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ON THE NONEXISTENCE OF UNKNOWN PERFECT 6- AND 8-CODES IN HAMMING SCHEMES $H(n, q)$ WITH q ARBITRARY

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0. Introduction

The first important study for the (non-)existence of perfect e-codes in the Hamming schemes $H(n, q)$ with arbitrary q was made by E. Bannai [1]. In his paper, Bannai determined the asymptotic locations of the zeros of Lloyd's polynomials as $\beta=\sqrt{(n-e)(q-1)}/q\to\infty$. (See 1.3.). Derived from this, he proved that, for each e, there exists a number $\beta_0(e)$ such that if $\beta\geqslant\beta_0(e)$, then there is no nontrivial perfect e-code in $H(n, q)$ for $q>2$. In this paper, we will use Bannai's idea and explictly calculate such numbers $\beta_0(e)$ for $e=6$ and 8. Namely, we will prove that we can take $\beta_0(6)=15$ and $\beta_0(8)=18$ under the assumption that $q\geqslant 30$. The remaining cases $\beta<\beta_0(e)$ (and $q\geqslant 30$) are also treated. Since the cases $q<30$ are already determined (see 1.2.), we then get the following theorem.

Theorem A. *There exists no nontrivial perfect e-code in Hamming schemes $H(n, q)$ for $e=6$ or 8 with q arbitrary.*

As explained in section 1.2., the nonexistence of nontrivial perfect e-codes in $H(n, q)$ for all $e\geqslant 3$ was almost completed by Best [2]. He used Bannai's idea [1] to prove this nonexistence for $e=7$ and $e\geqslant 9$. The cases $e=3$, 4, and 5 were previously solved by Reuvers [7]. Thus theorem A fills the gap (of $e=6$ and 8) and we get:

Theorem B (see 1.1.2 and 1.2). *For $e\geqslant 3$, the only perfect e-codes in $H(n, q)$ are the trivial codes* (of size 1 or 2) *and the binary Golay code* ($q=2$, $n=23$, $e=3$).

We conclude this section with the following open problem.

For $e=1$ or 2, the existence or classification of perfect e-codes still remains open. As far as the author knows, for $e=2$, only the ternary Golay code ($q=3$, $n=11$, $e=2$) is known. For $e=1$, there are many of them known [12], and the classification seems very difficult.

1. Definitions, previously known results, and important theorems

Definition 1.1.1. The Hamming schemes $H(n, q)$ are defined as follows. Let Q be a set of cardinality $q\geqslant 2$,

$V=Q^n$, the set of all n-tuples over Q, and

d_H: $V\times V\to N\cup\{0\}$ be the Hamming distance defined by $d_H(x, y)=|\{i: 1\leq i\leq n, x_i\neq y_i\}|$ for all n-tuples $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, \cdots, y_n)$ in V.

Then (V, d_H) forms a metric space and is equipped with the structure of association scheme. We call it the *Hamming scheme* and denote it by $H(n, q)$. A subset C of V is called a *perfect e-code* (in $H(n, q)$) if the set of closed e-balls $B_e(c)=\{x\in V: d_H(x, c)\leqslant e\}$, as c runs through C, forms a partition of V.

1.1.2. For any Hamming scheme $H(n, q)$, there is a trivial perfect e-code; namely, the code of size 1. Besides this kind of code, the only perfect e-codes known for $e\geqslant 2$ have the following parameters.

(i) $q=2$, $n=2e+1$, e is arbitrary (binary repetition codes, code size=2);
(ii) $q=2$, $n=23$, $e=3$ (binary Golay code); and
(iii) $q=3$, $n=11$, $e=2$ (ternary Golay code).

Codes (i), (ii), and (iii) are unique up to isomorphism. In this paper, we will call the codes of size 1 or 2 *trivial*. Thus, when $n\leqslant e$, the code is automatically trivial.

1.2. There are many papers concerning the nonexistence of perfect e-codes in $H(n, q)$. Here is a list of the major results.

No unknown perfect e-code exists when:

(i) $q=p^s$ where p is a prime and $e\geqslant 2$, Tietäväinen-van Lint (cf. [6], [9], [10]);
(ii) $q=p_1^s p_2^t$ and $e\geqslant 3$ (cf. Tietäväinen [11]);
(iii) $e=3$, 4, or 5, Reuvers [7];
(iv) $e=7$ or $e\geqslant 9$, Best [2].

Also

(v) For each $e\geqslant 3$ with $q\geqslant 3$ arbitrary, there are only finitely many non-trivial perfect e-codes, Bannai [1].

In this paper, we prove the nonexistence of perfect e-codes for $e=6$ or 8 under the assumption that $q\geqslant 30$. Thus, with the results (i), (ii), (iii), and (iv), the nonexistence of unknown perfect e-codes is settled for $e\geqslant 3$ with q arbitrary, which proves theorem B.

