N}_x^U$ and $\lambda_x(f) = \underline{\lim}_x H_f^U$.

REMARKS. In the Theorem 7, (1) was proved by O. Frostman [2] in the classical potential theory. The fundamental contribution to the behavior of normalized solution $H_f^{U,x}$ in the axiomatic potential theory is due to Constantinescu-Cornea [1], and when U is relatively compact open set the precise investigation was given by Lukeš-Malý [6].

(3) has a counterpart in a result of J. Hyvönen [3] (Cor. 1.6).

(4) is considered to be a refined variant of a theorem of Smyrnélis [7] (Cor. 2) and the maximal sequence $\{b_k\}$ corresponds to "une suite maximal".

It is plausible that λ_x is the vague limit of $\{\varepsilon_x^{C U_n}\}$, where $U_n = U \cap X_n$ with a compact exhaustion $\{X_n\}$ of X .

4. The structure of \mathcal{N}_x^U

In [6] Lukeš-Malý proved that in the case where U is a relatively compact open set \mathcal{N}_x^U has only four types: (1) $\mathcal{N}_x^U = \{\varepsilon_x\}$, (2) $\mathcal{N}_x^U = \{\varepsilon_x^{C U}\}$, (3) $\mathcal{N}_x^U = \{\varepsilon_x, \varepsilon_x^{C U}\}$, (4) $\mathcal{N}_x^U = \{t\varepsilon_x + (1-t)\varepsilon_x^{C U}; 0 \leq t \leq 1\}$. The situation is quite the same in our consideration. That is, in the same argument as in [6], we can prove

Theorem 8. *Let $x \in \partial U$. \mathcal{N}_x^U has the following four types:*

- (1) $\mathcal{N}_x^U = \{\varepsilon_x\}$, i.e., x is regular for \bar{U} ,
- (2) $\mathcal{N}_x^U = \{\lambda_x\}$, i.e., x is semi-regular for \bar{U} ,
- (3) $\mathcal{N}_x^U = \{\varepsilon_x, \lambda_x\}$, i.e., x is weak-irregular,
- (4) $\mathcal{N}_x^U = \{t\varepsilon_x + (1-t)\lambda_x; 0 \leq t \leq 1\}$.

The type of \mathcal{N}_x^U is a local property, i.e., the above types are unaltered if we con-

sider $\overline{U \cap V}$ in stead of \bar{U} , where V is a neighborhood of x . Therefore when X is elliptic no boundary point $x \in \partial U$ is weak-irregular.

To prove the theorem we need some lemmas.

Lemma 9. *In a resolutive compactification, let x be a point of the harmonic boundary Γ . If f is bounded, resolutive and continuous in a neighborhood of x , then we have*

$$\underline{\lim}_x H_f \leq f(x) \leq \overline{\lim}_x H_f.$$

Proof. Suppose that $|f| \leq M$. Let V_1, V_2 be neighborhoods of x such that $V_1 \subset V_2$, and f is continuous on $V_2 \cap \Delta$. And let φ_1, φ_2 be continuous on Δ such that

$$\varphi_1 = \begin{cases} f \text{ on } V_1 \\ M \text{ on } \Delta \setminus V_2 \end{cases} \quad \text{and} \quad \varphi_2 = \begin{cases} f \text{ on } V_1 \\ -M \text{ on } \Delta \setminus V_2. \end{cases}$$

And finally let $f_1 = \max(f, \varphi_1)$ and $f_2 = \min(f, \varphi_2)$. Then

$$\underline{\lim}_x H_f \leq \underline{\lim}_x H_{f_1} \leq f_1(x) = f(x) = f_2(x) \leq \overline{\lim}_x H_{f_2} \leq \overline{\lim}_x H_f.$$

Lemma 10. *$x \in \Gamma(U)$ if and only if $x \in \Gamma(U \cap V)$ for every neighborhood V of x .*

Proof. Suppose that $x \in \Gamma(U \cap V)$ for some V . Then by [6] Cor. 19, there is a sequence $\{a_k\}$ tending to x and $\lim_k H_\varphi^{U \cap V}(a_k) = \varphi(x)$ for every resolutive and bounded function φ which is continuous at x . Since $H_f^U = H_\varphi^{U \cap V}$ for every $f \in C(\Delta U)$, where $\varphi = f$ on $\partial U \cap V$ and $\varphi = H_f^U$ on $\partial V \cap U$, we conclude that $\lambda_{a_k} \rightarrow \varepsilon_x$ and $x \in \Gamma(U)$.

