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ON A QUASI EVERYWHERE EXISTENCE OF THE LOCAL TIME OF THE 1-DIMENSIONAL BROWNIAN MOTION

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1. Introduction

Recently quasi everywhere properties of the Brownian motion were discussed by many authors; Williams considered the quadratic variation (see [9]) and Fukushima [3] considered the nowhere differentiability, Lévy's Hölder continuity, the law of iterated logarithm etc. By the way, the *local time* plays an important role in stochastic analysis. The existence of the local time of the 1-dimensional Brownian motion was proved by Trotter [10]. He proved that the local time of the 1-dimensional Brownian motion exists almost everywhere (a.e.) with respect to the Wiener measure. In this paper we shall prove that it exists *quasi everywhere* (q.e.) with respect to the *Ornstein-Uhlenbeck process* on the Wiener space.

Fukushima's study is based on a concept of *capacity* related to the Ornstein-Uhlenbeck process. The term "quasi everywhere" means "except on a set of capacity 0". A set of capacity 0 is characterized by the Ornstein-Uhlenbeck process as follows (see [2], [6]). Let W_0^1 be a set of all continuous paths w: $[0, \infty)\to\boldsymbol{R}$ vanishing at 0 with the compact uniform topology and μ be the Wiener measure on W_0^1. Let $(X_\tau)_{\tau\geqq 0}$ be a W_0^1-valued Ornstein-Uhlenbeck process with the initial distribution μ defined on an auxiliary probability space $(\Omega, \mathcal{F}, P)$. Then for any $A\subset W_0^1$, A is of capacity 0 if and only if

$$P[X_\tau\notin A \qquad \text{for all } \tau>0]=1\,. \tag{1.1}$$

On the other hand, by the Tanaka formula the local time $(\phi(\tau, t, a))$ of a Brownian motion $(X_\tau(t))_{t\geqq 0}$ is given by

$$\phi(\tau, t, a)=(X_\tau(t)-a)^+-(X_\tau(0)-a)^+-\int_0^t 1_{(a,\infty)}(X_\tau(s))X_\tau(ds)$$

(cf. [4], [8]). Then our main theorem is stated as follows.

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Theorem. *There exists a continuous version of* $(\phi(\tau, t, a))$ *in* (τ, t, a) *with respect to the measure P.*

By the above theorem, we can show the quasi everywhere existence of the local time. To see this, denote a continuous version also by $\phi(\tau, t, a)$. Then, for fixed $\tau>0$,

$$\int_0^t f(X_\tau(s))ds = 2\int_{\boldsymbol{R}} \phi(\tau, t, a)f(a)da\,, \qquad \forall t\geqq 0,\ \forall f\in C_0^\infty(\boldsymbol{R}) \tag{1.2}$$

P-a.e. (the almost everywhere existence of the local time). By the continuity of $(\tau, t)\mapsto X_\tau(t)$ and $(\tau, t, a)\mapsto\phi(\tau, t, a)$, (1.2) holds for all $\tau>0$ P-.a.e. which asserts (1.1). Hence the local time exists quasi everywhere.

We will give a proof of the theorem in Section 2.

2. Proof of the theorem

First we give a realization of the Ornstein-Uhlenbeck process $(X_\tau)_{\tau\geqq 0}$ on the probability space $(\Omega, \mathcal{F}, P)$ as follows. Let $(X_0(t))_{t\geqq 0}$ be a 1-dimensional Brownian motion and $(W(\tau, t))_{\tau,t\geqq 0}$ be a two parameter Brownian motion on $(\Omega, \mathcal{F}, P)$, i.e., $(W(\tau, t))_{\tau,t\geqq 0}$ is a Gaussian process with mean 0 and the covariance given by

$$E[W(\tau, t)W(\sigma, s)] = (\tau\wedge\sigma)(t\wedge s)$$

where E denotes the expectation relative to P. Assume moreover that $(X_0(t))_{t\geqq 0}$ and $(W(\tau, t))_{\tau,t\geqq 0}$ are independent. Then the Ornstein-Uhlenbeck process (X_τ) is given by

$$X_\tau(t) = e^{-\tau/2}X_0(t)+\int_0^\tau e^{-(\tau-\sigma)/2}\,W(d\sigma, t) \tag{2.1}$$

where the integral is the stochastic integral with respect to a martingale $\tau\mapsto W(\tau, t)$. Set $\mathcal{F}_t=\sigma\{X_0(s), W(\tau, s)\,|\,s\leqq t, \tau\geqq 0\}$. Then for fixed $\tau\geqq 0$, $(X_\tau(t))_{t\geqq 0}$ is an $(\mathcal{F}_t)$-Brownian motion and for fixed τ, $\sigma\geqq 0$, the quadratic variation of $(X_\tau(t))_{t\geqq 0}$ and $(X_\sigma(t))_{t\geqq 0}$ is given by

$$\langle X_\tau, X_\sigma\rangle_t = \exp\left(-\frac{1}{2}|\tau-\sigma|\right)t\,. \tag{2.2}$$

