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Osaka J. Math.
21 (1984), 701-706

ON A CONSTRUCTION OF EXCEPTIONAL PSEUDOSYMMETRIC SETS

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(Received January 17, 1984)

1. Introduction

If we denote the conjugation $b^{-1}ab$ by $a\circ b$ or $a^{\sigma(b)}$ in a group, it satisfies (i) $\sigma(b)$ is a permutation on the group, (ii) $a\circ a=a$, and (iii) $(a\circ b)\circ c=(a\circ c)\circ(b\circ c)$. Under the condition (i), (iii) is equivalent with a fundamental identity: $\sigma[b^{\sigma(c)}]=\sigma(c)^{-1}\sigma(b)\sigma(c)$. The present author called a binary system satisfying these conditions a pseudosymmetric set in [2] and [3]. D. Joyce called it a quandle in [1]. When every $\sigma(b)$ is of order 2, a pseudosymmetric set is called a symmetric set (or an involutive quandle in [1], or a symmetric groupoid in [4].) A pseudosymmetric set is called *special* if it is isomorphic with a pseudosymmetric subset of a group (=a pseudosymmetric set in the above sense). It is easily seen that a special pseudosymmetric set is isomorphic with a union of conjugacy classes of some group. Its structure is determined to some extent by its enveloping group ([3]). A pseudosymmetric set which is not special is called *exceptional* by analogy with the thoery of Jordan algebras. The simplest exceptional pseudosymmetric set is a set $\{a, b, c\}$ with $\sigma(a)=\sigma(b)=1$ and $\sigma(c)\neq 1$ ([4], p. 72). The structure of general pseudosymmetric sets seems to be more complicate, and our consideration will be restricted to the transitive one; a pseudosymmetric set is called transitive when the group of permutations generated by all $\sigma(b)$ is a transitive permutation group on the set. The object of this paper is to show a construction method of transitive exceptional pseudosymmetric sets. In this direction, S. Doro discussed about the existence of transitive exceptional symmetric sets in his unpublished paper. (See the remark in the last part of this paper.) For the construction, we need an extension theory of pseudosymmetric sets, which will be given in **2**. Let S be a pseudosymmetric set and A a set (a trivial pseudosymmetric set). We intend to make $A\times S$ a pseudosymmetric set by introducing a suitable composition. For $(a, s), (b, t)\in A\times S$, we consider a composition $(a, s)\circ(b, t)=(c, s\circ t)$, where generally c depends on a, b, s and t, satisfying the three conditions of a pseudosymmetric set. But in this paper we condiser a simple but important case that c depends only on a, s and t. Denote $c=a^{\pi(s,t)}$. $\pi(s, t)$ is a permutation on A. In order that $A\times S$ becomes a pseudosymmetric set by this composition, it must satisfy two conditions (1) and (2) given in **2**. In this case, we call the mapping π of $S\times S$ into $P(A)$=the group of permutations on A *a factor set*. The pseudosymmetric set $A\times S$ given in this way is called a null extension (of A by S). ("null" because c does not depend on b.) We can develop some formalism on factor sets similar to that of group extension theory. This may be useful in developing the structure theory, but for the purpose of this paper we are rather interested in a special factor set. When φ is a homomorphism of S into $P(A)$, let $\pi(s, t)=\varphi(s)^{-1}\varphi(t)$. It can be shown that this π is a factor set. Especially, if S is a pseudosymmetric subset of $P(A)$, we can take the imbedding mapping as φ. In **4**, we specialize $A=\{1, 2, 3, 4, 5, 6\}$ and S=the subset of $P(A)$ consisting of all permutations of order 2 that move every element of A. We shall show that the null extension $A\times S$ is a transitive exceptional pseudosymmetric set of order 90. To show that it is not special, we need a criterion for a transitive null extension to be special, which will be given in **3**. Let $X=A\times S$ be a null extension. For $s\in S$, let $G_s=\{\tau\in G(X)\,|\,(A, s)^\tau=(A, s)\}$, where $G(X)$ is the group generated by $\sigma(x)$ $(x\in X)$. If $\tau\in G_s$, it induces a permutation $\bar\tau$ on A from $(a, s)^\tau=(a^{\bar\tau}, s)$. Then, $F_s=\{\bar\tau\,|\,\tau\in G_s\}$ is a group of permutations on A. The criterion we are looking for is that F_s is a regular permutation group if X is special and if S is an effective pseudosymmetric set, i.e., $\sigma(s)\neq\sigma(t)$ whenever $s\neq t$. This example can be generalized. In place of 6, we take an even number $m\geq 6$, and let $A=\{1, 2, \cdots, m\}$. S is the set of permutations of order 2 on A that move every element of A. The null extension $A\times S$ constructed as above can be shown to be a transitive exceptional pseudosymmetric set. Thus, we can construct infinitely many finite transitive exceptional pseudosymmetric sets.

