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THE HAUSDORFF DIMENSION OF QUASI-ALL BROWNIAN PATHS

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1. Introduction. Properties of Brownian paths have been main subjects of many papers and various results have been already well known. The size of sample path, for example, has been investigated by evaluating Lebesgue measure and Hausdorff measure etc. of path. Taylor [7] determined the Hausdorff dimension of almost all Brownian paths. On the other hand, Fukushima [2] recently studied some basic properties of path, such as nowhere-differentiability, Lévy's Hölder continuity, the law of iterated logarithm etc. in connection with the Dirichlet space theory on the Wiener space. He proved that these properties hold not only almost surely but also *quasi-everywhere.*

In this paper we shall present a refinement of Taylor's result from the viewpoint of the Dirichlet space theory. It is easily derived from a combination of the definition of Hausdorff measure and Fukushima's result corresponding to Lévy's Hölder continuity that the Hausdorff dimension of path is no more than 2 quasi-everywhere. We shall prove that the Hausdorff dimension is no less than 2 quasi-everywhere, by showing that a specific Wiener functional used in [7] belongs not only to L^2-space but also to the Dirichlet space.

2. Dirichlet space. We first prepare some notations and definitions on the Wiener space and a Dirichlet space on it (cf. Shigekawa [6] and Fukushima [2]).

Let W be the Banach space of all $\boldsymbol{R}^d$-valued continuous functions $w(t)$ on [0, 1] satisfying $w(0)=0$, with standard supremum norm, H be the Hilbert space of all absolutely continuous functions of W having square integrable derivatives, with inner product

$$\langle h, g\rangle_H = \int_0^1 \dot{h}(t)\cdot\dot{g}(t)\,dt\,,$$

and norm $\|h\|_H=\sqrt{\langle h, h\rangle_H}$, where $\dot{h}$ denotes the derivative of h and $a\cdot b$ denotes the inner product in $\boldsymbol{R}^d$. Let P be the Wiener measure on W. A Wiener functional is a real (or more generally Hilbert space) valued mapping defined on W, measurable with respect to the Borel field of W. Let $L^2(W)$ be the

Hilbert space of real valued Wiener functionals, square integrable with respect to P, with inner product

$$(F, G)_{L^2(W)} = \mathrm{E}[F(w)G(w)],$$

where E denotes the integration with respect to P. Let $L^2(W;H)$ be the space of H-valued Wiener functionals F satisfying $\mathrm{E}[\|F(w)\|_H^2]<\infty$.

We define a weak derivative of a Wiener functional after [6]. A real valued Wiener functional F is called H-differentiable if for every w of W there exists $DF(w)$ of H such that for any h of H,

$$F(w+h)-F(w) = \langle DF(w), h\rangle_H + o(\|h\|_H), \text{ as } \|h\|_H \to 0.$$

A real valued Wiener functional F is called to have a weak derivative if there exists a sequence of real valued smooth (in the sence of [6]) functionals F_k convergent to F in $L^2(W)$ such that the sequence of H-derivatives DF_k is a Cauchy sequence in $L^2(W;H)$. The weak derivative of F is defined by $L^2(W;H)$-limit of DF_k and denoted by DF (see also Ikeda and Watanabe [3]).

Next we define a Dirichlet space and a Dirichlet form after [2]. Let $\mathcal{F}$ be the space of Wiener functionals in $L^2(W)$ having weak derivatives in $L^2(W;H)$, and

$$\mathcal{E}(F, G) = \frac{1}{2}\mathrm{E}[\langle DF, DG\rangle_H],$$

and

$$\mathcal{E}_1(F, G) = \mathcal{E}(F, G)+(F, G)_{L^2(W)}, \quad F, G\in\mathcal{F}.$$

Then a capacity is defined by

$$\mathrm{Cap}(A) = \inf\{\mathcal{E}_1(F, F): F\in\mathcal{F}, F\geq 1, P\text{-a.e. on } A\}$$

for an open set A of W, and for any set B of W

$$\mathrm{Cap}(B) = \inf\{\mathrm{Cap}(A): A \text{ is open}, A\supset B\}.$$

It is well known that this capacity is a nonnegative, increasing set function on W satisfying the following properties:

(I) $P(A)\leq\mathrm{Cap}(A)$ for any Borel set A, in particular $\mathrm{Cap}(W) = 1$.