1.3. Important theorems.

Theorem 1.3.1 (Generalized) Lloyd's Theorem (cf. [3], [4], [5]). *Suppose there exists a perfect e-code in $H(n, q)$. Then the Lloyd's polynomial*

$$\varphi_e(x)=\sum_{i=0}^{e}(-1)^i\binom{n-x}{e-i}\binom{x-1}{i}(q-1)^{e-i}$$

has e distinct integral zeros in the interval $[1, n]$.

REMARK. In the remaining part of this paper, we will use the following expression for $\varphi_e(x)$ instead of the one in theorem 1.3.1. (cf. [8] p. 38).

$$\varphi_e(x)=\sum_{i=0}^{e}(-q)^i(q-1)^{e-i}\binom{n-1-i}{e-i}\binom{x-1}{i}.$$

Theorem 1.3.2 (E. Bannai [1], 1977, prop. 15). *Let the zeros of* $\varphi_e(x)$ *be*

$$\alpha_i=\alpha+\beta\xi_i+\lambda_i \quad \left(i=\pm 1, \pm 2, \cdots, \pm\left[\frac{e}{2}\right] \text{ and } i=0 \text{ if } e \text{ is odd}\right)$$

where $\alpha=\frac{1}{e}(\alpha_1+\cdots+\alpha_e)=\frac{(n-e)(q-1)}{q}+\frac{e+1}{2}$,

$$\beta=\frac{\sqrt{(n-e)(q-1)}}{q}, \text{ and}$$

ξ_i=*the zeros of the Hermite polynomial* $H_e(x)$ *defined by*

$$H_e(x)=(-1)^e\exp\left(\frac{x^2}{2}\right)\frac{d^e}{dx^e}\left\{\exp\left(-\frac{x^2}{2}\right)\right\}.$$

Then $\lambda_i\to\left(1-\frac{2}{q}\right)\left(\frac{e-1-\xi_i^2}{6}\right)$ as $\beta\to\infty$.

Theorem 1.3.3 (E. Bannai [1], 1977, prop. 16). *There exists a number* β_0 *(depending only on e) such that if* $\beta\geqslant\beta_0$, *then no perfect e-code exists in* $H(n, q)$ *for* $q>2$ *and* $n>e$.

Lloyd's theorem is the starting point for studying perfect codes. In the following sections, we will use Lloyd's theorem and the ideas given by E. Bannai in the proofs of theorems 1.3.2 and 1.3.3. to prove theorem A. In particular, we will prove that we can take

$$\beta_0=15 \text{ if } e=6 \text{ and}$$
$$\beta_0=18 \text{ if } e=8$$

under the assumption that $q\geqslant 30$.

2. β_0=15 when e=6 and $q\geqslant 30$.

2.1. We begin with some discussion of Hermite polynomials and their zeros. For every positive integer n, we define the Hermite polynomials $H_n(x)$ by

$$H_n(x)=(-1)^n\exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right).$$

This family of polynomials has the following recurrence relation.

$$H_{n+1}(x)=xH_n(x)-nH_{n-1}(x) \qquad (n\geqslant 1)$$
$$\text{with } H_0(x)=1 \text{ and } H_1(x)=x.$$

By using the recurrence relation, we can easily obtain

$$H_6(x)=x^6-15x^4+45x^2-15.$$

Let the zeros of $H_6(x)$ be ξ_{-3}, ξ_{-2}, ξ_{-1}, ξ_1, ξ_2, and ξ_3 in increasing order. Then, by applying the Cardano formula, we can get

$$0.3803274<\xi_1^2<0.3803276,$$
$$3.5689847<\xi_2^2<3.5689849, \text{ and}$$
$$11.050687<\xi_3^2<11.050689.$$

Notice that $\xi_i=-\xi_{-i}$ for $i=1$, 2, or 3.

2.2. Following theorem 1.3.2., we express the roots α_i $(i=\pm 1, \pm 2, \pm 3)$ of $\varphi_6(x)$ as follows.

$$\alpha_i=\alpha+\beta\xi_i+\lambda_i \quad \text{where} \quad \alpha=\frac{(n-6)(q-1)}{q}+\frac{7}{2} \quad \text{and}$$
$$\beta=\frac{\sqrt{(n-6)(q-1)}}{q}.$$

We also know that $\lambda_i\to\left(1-\frac{2}{q}\right)\left(\frac{5-\xi_i^2}{6}\right)$ as $\beta\to\infty$.