Next, suppose that $x \notin \Gamma(U \cap V)$ then x is irregular for $\overline{U \cap V}$, and, by Corollary 4, x is irregular for \bar{U} . On the other hand, every bounded harmonic function on $U \cap V$ is extended continuously at x , [3]. Cor. 1.6. Therefore every bounded harmonic function on U , in particular H_f^U , is extended continuously at x , which implies that $x \notin \Gamma(U)$.

Proof of Theorem 8. Suppose that $x \in \Gamma(U)$ and x is irregular, then $\lambda_x \neq \varepsilon_x$ and $\{\varepsilon_x, \lambda_x\} \subset \mathcal{N}_x^U$. If the case (4) does not occur, then, in view of Theorem 7 (1), there exist $t \in (0, 1)$ and $f \in C(\Delta U)$ such that $t\varepsilon_x + (1-t)\lambda_x \notin \mathcal{N}_x^U$ and $f(x) \neq \lambda_x(f)$. We may assume that $f(x) > \lambda_x(f)$. Then there is a neighborhood V of x such that $u = H_f^U \neq \alpha$ on $U \cap V$, where $\alpha = tf(x) + (1-t)\lambda_x(f)$. Denoting by

$$\begin{aligned} V_1 &= \{a \in U \cap V; u(a) > \alpha\}, \\ V_2 &= \{a \in U \cap V; u(a) < \alpha\}, \end{aligned}$$

we have $V_i \neq \emptyset$ ($i=1, 2$) and $V_1 \cup V_2 = U \cap V$.

Let

$$\varphi_i = \begin{cases} f & \text{on } \partial U \cap V_i \\ u & \text{on } \partial V_i \cap U. \end{cases} \quad (i = 1, 2)$$

The functions φ_i are bounded, resolutive and $u = H_{\varphi_i}^V$ on V_i .

Since

$$\overline{\lim}_x H_{\varphi_2}^V = \overline{\lim}_{\substack{a \rightarrow x \\ a \in V_2}} H_f^U(a) \leq \alpha < f(x) = \varphi_2(x)$$

we have $x \notin \Gamma(V_2)$ by Lemma 9. Then there exists $\lim_{\substack{a \rightarrow x \\ a \in V_2}} H_{\varphi_2}^V(a)$ ([3], Cor.

1.6), which means that $\lim_{\substack{a \rightarrow x \\ a \in V_2}} \lambda_a = \lambda_x$.

The fact that $x \in \partial V_1 \cap \partial V_2$ and x is irregular for V_2 implies that $X \setminus V_2$ is thin at x and V_2 is not thin at x ; Further $X \setminus V_1$ is not thin at x , for $V_2 \subset X \setminus V_1$, which means that x is regular for V_1 . Thus there exists $\lim_{\substack{a \rightarrow x \\ a \in V_1}} H_{\varphi_1}^V(a)$

([6], Cor. 19) and we can conclude that $\lim_{\substack{a \rightarrow x \\ a \in V_1}} \lambda_a = \varepsilon_x$.

The remaining part of the theorem is proved by the above consideration and Lemma 10, since regularity is a local property.

Finally, we shall remark on the cluster set \mathcal{M}_x^U of balayaged measures ε_x^U at x .

If f is continuous on ∂U and has a compact carrier then f is extended continuously on ΔU to be 0 on $\Delta \cap \bar{U}$, then we have

$$H_f^U = H_f^{U,x}$$

and this shows

Theorem 11. *Let $x \in \partial U$. We have*

- (1) $\mathcal{M}_x^U = \{\varepsilon_x\}$ if x is regular for \bar{U} ,
- (2) $\mathcal{M}_x^U = \{\varepsilon_x^U\}$ if x is semi-regular,
- (3) $\mathcal{M}_x^U = \{\varepsilon_x, \varepsilon_x^U\}$ if x is weak-irregular,
- (4) otherwise, $\mathcal{M}_x^U = \{t\varepsilon_x + (1-t)\varepsilon_x^U; 0 \leq t \leq 1\}$.

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