(II) $\mathrm{Cap}(\cup_{n=1}^{\infty}A_n) = \sup\{\mathrm{Cap}(A_n): n\geq 1\}$, for an increasing sequence A_n.

(III) $\mathrm{Cap}(\cup_{n=1}^{\infty}A_n)\leq\sum_{n=1}^{\infty}\mathrm{Cap}(A_n)$.

(see also [1]). The term '*quasi-everywhere*' or simply '*q.e.*' means '*except on a set of zero capacity*'.

The Hausdorff dimension $\dim(A)$ of a set of $\boldsymbol{R}^d$ is defined by

$$\dim(A) = \inf\{\alpha>0: \Lambda^{\alpha}(A) = 0\},$$

where Λ^{α} denotes the Hausdorff α-measure, i.e. Hausdorff measure with respect to measure function t^{α} (cf. Rogers [5]).

Let $w[0, 1]=\{w(t): 0\leq t\leq 1\}$ for w of W. Our result is the following.

Theorem. $\dim(w[0, 1]) = 2$, *q.e.* $(d\geq 2)$.

3. Proof of Theorem. i) First we shall prove that $\dim(w[0, 1])\leq 2$ q.e. Let $w^i(t)$, $1\leq i\leq d$, be the i-th component of $w(t)$. Theorem 3 of [2] tells us that

$$\lim_{\delta\downarrow 0}\sup\{|w^i(t_2)-w^i(t_1)|(2t\log 1/t)^{-1/2}: 0\leq t_1<t_2\leq 1,\ t = t_2-t_1<\delta\} = 1, \text{ q.e.}$$

Let A_i be the set of w of W such that w^i has the above continuity property and B be the set of w which has λ-Hölder continuity peoperties for all $0<\lambda<1/2$. Then B includes $\cap_{i=1}^{d}A_i$ and by (II) and (III),

$$\mathrm{Cap}(B^c)\leq \mathrm{Cap}(\cup_{i=1}^{d}A_i^c)\leq\sum_{i=1}^{d}\mathrm{Cap}(A_i^c) = 0.$$

On the other hand, by the same argument as in [7], it is easily seen that for a λ-Hölder continuous w, $w[0, 1]$ has zero Hausdorff β-measure for $\beta>1/\lambda$. Thus for any w of B, $w[0, 1]$ has zero Hausdorff β-measure for any $\beta>2$. This implies the desired result.

ii) We shall next prove that $\dim(w[0, 1])\geq 2$ q.e. [7] showed that for any w of W and $1<\alpha<2$,

$$\int_0^1\int_0^1 |w(s)-w(t)|^{-\alpha}\,ds\,dt<\infty$$

implies that $w[0, 1]$ has positive Hausdorff α-measure. We define Wiener functionals $F^{(\alpha)}$, $1\leq\alpha<2$, by

$$F^{(\alpha)}(w) = \int_0^1\int_0^1 |w(s)-w(t)|^{-\alpha}\,ds\,dt,$$

and $F_n=F^{(2-1/n)}$, $n=1, 2, \cdots$. If we can show that for any n, $\mathrm{Cap}(\{F_n=\infty\})=0$, then by (III)

$$\mathrm{Cap}(\cup_{n=1}^{\infty}\{F_n = \infty\})\leq\sum_{n=1}^{\infty}\mathrm{Cap}(\{F_n = \infty\}) = 0,$$

and for any w of $\cap_{n=1}^{\infty}\{F_n<\infty\}$, $w[0, 1]$ has positive Hausdorff $(2-1/n)$-measure. This shows that $\dim(w[0, 1])\geq 2$ q.e.