By Lloyd's theorem, we know if there exists any perfect 6-code, then

$$(\alpha_2+\alpha_{-2})-(\alpha_3+\alpha_{-3})=(\lambda_2+\lambda_{-2})-(\lambda_3+\lambda_{-3}) \text{ is an integer.}$$

We also have

$$(\lambda_2+\lambda_{-2})-(\lambda_3+\lambda_{-3})\to\left(1-\frac{2}{q}\right)\left(\frac{\xi_3^2-\xi_2^2}{3}\right) \text{ as } \beta\to\infty.$$

Calculation shows

$$2.3276406<\left(1-\frac{2}{q}\right)\left(\frac{\xi_3^2-\xi_2^2}{3}\right)<2.4939015 \text{ if we assume } q\geqslant 30.$$

Therefore if we can get β_0 such that, when $\beta\geqslant\beta_0$ and $q\geqslant 30$,

$$2<(\lambda_2+\lambda_{-2})-(\lambda_3+\lambda_{-3})<3,$$

then the nonexistence problem will be proved for $\beta\geqslant\beta_0$.

Suppose, for $i=\pm 2$ and ± 3,

$$\lambda_i \in B\left(\left(1-\frac{2}{q}\right)\left(\frac{5-\xi_i^2}{6}\right); \varepsilon\right). \text{ Here } B(a;\varepsilon) \text{ means } \{x\in \boldsymbol{R}: |x-a|<\varepsilon\}.$$

Then $(\lambda_2+\lambda_{-2})-(\lambda_3+\lambda_{-3})\in B\left(\left(1-\frac{2}{q}\right)\left(\frac{\xi_3^2-\xi_2^2}{3}\right); 4\varepsilon\right)$. It is easy to see that if we choose $\varepsilon=0.08191$, then

$$2<(\lambda_2+\lambda_{-2})-(\lambda_3+\lambda_{-3})<3\,.$$

Lemma 2.3.1. *If $\beta\geqslant 15$ and $q\geqslant 30$, then $\lambda_i\in B\left(\left(1-\frac{2}{q}\right)\left(\frac{5-\xi_i^2}{6}\right); 0.08191\right)$ for $i=\pm 2$ and ± 3.*

Corollary 2.3.2. *There is no perfect 6-code in $H(n, q)$ if $\beta\geqslant 15$ and $q\geqslant 30$.*

Proof of Lemma 2.3.1. It is enough to show that $\varphi_6(x)/q^6$ changes its sign at $x=\alpha+\beta\xi_i+\left(1-\frac{2}{q}\right)\left(\frac{5-\xi_i^2}{6}\right)\pm 0.08191$ for $i=\pm 2$ ± 3. To begin with, we rewrite $\varphi_6(x)$ by the substitutions

$$x=(\alpha+\beta\xi+\lambda=)\beta^2 q+\frac{7}{2}+\beta\xi+\lambda \quad\text{and}\quad n-6=\frac{\beta^2q^2}{q-1}\,.$$

Thus $\varphi_6(x)$ can be written in terms of q, β, ξ, and λ. Let $\frac{\varphi_6(x)}{q^6}=\sum_{k=0}^{\infty} A_k\beta^k$ where A_k's are expressions in q, ξ, and λ. Then, by straightforward calculation, we have $A_k=0$ for $k\geqslant 7$. If we further substitute λ by $\left(1-\frac{2}{q}\right)\left(\frac{5-\xi^2}{6}\right)\pm\varepsilon$ in the expressions A_5 and A_4, then we get

$$A_6=\frac{1}{720}H_6(\xi)\,,$$

$$A_5=\left(1-\frac{2}{q}\right)\left(\frac{-\xi}{720}\right)H_6(\xi)\pm\varepsilon\left(\frac{1}{8}\xi-\frac{1}{12}\xi^3+\frac{1}{120}\xi^5\right),$$

$$\begin{aligned}A_4=&\left[\frac{1}{48}(3-6\xi^2+\xi^4)\left(\frac{5-\xi^2}{6}\right)+\frac{1}{12}(\xi^2-1)\left(\frac{5-\xi^2}{6}\right)^2-\frac{1}{576}(18\xi^2-26)\right]\\&\left(1-\frac{2}{q}\right)^2\pm\varepsilon\left[\frac{1}{24}(3-6\xi^2+\xi^4)\left(\frac{5-\xi^2}{6}\right)+\frac{1}{12}(\xi^2-1)\right]\left(1-\frac{2}{q}\right)\\&+\frac{1}{48}\varepsilon^2(3-6\xi^2+\xi^4)-\frac{1}{576}(7\xi^4-48\xi^2+27)\,,\end{aligned}$$

$$\begin{aligned}A_3=&\frac{1}{144}\xi\lambda\left[-12\lambda^2+4\xi^2\lambda^2+12\left(1-\frac{2}{q}\right)\lambda-7\xi^2+3\left(5+\frac{12}{q}-\frac{12}{q^2}\right)\right]\\&-\frac{1}{720}\left(29+\frac{14}{q}-\frac{216}{q^2}+\frac{144}{q^3}\right)\xi\,,\end{aligned}$$