Hereafter we consider this specific Ornstein-Uhlenbeck process.

Define $\psi(\tau, t, a)$ by

$$\psi(\tau, t, a) = \int_0^t 1_{(a,\infty)}(X_\tau(s))X_\tau(ds)\,.$$

To prove the theorem it suffices to show that there exists a continuous version of $\psi(\tau, t, a)$, i.e., there exists a continuous process $(\hat{\psi}(\tau, t, a))$ in (τ, t, a) P-a.e. such that for fixed $(\tau, t, a)\in[0, \infty)\times[0, \infty)\times \boldsymbol{R}$, $\psi(\tau, t, a)=\hat{\psi}(\tau, t, a)$ P-a.e. Since for fixed $\tau\geqq 0$, $a\in\boldsymbol{R}$, $t\mapsto\psi(\tau, t, a)$ is a continuous process, we regard $\psi(\tau, \cdot, a)$ as a $C([0, \infty)\to\boldsymbol{R})$-valued random variable. Hence if we show the following proposition, we can get a desired result by Kolmogorov's theorem ([1]).

Proposition. *For any $T>0$, there exists a constant $K=K(T)>0$ such that for any $a, b\in\boldsymbol{R}$ and $0\leqq\tau, \sigma\leqq T$,*

$$E[\sup_{0\leqq t\leqq T}|\psi(\tau, t, a)-\psi(\sigma, t, b)|^{10}]\leqq K(|\tau-\sigma|^{9/4}+|a-b|^5)\,.$$

For the proof, we need the following lemmas.

Lemma 1. *Let $(B(t))$ be a 1-dimensional Brownian motion. Then for any $p\geqq 1$ and any $T>0$, there exists a constant $C=C(p, T)>0$ such that for any $a, b\in\boldsymbol{R}$, $a\leqq b$,*

$$E[\{\int_0^T 1_{(a,b]}(B(t))dt\}^p]\leqq C(b-a)^p.$$

The proof is easy and will be omitted.

Lemma 2.*) *Let $(B_1(t), B_2(t))$ be a 2-dimensional Brownian motion starting at 0, m be a positive constant and C_a, $a\in\boldsymbol{R}$, be a subset of $\boldsymbol{R}^2$ defined by*

$$C_a=\{(x, y)\in\boldsymbol{R}^2|(y+mx+a)(y-mx+a)\leqq 0\}\,.$$

Then it holds that for $T>0$, $p\geqq 1$,

$$E[\{\int_0^T 1_{C_a}(B_1(t), B_2(t))dt\}^p]\leqq E[\{\int_0^T 1_{C_0}(B_1(t), B_2(t))dt\}^p]\,.$$

Proof. Define a stopping time η by

$$\eta=\inf\left\{t\geqq 0\,|\,B_2(t)=\frac{1}{2}a\right\}$$

and a continuous process $(\tilde{B}_2(t))$ by

$$\tilde{B}_2(t)=\begin{cases} B_2(t) & \text{if } t<\eta \\ a-B_2(t) & \text{if } t\geqq\eta. \end{cases}$$

Then by the reflection principle of the Brownian motion, $(B_1(t), \tilde{B}_2(t))$ is also a 2-dimensional Brownian motion starting at 0. Moreover it holds that

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*) The proof is due to H. Kaneko, who simplified the author's original one.

$$1_{C_a}(B_1(t), B_2(t)) \leqq 1_{C_0}(B_1(t), \tilde{B}_2(t)).$$

Hence we can easily get a desired result. $\square$

Proof of the Proposition. Set

$$I = E[\sup_{0 \leqq t \leqq T} |\psi(\tau, t, a) - \psi(\tau, t, b)|^{10}] \quad \text{and}$$
$$J = E[\sup_{0 \leqq t \leqq T} |\psi(\tau, t, b) - \psi(\sigma, t, b)|^{10}].$$