Therefore we have only to prove that $\mathrm{Cap}(\{F^{(\alpha)}=\infty\})=0$ for any $1\leq\alpha<2$. Fix α, $1\leq\alpha<2$. Let

$$F_{\varepsilon}(w) = \int_0^1\int_0^1 (|w(s)-w(t)|^d+\varepsilon)^{-\alpha/d}\,ds\,dt,\quad \varepsilon>0.$$

Then F_ε is continuous on W and for any w, $F_\varepsilon(w) \uparrow F^{(\alpha)}(w)=F(w)$ as $\varepsilon \downarrow 0$. If we can verify that $F_\varepsilon \in \mathcal{F}$ and $K=\sup_{\varepsilon>0}\mathcal{E}_1(F_\varepsilon, F_\varepsilon)$ is finite, then the next well-known inequality (cf. [1])

$$\mathrm{Cap}(\{F_\varepsilon>\lambda\}) \leq \lambda^{-2}\mathcal{E}_1(F_\varepsilon, F_\varepsilon), \quad \lambda>0,$$

and (II) imply that for $\lambda>0$,

$$\mathrm{Cap}(\{F>\lambda\}) \leq \sup_{\varepsilon>0}\mathrm{Cap}(\{F_\varepsilon>\lambda\}) \leq \lambda^{-2}K.$$

Letting $\lambda \to \infty$, we have $\mathrm{Cap}(\{F=\infty\})=0$.

The finiteness of K is derived from the following two lemmas and the proof of Theorem is completed.

Lemma 1. $\sup_{\varepsilon>0}(F_\varepsilon, F_\varepsilon)_{L^2(W)}<\infty$.

Lemma 2. $\sup_{\varepsilon>0}\mathcal{E}(F_\varepsilon, F_\varepsilon)<\infty$.

Proof of Lemma 1. Let $\phi_\varepsilon(x)=(|x|^d+\varepsilon)^{-\alpha/d}$ and

$$\begin{aligned} f_\varepsilon(x) &= \int_0^{|x|} r^{-d+1}\,dr \int_0^r s^{d-1}(s^d+\varepsilon)^{-\alpha/d}\,ds \\ &= d^{-1}\int_0^{|x|} r^{-d+1}\,dr \int_0^{r^d} (s+\delta)^{-\alpha/d}\,ds \\ &= (d-\alpha)^{-1}\int_0^{|x|} r^{-d+1}\{(r^d+\varepsilon)^{1-\alpha/d}-\varepsilon^{1-\alpha/d}\}\,dr, \quad x\in \boldsymbol{R}^d. \end{aligned}$$

Then $\Delta f_\varepsilon(x)=\phi_\varepsilon(x)$, where Δ is the Laplacian on $\boldsymbol{R}^d$.

$$\begin{aligned} &(F_\varepsilon, F_\varepsilon)_{L^2(W)} = \mathrm{E}[(F_\varepsilon)^2] \\ &= 4\mathrm{E}[(\int_0^1 dt \int_t^1 \phi_\varepsilon(w(s)-w(t))\,ds)^2] \\ &\leq 4\mathrm{E}[\int_0^1 dt\,(\int_t^1 \phi_\varepsilon(w(s)-w(t))\,ds)^2] \\ &= 4\int_0^1 dt\,\mathrm{E}[(\int_0^{1-t} \phi_\varepsilon(w(s))\,ds)^2] \\ &\leq 4\mathrm{E}[(\int_0^1 \phi_\varepsilon(w(s))\,ds)^2]. \end{aligned}$$

In the above we have used the time-homogeneity of Brownian motion. To estimate the right-hand side, we apply the Ito formula to $f_\varepsilon(x)$,

$$f_\varepsilon(w(1)) = \int_0^1 \nabla f_\varepsilon(w(s))\cdot dw(s) + 1/2\int_0^1 \phi_\varepsilon(w(s))\,ds,$$

where

$$\nabla f_\varepsilon(x) = \left(\frac{\partial}{\partial x_j} f_\varepsilon(x)\right)_{1\leq j\leq d}$$

$$= ((d-\alpha)^{-1}|x|^{-d}\{(|x|^d+\varepsilon)^{1-\alpha/d}-\varepsilon^{1-\alpha/d}\}x_j)_{1\le j\le d}\,.$$