$$A_2=\frac{1}{48}\lambda^4(\xi^2-1)+\frac{1}{36}\lambda^3\left(1-\frac{2}{q}\right)+\frac{1}{96}\lambda^2\left(5+\frac{12}{q}-\frac{12}{q^2}-7\xi^2\right)$$
$$-\frac{1}{720}\lambda\left(29+\frac{14}{q}-\frac{216}{q^2}+\frac{144}{q^3}\right)+\frac{259}{11520}\xi^2-\frac{1}{768}\left(9+\frac{24}{q}+\frac{104}{q^2}\right.$$
$$\left.-\frac{256}{q^3}+\frac{128}{q^4}\right),$$
$$A_1=\frac{1}{5760}\xi\lambda(48\lambda^4-280\lambda^2+259),$$
$$A_0=\frac{1}{11520}\lambda^2(16\lambda^4-140\lambda^2+259)-\frac{5}{1024}.$$

By using

$$3.5689847<\xi_2^2<3.5689849,\quad 11.050687<\xi_3^2<11.050689,$$
$$q\geqslant 30,\quad \varepsilon=0.08191,\quad \text{and}$$
$$|\lambda|<\begin{cases}0.3204127 & \text{if } \lambda=\left(1-\frac{2}{q}\right)\left(\frac{5-\xi_2^2}{6}\right)\pm\varepsilon,\\ 1.090357 & \text{if } \lambda=\left(1-\frac{2}{q}\right)\left(\frac{5-\xi_3^2}{6}\right)\pm\varepsilon,\end{cases}$$

we get

$$A_6=0 \quad \text{if } \xi=\xi_{\pm2} \text{ or } \xi_{\pm3};$$
$$A_5=\pm\varepsilon\left(\frac{1}{8}\xi-\frac{1}{12}\xi^3+\frac{1}{120}\xi^5\right) \quad \text{if } \xi=\xi_{\pm2} \text{ or } \xi_{\pm3};$$
$$|A_5|>\begin{cases}0.0102545 & \text{if } \xi=\xi_{\pm2},\\ 0.0638226 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_4|<\begin{cases}0.08778691 & \text{if } \xi=\xi_{\pm2},\\ 0.67601801 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_3|<\begin{cases}0.267625 & \text{if } \xi=\xi_{\pm2},\\ 3.784096 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_2|<\begin{cases}0.1045 & \text{if } \xi=\xi_{\pm2},\\ 1.5095 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_1|<\begin{cases}0.026 & \text{if } \xi=\xi_{\pm2},\\ 0.416 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_0|<\begin{cases}0.008 & \text{if } \xi=\xi_{\pm2},\\ 0.049 & \text{if } \xi=\xi_{\pm3}.\end{cases}$$

When $\beta\geqslant 15$, we get

$$|A_5| > |A_4|/\beta + |A_3|/\beta^2 + |A_2|/\beta^3 + |A_1|/\beta^4 + |A_0|/\beta^5 .$$

Therefore $\frac{\varphi_6(x)}{q^6} = A_6\beta^6 + A_5\beta^5 + \cdots + A_0$ (and hence $\varphi_6(x)$) changes its sign at

$$x = \alpha + \beta\xi_i + \left(1 - \frac{2}{q}\right)\left(\frac{5 - \xi_i^2}{6}\right) \pm 0.08191 \quad \text{for } i = \pm 2, \pm 3 .$$

And lemma 2.3.1. is proved.

3. The cases $\beta < 15$, $e = 6$, and $q \geqslant 30$

Lemma 3.1. *Suppose there exist perfect e-codes in $H(n, q)$. Then* $(1/q)^{e-k}\binom{e}{k}(n-e)\cdots(n-1-k) \in \boldsymbol{Z}$ *for* $0 \leqslant k \leqslant e-1$.

Proof. Let the Lloyd's ploynomial $\varphi_e(x) = a_e x^e + \cdots + a_0$.

Then $a_e = (-q)^e/e!$ and

$$\varphi_e(x)/a_e = \sum_{k=0}^{e} b_k \frac{x(x-1)\cdots(x-k)}{x} \quad \text{where } b_e = 1 \text{ and}$$

$$b_k = (q-1/-q)^{e-k}\binom{e}{k}(n-e)\cdots(n-1-k)$$
$$\text{for } 0 \leqslant k \leqslant e-1 .$$

By Lloyd's theorem, if there exist perfect e-codes, then $\varphi_e(x)/a_e \in \boldsymbol{Z}[x]$. This implies

$$b_k = (q-1/q)^{e-k}\binom{e}{k}(n-e)\cdots(n-1-k) \in \boldsymbol{Z} \quad \text{for } 0 \leqslant k \leqslant e-1 .$$