Then we easily have

$$E[\sup_{0 \leqq t \leqq T} |\psi(\tau, t, a) - \psi(\sigma, t, b)|^{10}] \leqq 2^9(I+J).$$

Firstly we give the estimate for I. Without loss of generality, we may assume $a \leqq b$. Then by the Burkholder-Davis-Gundy inequality and Lemma 1, we have

$$\begin{aligned} I &= E[\sup_{0 \leqq t \leqq T} |\int_0^t 1_{(a,b]}(X_\tau(s)) X_\tau(ds)|^{10}] \\ &\leqq c_1 E[\{\int_0^T 1_{(a,b]}(X_\tau(t)) dt\}^5]^{*)} \\ &\leqq c_2 |a-b|^5. \end{aligned}$$

Secondly we give the estimate for J. By (2.2) and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} J &\leqq c_3 E[\langle \psi(\tau, \cdot, b) - \psi(\sigma, \cdot, b)\rangle_T^5] \\ &= c_3 E[\{\langle \psi(\tau, \cdot, b)\rangle_T - 2\langle \psi(\tau, \cdot, b), \psi(\sigma, \cdot, b)\rangle_T + \langle \psi(\sigma, \cdot, b)\rangle_T\}^5] \\ &= c_3 E[\{\int_0^T 1_{(b,\infty)}(X_\tau(t))^2 dt - 2e^{-|\tau-\sigma|/2} \\ &\qquad \times \int_0^T 1_{(b,\infty)}(X_\tau(t)) 1_{(b,\infty)}(X_\sigma(t)) dt + \int_0^T 1_{(b,\infty)}(X_\sigma(t))^2 dt\}^5] \\ &= c_3 E[\{\int_0^T (1_{(b,\infty)}(X_\tau(t)) - 1_{(b,\infty)}(X_\sigma(t)))^2 dt \\ &\qquad + 2(1-e^{-|\tau-\sigma|/2}) \int_0^T 1_{(b,\infty)}(X_\tau(t)) 1_{(b,\infty)}(X_\sigma(t)) dt\}^5] \\ &\leqq c_4 E[\{\int_0^T (1_{(b,\infty)}(X_\tau(t)) - 1_{(b,\infty)}(X_\sigma(t)))^2 dt\}^5] \\ &\qquad + c_4 (1-e^{-|\tau-\sigma|/2})^5 E[\{\int_0^T 1_{(b,\infty)}(X_\tau(t)) 1_{(b,\infty)}(X_\sigma(t)) dt\}^5] \\ &:= J_1 + J_2. \end{aligned}$$

As for J_2, we easily obtain

*) $c_1, c_2, \cdots$ are positive constants which depend only on T.

$$J_2 \leqq c_5 |\tau-\sigma|^5 .$$

As for J_1, we define a subset $A_b \subseteq \boldsymbol{R}^2$ by

$$A_b = \{(x, y) \in \boldsymbol{R}^2 \mid (x-b)(y-b) \leqq 0\} .$$

Then we get

$$J_1 = c_4 E[\{\int_0^T 1_{A_b}(X_\tau(t), X_\sigma(t))dt\}^5] .$$

On the other hand, we define a matrix $U(\tau, \sigma)$ by

$$\begin{aligned} &U(\tau, \sigma) \\ &=(U_{ij}(\tau, \sigma)) \\ &=\frac{1}{2\sqrt{1-e^{-|\tau-\sigma|}}}\begin{pmatrix} \sqrt{1+e^{-|\tau-\sigma|/2}}+\sqrt{1-e^{-|\tau-\sigma|/2}} & -\sqrt{1+e^{-|\tau-\sigma|/2}}+\sqrt{1-e^{-|\tau-\sigma|/2}} \\ -\sqrt{1+e^{-|\tau-\sigma|/2}}+\sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}}+\sqrt{1-e^{-|\tau-\sigma|/2}} \end{pmatrix} \end{aligned}$$

and 2-dimensional continuous process $(B_1(t), B_2(t))$ by

$$B_i(t) = U_{i1}(\tau, \sigma)X_\tau(t) + U_{i2}(\tau, \sigma)X_\sigma(t) , \qquad i = 1, 2 .$$