2. Null extensions and factor sets

Let $S=\{s, t, \cdots\}$ be a pseudosymmetric set and $A=\{a, b, \cdots\}$ a set. Let π be a mapping of $S\times S$ into $P(A)$, the group of permutations on A. When π satisfies the following two conditions (1) and (2), we say that π is a factor set (with respect to A and S).

$$(1)\qquad \pi(s, s)=1\,,$$

$$(2)\qquad \pi(s, t)\pi(s\circ t, u)=\pi(s, u)\pi(s\circ u, t\circ u)\,.$$

Let π be a factor set. We define a composition on $A\times S=\{(a, s)\,|\,a\in A$ and $s\in S\}$ by $(a, s)\circ(b, t)=(a^{\pi(s,t)}, s\circ t)$. It can be checked that $A\times S$ satisfies the three conditions of a pseudosymmetric set. We call it a null extension (of A by S) with the factor set π and denote it by $[A\times S; \pi]$. We have an example: Let φ be a homomorphism of S into $P(A)$. Define π by $\pi(s, t)=\varphi(s)^{-1}\varphi(t)$.

π is a factor set. Since φ is a homomorphism, we have $\varphi(s\circ t)=\varphi(s)\circ\varphi(t)=\varphi(t)^{-1}\varphi(s)\varphi(t)$. Using it, we can check that both terms of (2) coincide with $\varphi(s)^{-2}\varphi(t)\varphi(u)$.

We derive some formalism on factor sets. Let $[A\times S;\ \pi_1]$ and $[A\times S;\ \pi_2]$ be nullexten sions with factor sets π_1 and π_2 respectively. Denote elements of $[A\times S;\ \pi_i]$ by $(a,\ s)_i$, $i=1,\ 2$, and also let $(A,\ s)_i=\{(a,\ s)_i\,|\,a\in A\}$. When there is an isomorphism f of $[A\times S;\ \pi_1]$ onto $[A\times S;\ \pi_2]$ such that $f[(A,\ s)_1]=(A,\ s)_2$ for every s, we say that $[A\times S;\ \pi_1]$ and $[A\times S;\ \pi_2]$ are equivalent, or $\pi_1\sim\pi_2$. In this case, $f[(a,\ s)_1]=(a^{\theta(s)},\ s)_2$, where $\theta(s)\in P(A)$. From $f[(a,\ s)_1\circ(b,\ t)_1]=f[(a,\ s)_1]\circ f[(b,\ t)_1]$, we can obtain $\pi_1(s,\ t)\theta(s\circ t)=\theta(s)\pi_2(s,\ t)$, or

$$(3)\qquad \pi_2(s,\ t)=\theta(s)^{-1}\pi_1(s,\ t)\theta(s\circ t)\,.$$

Conversely, if (3) is satisfied for a mapping θ of S into $P(A)$, then $\pi_1\sim\pi_2$ as we can see easily. We can also see that "$\sim$" is an equivalent relation and hence all factor sets with respect to A and S can be classified to factor set classes. Also note that if a factor set π_1 is given, define π_2 by (3), and we can check that π_2 is a factor set. Let $\mathcal{E}(s,\ t)=1$ for any s and t. $\mathcal{E}$ is a factor set. When $\pi\sim\mathcal{E}$, we say that π is trivial or that $[A\times S;\ \pi]$ is splitting. The condition for $\pi\sim\mathcal{E}$ is given by