Since $q-1$ and q are relatively prime, we get

$$(1/q)^{e-k}\binom{e}{k}(n-e)\cdots(n-1-k) \in \boldsymbol{Z} \quad \text{for } 0 \leqslant k \leqslant e-1 .$$

3.2. Suppose there exist perfect 6-codes, then by lemma 3.1.1., we have

(1) $6(n-6)/q \in \boldsymbol{Z}$,
(2) $15(n-6)(n-5)/q^2 \in \boldsymbol{Z}$,
(3) $20(n-6)(n-5)(n-4)/q^3 \in \boldsymbol{Z}$,
(4) $15(n-6)(n-5)(n-4)(n-3)/q^4 \in \boldsymbol{Z}$,
(5) $6(n-6)(n-5)(n-4)(n-3)(n-2)/q^5 \in \boldsymbol{Z}$, and
(6) $(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)/q^6 \in \boldsymbol{Z}$.

Assume that $\beta < 15$, $q \geqslant 30$, and $n \geqslant 7$. Then we have

Lemma 3.2.1. *If there exist perfect 6-codes in $H(n, q)$, then $q = 2^s 3^t 5$ for some positive integers s and t.*

Proof. First, we show that, for any prime number $p \geqslant 7$, $p \not| q$. Suppose there is a prime number $p \geqslant 7$ such that $p^k | q$ and $p^{k+1} \not| q$ for some $k \geqslant 1$. Then, by (1) $6(n-6)/q \in \boldsymbol{Z}$, we have $p^k | (n-6)$. By (6), $(n-6)\cdots(n-1)/q^6 \in \boldsymbol{Z}$, and the assumption that $p \geqslant 7$, we get $p^{6k} | (n-6)$. This implies

$$6(n-6)/q = p^{5k}h \geqslant 7^5 > 1400 \quad \text{for some positive integer } h.$$

But, $\beta = \sqrt{(n-6)\,(q-1)}/q < 15$ and $q \geqslant 30$ imply

$$(n-6)/q < 225(q/q-1) \leqslant 225(30/29) < 233 \text{ and hence } 6(n-6)/q < 1400.$$

Contradiction.

Second, we show that $5^k \not| q$ for $k \geqslant 2$.

Suppose $5^k | q$ and $5^{k+1} \not| q$ for some $k \geqslant 2$. Then, by (1), $5^k | (n-6)$ and, by (5), $5^{5k} | (n-6)$. This implies

$$6(n-6)/q = 5^{4k}h \geqslant 5^8 > 1400 \quad \text{for some positive integer } h. \text{ Again, a contradiction.}$$

Therefore, if there exist perfect 6-codes, then

$$q = 2^s 3^t 5 \quad \text{for some positive integers } s \text{ and } t. \quad \text{(cf. section 1.2.)}$$

Lemma 3.2.2 (See [13] also.). *q cannot be 2^s3^t5 if there exist perfect 6-codes.*

Proof. Suppose $q = 2^s3^t5$ for some positive integers s and t. Then $6(n-6)/q \in \boldsymbol{Z}$ implies

$$n-6 = 2^{s-1}3^{t-1}5h \quad \text{for some positive integer } h.$$

From (5), we get $5^5 | (n-6)$. Hence

$$n-6 = 2^{s-1}3^{t-1}5^5k \quad \text{for some positive integer } k.$$

Since $6(n-6)/q = 5^4k < 1400$ (see the proof in the previous lemma), we get $k=1$ or 2. Thus

$$(n-6) = 2^{s-1}3^{t-1}5^5 \text{ or } 2^s3^{t-1}5^5.$$

Case 1. Let $(n-6) = 2^{s-1}3^{t-1}5^5$ and $t \geqslant 2$. Then expression (3) becomes

$$\frac{20(n-6)(n-5)(n-4)}{q^3} = \frac{3^{t-1}k}{3^{3t}h} \notin \boldsymbol{Z}$$ where k and h are positive integers not divisible by 3.

Contradiction.

Case 2. Let $n-6 = 2^s3^{t-1}5^5$ with $s \geqslant 1$ or $2^{s-1}3^{t-1}5^5$ with $s \geqslant 2$.

Then expression (2) becomes

$$\frac{15(n-6)(n-5)}{q^2} = \frac{(2^s \text{ or } 2^{s-1})k}{2^{2s}h} \notin \boldsymbol{Z}$$ for some odd integers k and h. Contradiction again.

Case 3. When $(n-6) = 2^0 3^0 5^5$ and $q = 2^1 3^1 5$, we can easily see expression (2) becomes

$$\frac{15(n-6)(n-5)}{q^2} = 162812.5 \notin \boldsymbol{Z}$$

Thus the lemma is proved.

Proposition 3.3. *There is no nontrivial perfect* 6-*code in* $H(n, q)$.

Proof. It is a direct consequence of 1.2 (i) and (ii), corollary 2.3.2., lemma 3.2.1., and lemma 3.2.2.