Then it is easy to see that $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion starting at 0. Moreover set

$$\hat{B}_i(t) = \sum_{j=1}^{2} O_{ij} B_j(t) , \qquad i = 1, 2 ,$$

where $O=(O_{ij})$ is an orthogonal matrix defined by

$$O = (O_{ij}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} .$$

Then $(\hat{B}_1(t), \hat{B}_2(t))$ is also a 2-dimensional Brownian motion starting at 0 by the rotation invariance. Thus we have

$$J_1 = c_4 E[\{\int_0^T 1_{OU(\tau,\sigma)A_b}(\hat{B}_1(t), \hat{B}_2(t))dt\}^5] .$$

Since

$$(OU(\tau, \sigma))^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} \\ -\sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} \end{pmatrix} ,$$

a subset $OU(\tau, \sigma)A_b$ is given by

$$OU(\tau, \sigma)A_b = \left\{(x, y)\in \boldsymbol{R}^2 \mid \left(y+\frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}}x-\frac{\sqrt{2}}{\sqrt{1+e^{-|\tau-\sigma|/2}}}b\right)\right.$$
$$\left.\times\left(y-\frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}}x-\frac{\sqrt{2}}{\sqrt{1+e^{-|\tau-\sigma|/2}}}b\right)\leqq 0\right\}.$$

By Lemma 2, we obtain

$$J_1\leqq c_4 E[\{\int_0^T 1_{OU(\tau,\sigma)A_0}(\hat{B}_1(t), \hat{B}_2(t))dt\}^5].$$

For any $R>0$, we define S_R, $V_R\subseteq \boldsymbol{R}^2$ by

$$S_R = \{(x, y)\in \boldsymbol{R}^2 \mid \sqrt{x^2+y^2}\geqq R\}$$
$$V_R = \left\{(x, y)\in \boldsymbol{R}^2 \mid |y|<\frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}}R\right\},$$

then it is easy to see that $OU(\tau, \sigma)A_0\subseteq S_R\cup V_R$. Let η_R be a stopping time given by

$$\eta_R = \inf\{t\geqq 0 \mid (\hat{B}_1(t), \hat{B}_2(t))\in S_R\},$$

then we have

$$\begin{aligned} J_1 &\leqq c_5 E[\{\int_0^T 1_{S_R}(\hat{B}_1(t), \hat{B}_2(t))dt\}^5]+c_5 E[\{\int_0^T 1_{V_R}(\hat{B}_1(t), \hat{B}_2(t))dt\}^5] \\ &\leqq c_5 T^5 P[\eta_R\leqq T]+c_6\left(\frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}}R\right)^5 \\ &\leqq c_7(P[\eta_R\leqq T]+|\tau-\sigma|^{5/2}R^5). \end{aligned}$$

Since $(\sqrt{\hat{B}_1(t)^2+\hat{B}_2(t)^2})$ is a Bessel process with index 2, the following fact is well-known (see e.g., Itô-McKean [5]). For any $\alpha>0$,

$$E[e^{-\alpha\eta_R}]=\frac{I_0(0)}{I_0(\sqrt{2\alpha}R)},$$

where I_0 is the modified Bessel function:

$$I_0(x) = \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}.$$

Hence by the Chebyshev inequality,

$$P[\eta_R\leqq T]\leqq e^T E[e^{-\eta_R}] = e^T\frac{1}{I_0(\sqrt{2}R)}\leqq c_8 R^{-45}.$$

Then, by setting $R=|\tau-\sigma|^{-1/20}$, we get

$$J_1\leqq c_9(R^{-45}+|\tau-\sigma|^{5/2}R^5)\leqq 2c_9|\tau-\sigma|^{9/4}.$$

This completes the proof. $\square$

Corollary. *For $w\in W_0^1$, set $Z_w=\{t>0\,|\,w(t)=0\}$. Then Z_w has a continuum cardinal number and its Lebesgue measure is 0 for q.e.w.*

Proof. Let Ω_0 be a set of all $\omega\in\Omega$ which satisfy (1.2) for all $\tau>0$. Then on Ω_0, we have

$$\phi(\tau, t, 0) = \lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(X_\tau(s))ds\,.$$

Hence it holds that

$$\phi(\tau, t, 0) = \int_0^t 1_{\{0\}}(X_\tau(s))\phi(\tau, ds, 0)$$

and

$$\int_0^t 1_{\{0\}}(X_\tau(s))ds \leqq 4\varepsilon\frac{1}{4\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(X_\tau(s))ds \to 0 \qquad (\text{as } \varepsilon\to 0)\,.$$

Then the rest is easy. $\square$

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