$$(4)\qquad \pi(s,\ t)=\theta(s)^{-1}\theta(s\circ t)$$

for some θ. Especially, for a homomorphism φ of S into $P(A)$, put $\pi(s,\ t)=\varphi(s)^{-1}\varphi(s\circ t)$. π is a trivial factor set. Since $\varphi(s\circ t)=\varphi(s)\circ\varphi(t)=\varphi(t)^{-1}\varphi(s)\varphi(t)$, we have $\pi(s,\ t)=[\varphi(s),\ \varphi(t)]$ (a commutator). It is interesting to note that if we use this π in (2), we get an identity in a group: $[x,\ y][x^y,\ z]=[x,\ z][x^z,\ y^z]$. Lastly, we define F_s for $A\times S$. Let $X=A\times S$, a null extension, and $G(X)$ the group generated by all $\sigma(x)$, $x\in X$. If $\tau\in G(X)$ and $(A,\ s)^\tau=(A,\ s)$, then τ induces a permutation $\bar{\tau}$ on A by $(a,\ s)^\tau=(a^{\bar{\tau}},\ s)$. Let F_s be the set of all such $\bar{\tau}$. It is a permutation group on A. Suppose that $\pi_1\sim\pi_2$ via (3). Let $F_s^{(i)}$ be F_s for $[A\times S;\ \pi_i]$, $i=1,\ 2$. We show that $F_s^{(2)}=\theta(s)^{-1}F_s^{(1)}\theta(s)$. Let $X_i=[A\times S;\ \pi_i]$. The isomorphism f of X_1 onto X_2 induces in a natural sense an isomorphism of $G(X_1)$ onto $G(X_2)$, which we denote by g, i.e., $f(x^\tau)=f(x)^{g(\tau)}$ for $x\in X_1$ and $\tau\in G(X_1)$. Let $\tau\in G(X_1)$ such that $(A,\ s)_1^\tau=(A,\ s)_1$. Then $(a^{\bar{\tau}},\ s)_1=(a,\ s)_1^\tau$. So, $f[(a^{\bar{\tau}},\ s)_1]=f[(a,\ s)_1]^{g(\tau)}$, and hence $(a^{\bar{\tau}\theta(s)},\ s)_2=(a^{\theta(s)},\ s)_2^{g(\sigma)}=(a^{\theta(s)\bar{\mu}},\ s)_2$, where $\bar{\mu}=\overline{g(\tau)}\in F_s^{(2)}$. Thus, $\bar{\mu}=\theta(s)^{-1}\bar{\tau}\theta(s)$, or $F_s^{(2)}=\theta(s)^{-1}F_s^{(1)}\theta(s)$ as we asserted.

3. Transitive pseudosymmetric sets

Let $X=\{x,\ y,\ \cdots\}$ be a transitive pseudosymmetric set, and let $S=\{\sigma(x)\,|\,x\in X\}=\sigma(X)$. S is a special pseudosymmetric set and σ is considered as a homomorphism of X onto S. Corresponding to the homomorphism σ, we have a coset-decomposition (or a normal decomposition in [3]) of X; $X=\cup X_i$, where X_i is the set of inverse images of an element in S by σ. From now on, x_i indicate representatives of X_i. So, $X_i=\{x\in X\,|\,\sigma(x)=\sigma(x_i)\}$. Every element of $G(X)$ induces a permutation on $\{X_i\}$, the set of cosets. That X is transitive implies that all X_i are bijective each other. We choose a set A to which all X_i are bijective. Then, X is bijective to $A\times S$, where X_i is mapped to (A, s_i) for $s_i=\sigma(x_i)\in S$. Through this bijection, we can make $A\times S$ a pseudosymmetric set isomorphic with X. Then, $A\times S$ is a null extension. The different choice of bijections in the above only leads to an equivalent null extension. Now suppose that X is special, and let K be its enveloping group (the group generated by X). We show that A is a subgroup of $C(K)$, the center of K. If $\sigma(x)=\sigma(y)$, then $x^{-1}zx=y^{-1}zy$ for every z in X hence in K. Thus, $xy^{-1}\in C(K)$. So, every element y of X_1 is expressed as $y=ux_1$ with $u\in C(K)$. Let $N=\{u\in C(K)\,|\,ux_1\in X_1\}$. $X_1=Nx_1$. We show that N is a subgroup. Let u and $v\in N$. Take τ in $G(X)$ such that $x_1^\tau=ux_1$. Then, $(vx_1)^\tau=v(x_1^\tau)$. Note that $G(X)$ is the inner automorphism group of K. So, $(vx_1)^\tau=vux_1\in X_1$, and hence $vu\in N$. In the above, $x_1^{\tau^{-1}}=u^{-1}x_1$, and hence $u^{-1}\in N$. Thus, N is a subgroup. We can also show $X_i=Nx_i$ in the same way. We can take $A=N$. Let $s=\sigma(x_1)$. We consider F_s for our $N\times S$. The effect of elements of F_s on N is determined as follows. Let $G_{X_1}=\{\tau\in G(X)\,|\,X_1^\tau=X_1\}$. If $\tau\in G_{X_1}$, then τ induces a permutation $\bar{\tau}$ on N defined from $(ux_1)^\tau=u^{\bar{\tau}}x_1$. $F_s=\{\bar{\tau}\,|\,\tau\in G_{X_1}\}$. For $\tau\in G_{X_1}$, let $x_1^\tau=wx_1$ with w in N. Then, $u^{\bar{\tau}}=uw$. Conversely, for any element w in N, choose τ in $G(X)$ such that $x_1^\tau=wx_1$. Then $\bar{\tau}\in F_s$. So, F_s is isomorphic with N and is of course a regular permutation group on N. We have obtained