4. The nonexistence of perfect 8-codes

The discussion for the case $e=8$ is basically similar and parallel to that for $e=6$.

4.1. Again we start with the Hermite polynomial

$$H_8(x) = x^8-28x^6+210x^4-420x^2+105\,.$$

By applying the Cardano formula, the zeros ξ_i ($i=\pm1, \pm2, \pm3, \pm4$) of $H_8(x)$ are located as follows.

$$0.2906070<\xi_1^2<0.2906071\,,$$
$$2.6781945<\xi_2^2<2.6781946\,,$$
$$7.8539270<\xi_3^2<7.8539271\,,$$
$$17.177271<\xi_4^2<17.177272\,, \quad \text{and}$$
$$\xi_{-i} = -\xi_i \quad \text{for} \quad i = 1, 2, 3, 4\,.$$

4.2. The zeros of $\varphi_8(x)$ can be expressed as

$\alpha_i = \alpha+\beta\xi_i+\lambda_i$ ($i = \pm1, \pm2, \pm3, \pm4$) where

$$\alpha = \frac{(n-8)(q-1)}{q}+\frac{9}{2} = \beta^2 q+\frac{9}{2}\,,$$
$$\beta = \frac{\sqrt{(n-8)(q-1)}}{q}\,, \quad \text{and}$$
$$\lambda_i \to \left(1-\frac{2}{q}\right)\left(\frac{7-\xi_i^2}{6}\right) \quad \text{as } \beta \to \infty\,.$$

If there exist perfect 8-codes, then

$$(\alpha_1+\alpha_{-1})-(\alpha_3+\alpha_{-3}) = (\lambda_1+\lambda_{-1})-(\lambda_3+\lambda_{-3}) \in \boldsymbol{Z} \quad \text{where}$$
$$(\lambda_1+\lambda_{-1})-(\lambda_3+\lambda_{-3}) \to \left(1-\frac{2}{q}\right)\left(\frac{\xi_3^2-\xi_1^2}{3}\right) \quad \text{as } \beta \to \infty\,, \quad \text{and}$$
$$2.3530329<\left(1-\frac{2}{q}\right)\left(\frac{\xi_3^2-\xi_1^2}{3}\right)<2.5211067\,, \quad \text{assuming that } q \geqslant 30.$$

Suppose $\lambda_i \in B\left(\left(1-\frac{2}{q}\right)\left(\frac{7-\xi_i^2}{6}\right);\ \varepsilon\right)$ for $i=\pm 1, \pm 3$.

Then $(\lambda_1+\lambda_{-1})-(\lambda_3+\lambda_{-3}) \in B\left(\left(1-\frac{2}{q}\right)\left(\frac{\xi_3^2-\xi_1^2}{3}\right);\ 4\varepsilon\right)$.

If we choose $\varepsilon=0.088258$, then $2<(\lambda_1+\lambda_{-1})-(\lambda_3+\lambda_{-3})<3$. Now, by fixing $\varepsilon=0.088258$, we are ready for obtaining β_0.

Lemma 4.3.1. *If $\beta \geqslant 18$ and $q \geqslant 30$, then $\lambda_i \in B\left(\left(1-\frac{2}{q}\right)\left(\frac{7-\xi_i^2}{6}\right);\ 0.088258\right)$ for $i=\pm 1$ and ± 3.*

Corollary 4.3.2. *There is no perfect 8-code in $H(n, q)$ if $\beta \geqslant 18$ and $q \geqslant 30$.*

Proof of lemma 4.3.1. We will show that $\varphi_8(x)$ changes its sign at

$$x=\alpha+\beta\xi_i+\left(1-\frac{2}{q}\right)\left(\frac{7-\xi_i^2}{6}\right)\pm 0.088258 \quad \text{for } i=\pm 1 \text{ and } \pm 3\,.$$

First, we rewrite $\varphi_8(x)$ by the substitutions

$$x=(\alpha+\beta\xi+\lambda=)\beta^2 q+\frac{9}{2}+\beta\xi+\lambda \quad \text{and} \quad n-8=\frac{\beta^2 q^2}{q-1}\,.$$

Then $\varphi_8(x)$ is rewritten in terms of q, β, ξ, and λ. Let $\varphi_8(x)8!/q^8=\sum_{k=0}^{\infty} A_k\beta^k$ where A_k's are expressions in q, ξ, and λ. Also substitute λ by $\left(1-\frac{2}{q}\right)\left(\frac{7-\xi^2}{6}\right)\pm\varepsilon$ in the expressions A_7 and A_6.