Since $|f_\varepsilon(x)|\le(d-\alpha)^{-1}(2-\alpha)^{-1}|x|^{2-\alpha}$ and $|\nabla f_\varepsilon(x)|\le(d-\alpha)^{-1}|x|^{1-\alpha}$,

$$\mathrm{E}[(\int_0^1\phi_\varepsilon(w(s))\,ds)^2]$$

$$\le 4\mathrm{E}[(f_\varepsilon(w(1))-\int_0^1\nabla f_\varepsilon(w(s))\cdot dw(s))^2]$$

$$\le 8\mathrm{E}[f_\varepsilon(w(1))^2]+8\mathrm{E}[\int_0^1|\nabla f_\varepsilon(w(s))|^2ds]$$

$$\le 8(d-\alpha)^{-2}(2-\alpha)^{-2}\mathrm{E}[|w(1)|^{2(2-\alpha)}]$$

$$+8(d-\alpha)^{-2}\int_0^1\mathrm{E}[|w(s)|^{2(1-\alpha)}]ds \quad <\infty\,.$$

This completes the proof of Lemma 1.

Corollary. *The Ito formula is applicable to* $|x|^{2-\alpha}$, $\alpha<2$, *i.e.*

$$|w(t)|^{2-\alpha}=\int_0^t(2-\alpha)|w(s)|^{-\alpha}w(s)\cdot dw(s)+\int_0^t(d-\alpha)\,(2-\alpha)\,|w(s)|^{-\alpha}ds\,.$$

Furthermore, $\int_0^1|w(s)|^{-\alpha}ds\in L^2(W)$.

Proof of Lemma 2. We use the same notations as in the proof of Lemma 1. By the mean value theorem, for w of W and h of H,

$$F_\varepsilon(w+h)-F_\varepsilon(w)=2\int_0^1dt\int_t^1\nabla\phi_\varepsilon(w(s)-w(t))\cdot(\int_t^s\dot h(u)du)\,ds+o(\|h\|_H)$$

$$=\int_0^1du\,\dot h(u)\cdot(2\int_0^u dt\int_u^1\nabla\phi_\varepsilon(w(s)-w(t))\,ds)+o(\|h\|_H)$$

as $\|h\|_H\to 0$. This implies the H-differentiability of F_ε and its H-derivative DF_ε is given by

$$\langle DF_\varepsilon(w),h\rangle_H=\int_0^1du\,\dot h(u)\cdot(2\int_0^u dt\int_u^1\nabla\phi_\varepsilon(w(s)-w(t))\,ds)\,.$$

Since the boundedness of $\|DF_\varepsilon(w)\|_H$ on W is easily derived from that of $|\nabla\phi_\varepsilon(x)|$, by the same argument as in Section 2 and 3 of Kusuoka [4], it is verified that the H-derivative of F_ε coincides with the weak derivative of F_ε a.s. Thus

$$2\mathcal{E}(F_\varepsilon,F_\varepsilon)=\mathrm{E}[\|DF_\varepsilon\|_H^2]$$

$$=4\int_0^1du\sum_{j=1}^d\mathrm{E}[(\int_0^u dt\int_u^1\psi_{\varepsilon j}(w(s)-w(t))\,ds)^2]\,,$$

where $\psi_{\varepsilon j}(x)$ is the j-th component of $\nabla\phi_\varepsilon(x)$. Let $g_{\varepsilon j}(x)$ be the j-th component of $\nabla f_\varepsilon(x)$. Then $g_{\varepsilon j}(x)$ is twice continuously differentiable and $\Delta g_{\varepsilon j}(x)=\psi_{\varepsilon j}(x)$. To estimate the above, we again apply the Ito formula to $g_{\varepsilon j}(x)$,

$$g_{\varepsilon j}(w(1)-w(t))-g_{\varepsilon j}(w(u)-w(t))$$
$$=\int_u^1 \nabla g_{\varepsilon j}(w(s)-w(t))\cdot dw(s)+1/2\int_u^1 \psi_{\varepsilon j}(w(s)-w(t))\,ds\,.$$

Hence we have

$$(*)\qquad \mathrm{E}[(\int_0^u dt\int_u^1 \psi_{\varepsilon j}(w(s)-w(t))\,ds)^2]$$
$$\leq 8\,\mathrm{E}[(\int_0^u \{g_{\varepsilon j}(w(1)-w(t))-g_{\varepsilon j}(w(u)-w(t))\}dt)^2]$$
$$+8\,\mathrm{E}[(\int_0^u dt\int_u^1 \nabla g_{\varepsilon j}(w(s)-w(t))\cdot dw(s))^2]$$