Theorem. *Let X be a transitive special pseudosymmetric set. Then, X is isomorphic with a null extension $N\times S$, where N is a subgroup of $C(K)$ and $S=\sigma(X)$. In this case, F_s is isomorphic with N and is a regular permutation group on N.*

In the above, we found that every transitive pseudosymmetric set X is isomorphic with a null extension $[A\times S;\ \pi]$. However, even if X is a null extension itself, i.e., $X=[A'\times S';\ \pi']$, we can not say that S' is isomorphic with S and A' is bijective to A. But if S' is effective, i.e., $S'\overset{\sigma}{\cong}\sigma(S')$, we can conclude it. For, let $X=[A'\times S';\ \pi']$ be transitive and suppose S' is effective. Then, $\sigma(X)$ is isomorphic with S'. As a matter of fact, the mapping $\sigma[(a', s')]\to s'$ gives the isomorphism, where $a'\in A'$ and $s'\in S'$. On the other hand, $S=\sigma(X)$. So, $S'\cong S$. Also, (A', s') corresponds to (A, s) in the isomorphism $X\cong[A\times S;\ \pi]$, where s' corresponds to s in the above isomorphism. So, A' is bijective to A. We may replace S' by S and A' by A, and we have $X=[A\times S;\ \pi']$. The isomorphism $X\cong[A\times S;\ \pi]$ gives $\pi'\sim\pi$. Especially, if

X is special, we have $X=[N\times S;\pi']\cong[N\times S;\pi]$ where $\pi'\sim\pi$. Using the last discussion of the previous section, we can conclude that F_s for $[N\times S;\pi']$ is a regular permutation group on N. We obtained

Corollary. *Suppose that a null extension $[A\times S;\pi]$ is transitive and special. Suppose also that S is effective. Then, F_s is a regular permutation group on A for every s.*

4. An exceptional pseudosymmetric set

Suppose that S is a pseudosymmetric subset of $P(A)$ for a set A. Define a mapping μ of $S\times S$ into $P(A)$ by $\mu(s,t)=s^{-1}t$. We have shown in **2** that $[A\times S;\mu]$ is a null extension. Denote $(s\circ t)\circ u$ by $s\circ t\circ u$, etc. It can be shown by induction that, in $[A\times S;\mu]$, we have $(a,s)\circ(b,t_1)\circ\cdots\circ(c,t_n)=(a^{\rho},s\circ t_1\circ\cdots\circ t_n)$, where $\rho=s^{-n}t_1\cdots t_n$. Now we specialize A and S as follows. Let $A=\{1,2,3,4,5,6\}$ and S=the set of all permutations of order 2 that move every element of A. S consists of all conjugates of $s=(12)(34)(56)$ and contains 15 elements. So, the order of $A\times S$ is 90. Denote this $[A\times S;\mu]$ by X_6. We show that X_6 is transitive and exceptional. First, note that S is transitive and effective. To show that X_6 is transitive, it is enough to show that F_s is transitive. Let $t_1=(13)(24)(56)$, $t_2=(14)(23)(56)$, $t_3=(12)(35)(46)$ and $t_4=(12)(36)(45)$. Then, $(1,s)\circ(1,t_1)\circ(1,t_2)=(1^{\rho},s\circ t_1\circ t_2)=(2,s)$, where $\rho=s^{-2}t_1t_2=t_1t_2$ and $1^{\rho}=2$. We have also $(1,s)\circ(1,t_1)\circ(1,t_3)\circ(1,s)\circ(1,t_4)=(3,s)$. In this way, we can show that 1 is mapped to any elements by elements of F_s and F_s is transitive. Next, we show that X_6 is exceptional. Due to Corollary, it is enough to show that F_s is not regular. We have already shown that $(1,s)\circ(1,t_1)\circ(1,t_2)=(2,s)$. On the other hand, $(5,s)\circ(1,t_1)\circ(1,t_2)=(5,s)$ as we can check. This implies that there is an element τ in F_s such that $\tau\neq 1$ and $5^{\tau}=5$, and hence F_s is not regular. Therefore, X_6 is transitive and exceptional. It is not hard to generalize this example. In place of 6 in the above, we take an even number $m\geq 6$, and construct X_m in the similar way. We can show that X_m is transitive and exceptional.

REMARK. X_6 is actually a symmetric set. However, we do not call it an exceptional symmetric set. We did not define an exceptional symmetric set in this paper. Usually, a group is considered as a symmetric set via the composition $a\circ b=ba^{-1}b$. So, the structure of a group considered as a pseudosymmetric set is different from that considered as a symmetric set. For the latter, see [5].

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