Then, by straightforward calculation, we get

$$\begin{aligned}
A_k &= 0 \quad \text{for } k\geqslant 9\,,\\
A_8 &= H_8(\xi)\,,\\
A_7 &= (1-2/q)\,(-4/3)\xi H_8(\xi)\pm\varepsilon\,[8\xi(\xi^6-21\xi^4+105\xi^2-105)]\,,\\
A_6 &= (1-2/q)^2\left[\left(\frac{7-\xi^2}{6}\right)^2(28\xi^6-420\xi^4+1260\xi^2-420)\right.\\
&\quad \left.+\left(\frac{7-\xi^2}{6}\right)(280\xi^4-1680\xi^2+840)-105\xi^4+910\xi^2-595\right]\\
&\quad \pm(1-2/q)\varepsilon\left[\left(\frac{7-\xi^2}{6}\right)(56\xi^6-840\xi^4+2520\xi^2-840)\right.\\
&\quad \left.+280\xi^4-1680\xi^2+840\right]+\varepsilon^2[28\xi^6-420\xi^4+1260\xi^2-420]\\
&\quad -21\xi^6+350\xi^4-1155\xi^2+420\,,\\
A_5 &= (840-560\xi^2+56\xi^4)\xi\lambda^3+(1-2/q)(-1680+56\xi^2)\xi\lambda^2\\
&\quad -70(7+104/q-104/q^2)\xi\lambda+140(7+12/q-12/q^2)\xi^3\lambda\\
&\quad -126\xi^5\lambda+28(1-2/q)(29+132/q-132/q^2)\xi\\
&\quad -28(1-2/q)(13+24/q-24/q^2)\xi^3\,,
\end{aligned}$$

$$A_4 = 70\lambda^4(3-6\xi^2+\xi^4)+(1-2/q)\lambda^3(-560+560\xi^2)$$
$$+35\lambda^2[-(7+104/q-104/q^2)+(42+72/q-72/q^2)\xi^2-9\xi^4]$$
$$+28(1-2/q)\lambda[(29+132/q-132/q^2)-(39+72/q-72/q^2)\xi^2]$$
$$+[-35.875+(1-1/q)(1/q)(574+7308/q-7308/q^2)]$$
$$-7[64.75+(1-1/q)(1/q)(150+480/q-480/q^2)]\xi^2+123.375\xi^4,$$
$$A_3 = 56\xi\lambda^5(-3+\xi^2)+280\xi\lambda^4(1-2/q)+140\xi\lambda^3(7+12/q-12/q^2-3\xi^2)$$
$$-84\xi\lambda^2(1-2/q)(13+24/q-24/q^2)-14\xi\lambda[64.75$$
$$+(1-1/q)(1/q)(150+480/q-480/q^2)]+493.5\xi^3\lambda+(375.5$$
$$+281/q-216/q^2-9456/q^3+14400/q^4-5760/q^5)\xi,$$
$$A_2 = 28\lambda^6(-1+\xi^2)+56\lambda^5(1-2/q)+35\lambda^4(7+12/q-12/q^2-9\xi^2)$$
$$-28\lambda^3(1-2/q)(13+24/q-24/q^2)-7\lambda^2[64.75+(1-1/q)(1/q)(150$$
$$+480/q-480/q^2)]+740.25\xi^2\lambda^2+\lambda(375.5+281/q-216/q^2$$
$$-9456/q^3+14400/q^4-5760/q^5)+98.4375+236.25/q$$
$$+840(1/q^2)(1+6/q-6/q^2)-201.8125\xi^2,$$
$$A_1 = 8\xi\lambda^7-126\xi\lambda^5+493.5\xi\lambda^3-403.625\xi\lambda, \quad \text{and}$$
$$A_0 = \lambda^8-21\lambda^6+123.375\lambda^4-201.8125\lambda^2+43.06640625.$$

Now, by using $0.2906070<\xi_1^2<0.2906071$, $7.8539270<\xi_3^2<7.8539271$,

$$q\geqslant 30, \quad \varepsilon=0.088258, \quad \text{and} \quad |\lambda_i|<\begin{cases}1.2065 & \text{if } i=1,\\ 0.23058 & \text{if } i=3,\end{cases}$$

we get

$$A_8 = 0 \quad \text{if } \xi=\xi_{\pm1} \text{ or } \xi_{\pm3};$$
$$A_7 = \pm\varepsilon[8\xi(\xi^6-21\xi^4+105\xi^2-105)] \quad \text{if } \xi=\xi_{\pm1} \text{ or } \xi_{\pm3};$$
$$|A_7|>\begin{cases}29.0211 & \text{if } \xi=\xi_{\pm1},\\ 180.544 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_6|<\begin{cases}100.80861 & \text{if } \xi=\xi_{\pm1},\\ 2619.7319 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_5|<\begin{cases}2578 & \text{if } \xi=\xi_{\pm1},\\ 7291 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_4|<\begin{cases}1794 & \text{if } \xi=\xi_{\pm1},\\ 6452 & \text{if } \xi=\xi_{\pm3};\end{cases}$$
$$|A_3|< 3300 \quad \text{if } \xi=\xi_{\pm1} \text{ or } \xi_{\pm3};$$
$$|A_2|< 3000 \quad \text{if } \xi=\xi_{\pm1} \text{ or } \xi_{\pm3};$$
$$|A_1|< 1000 \quad \text{if } \xi=\xi_{\pm1} \text{ or } \xi_{\pm3};$$
$$|A_0|< 700 \quad \text{if } \xi=\xi_{\pm1} \text{ or } \xi_{\pm3}.$$