As for the first term in the right-hand side, it is less than

$$16\mathrm{E}[\int_0^u \{g_{\varepsilon j}(w(1)-w(t))^2+g_{\varepsilon j}(w(u)-w(t))^2\}dt]$$
$$\leq 32\mathrm{E}[\int_0^1 g_{\varepsilon j}(w(t))^2 dt]$$
$$\leq 32(d-\alpha)^{-2}\mathrm{E}[\int_0^1 |w(t)|^{2(1-\alpha)}\,dt]\,,$$

since $|g_{\varepsilon j}(x)|\leq(d-\alpha)^{-1}\ |x|^{1-\alpha}$, $|x|\neq 0$. Thus the first term is bounded independently of ε by Corollary of Lemma 1. As for the second term, by standard calculation we have

$$\mathrm{E}[(\int_0^u dt\int_u^1 \nabla g_{\varepsilon j}(w(s)-w(t))\cdot dw(s))^2]$$
$$=\mathrm{E}[\int_0^u dt\int_0^u dv(\int_u^1 \nabla g_{\varepsilon j}(w(s)-w(t))\cdot dw(s))$$
$$\times(\int_u^1 \nabla g_{\varepsilon j}(w(s)-w(v))\cdot dw(s))]$$
$$\leq \int_u^1 ds\,\mathrm{E}[(\int_0^u |\nabla g_{\varepsilon j}(w(s)-w(t))|\,dt)^2]$$
$$\leq \int_0^1 ds\,\mathrm{E}[(\int_0^1 |\nabla g_{\varepsilon j}(w(t))|\,dt)^2]$$
$$=\mathrm{E}[(\int_0^1 |\nabla g_{\varepsilon j}(w(t))|\,dt)^2]\,.$$

Note that

$$\nabla g_{\varepsilon j}(x)=\nabla((d-\alpha)^{-1}x_j|x|^{-d}\{(|x|^d+\varepsilon)^{1-\alpha/d}-\varepsilon^{1-\alpha/d}\})$$
$$=((d-\alpha)^{-1}|x|^{-d}\{(|x|^d+\varepsilon)^{1-\alpha/d}-\varepsilon^{1-\alpha/d}\}\,(-d\,x_j\,x_k|x|^{-2}+\delta_{jk})$$
$$+x_j\,x_k|x|^{-2}\phi_\varepsilon(x))_{1\leq k\leq d}\,,$$

where δ_{jk} is Kronecker's symbol. Hence we have

$$|\nabla g_{\varepsilon j}(x)|\leq((1+d)/(d-\alpha)+1)|x|^{-\alpha}\,.$$

Therefore the second term in the right-hand side of (*) is less than

$$8\{(1+d)/(d-\alpha)+1\}^2 \mathrm{E}[(\int_0^1 |w(t)|^{-\alpha} dt)^2]$$

and is bounded independently of ε by Corollary of Lemma 1. This completes the proof of Lemma 2.

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References

[1] M. Fukushima: Dirichlet forms and Markov processes, North Holland and Kodansha, 1980.

[2] M. Fukushima: *Basic properties of Brownian motion and a capacity on the Wiener space*, J. Math. Soc. Japan **36** (1984), 161–176.

[3] N. Ikeda and S. Watanabe: Stochastic differential equations and diffusion processes, North Holland and Kodansha, 1981.

[4] S. Kusuoka: *Dirichlet forms and diffusion processes on Banach spaces*, J. Fac. Sci. Univ. Tokyo, Sec. IA **29** (1982), 79–95.

[5] C.A. Rogers: Hausdorff measures, Cambridge University Press, 1970.

[6] I. Shigekawa: *Derivatives of Wiener functionals and absolute continuity of induced measures*, J. Math. Kyoto Univ. **20** (1980), 263–289.

[7] S.J. Taylor: *The Hausdorff α-dimensional measure of Brownian path in n-space*, Proc. Cambridge Philos. Soc. **49** (1953), 31–39.

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