When $\beta \geqslant 18$, $|A_7|>|A_6|/\beta+|A_5|/\beta^2+\cdots+|A_0|/\beta^7$. Therefore $\varphi_8(x)8!/q^8=A_8\beta^8+A_7\beta^7+\cdots+A_0$, and hence $\varphi_8(x)$, changes its sign at

$$x=\alpha+\beta\xi_i+(1-2/q)\left(\frac{7-\xi_i^2}{6}\right)\pm 0.088258 \quad \text{for } i=\pm 1 \text{ and } \pm 3 .$$

This completes the proof of lemma 4.3.1.

4.4. The remaining cases $\beta<18$, $q\geqslant 30$, and $e=8$.

By lemma 3.1.1., we have the following.

If there exist perfect 8-codes in $H(n, q)$, then

(1) $8(n-8)/q \in \boldsymbol{Z}$,

(2) $28(n-8)(n-7)/q^2 \in \boldsymbol{Z}$,

(3) $56(n-8)(n-7)(n-6)/q^3 \in \boldsymbol{Z}$,

(4) $70(n-8)(n-7)(n-6)(n-5)/q^4 \in \boldsymbol{Z}$,

(5) $56(n-8)(n-7)(n-6)(n-5)(n-4)/q^5 \in \boldsymbol{Z}$,

(6) $28(n-8)(n-7)(n-6)(n-5)(n-4)(n-3)/q^6 \in \boldsymbol{Z}$,

(7) $8(n-8)(n-7)(n-6)(n-5)(n-4)(n-3)(n-2)/q^7 \in \boldsymbol{Z}$,

(8) $(n-8)(n-7)(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)/q^8 \in \boldsymbol{Z}$.

Assume $\beta<18$, $q\geqslant 30$, and $n\geqslant 9$.

Then $\beta=\dfrac{\sqrt{(n-8)(q-1)}}{q}<18$ and $q\geqslant 30$ imply that

$$(n-8)/q<324(30/29)<336 \quad \text{and} \quad 8(n-8)/q<2688 .$$

Lemma 4.4.1. *Suppose there exist perfect* 8-*codes in* $H(n, q)$. *Then* $q=2^s3^t5$ *for some positive integers* s *and* t.

Proof. First, let p be a prime number $\geqslant 7$ such that $p^k|q$ and $p^{k+1}\nmid q$ for some $k\geqslant 1$. Then, by (1), $p^k|(n-8)$ and, by (7), $p^{7k}|(n-8)$. So we get

$$8(n-8)/q=p^{6k}h\geqslant 7^6>2688 \quad \text{for some positive integer } h.$$

This contradicts $8(n-8)/q<2688$. Therefore, if p is a prime dividing q, p is 5, 3, or 2.

Second, we want to show that $5^2\nmid q$.

Suppose $5^k|q$ and $5^{k+1}\nmid q$ for some $k\geqslant 2$. Then by (1) and (5), we have $5^{5k}|(n-8)$. But then we will get

$$8(n-8)/q=5^{4k}h\geqslant 5^8>2688 \quad \text{for some positive integer } h.$$

Again, a contradiction.

Therefore, without loss of generality, we can say

$q=2^s3^t5$ for some positive integers s and t, if there is any perfect 8-code.

Lemma 4.4.2 (See [13].). *q cannot be 2^s3^t5 if there exist perfect* 8*-code in* $H(n, q)$.

Proof. Assume that $q=2^s3^t5$ for some positive integers s and t. Then by (1) and (5), $5^5|(n-8)$. Therefore

$$8(n-8)/q = 5^4k \quad \text{for some positive integer } k.$$

Since $8(n-8)/q<2688$, $k=1$, 2, 3, or 4 and

$$n-8 = 2^{s-3}3^t5^5\,, \quad 2^{s-2}3^t5^5\,, \quad 2^{s-3}3^{t+1}5^5\,, \quad \text{or} \quad 2^{s-1}3^t5^5\,.$$

But this is impossible if we compare the powers of 3 in the denominator and in the numerator of expression (3).

Proposition 4.5. *There is no nontrivial perfect* 8*-code in* $H(n, q)$.

Proof. It is an immediate consequence of 1.2. (i) and (ii), corollary 4.3.2., lemma 4.4.1., and lemma 4.4.2.

Thus theorem A is proved.

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