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ON THE RATE OF CONVERGENCE FOR MAXIMUM LIKELIHOOD ESTIMATES IN A TRUNCATED CASE

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1. Introduction. Let $X_1, \cdots, X_n$ be independent random variables with common density $f(x-\theta)$, $-\infty<x,\ \theta<\infty$, where θ is an unknown translation parameter. We shall consider here the case that $f(x)$ is a uniformly continuous density which vanishes on the interval $(-\infty, 0]$ and is positive on the interval $(0, \infty)$ and particularly

$$(1.1) \qquad f(x)\sim\alpha\beta x^{\alpha-1} \quad \text{as } x\to+0$$

with $1<\alpha<2$ and $\beta>0$.

Let $\hat{\theta}_n=\hat{\theta}_n(X_1, \cdots, X_n)$ denote the MLE (maximum likelihood estimate) of θ for the sample size n. Woodroofe [7] showed that $(\beta n)^{1/\alpha}(\hat{\theta}_n-\theta)$ has a limiting distribution which is not the normal distribution. Furthermore, he studied asymptotic properties of this limiting distribution. In this paper, we shall investigate the rate of convergence to the limiting distribution for MLE. This result is applied to estimate the probability of moderate deviations for the distribution of MLE.

The case that (1.1) is satisfied with $\alpha\geqq 2$ has already studied by the author ([2], [3]). In [2], the author showed that if $\alpha>3$, $\sqrt{nB}\ (\hat{\theta}_n-\theta)$ converges uniformly to the standard normal distribution with the convergence order $O(n^{-1/2})$, and that if $2<\alpha\leqq 3$, the order of convergence to normality is $o(n^{-\nu/2})$ for every $\nu<(\alpha-2)/2$. Here B denotes Fisher's information number. In the case $\alpha=2$, it was shown in [3] that for every real number t there exists $C>0$ such that for all θ and $n\geqq 1$

$$|P_\theta\{\sqrt{\beta n(\log n+\log\log n)}\ (\hat{\theta}_n-\theta)\leqq t\}-\Phi(t)|\leqq C(\log n)^{-1},$$

where Φ denotes the standard normal distribution. It is noticed that the rate of convergence, which is uniform in t, is a little slower than the order $(\log n)^{-1}$.

In the regular case, it is well known that the same result as in the case of (1.1) with $\alpha>3$ holds (see Pfanzagl [5]). It is interesting to note that Takeuchi [6] has studied Edgeworth type expansion of the distribution of the sum of independent random variables in some non-regular cases.

2. Conditions and results. We shall impose the following regularity conditions on $f(x)$. These conditions are stronger than those made by Woodroofe [7].

CONDITIONS

(i) $f(x)$ is a uniformly continuous density which vanishes on $(-\infty, 0]$ and is positive on $(0, \infty)$.

(ii) $f(x)$ is continuously differentiable on $(0, \infty)$ with derivative $f'(x)$ and $f'(x)$ is absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $f''(x)$.

Let $g(x)=\log f(x)$ for $x>0$. Then $g(x)$ will be continuously differentiable on $(0, \infty)$ with derivative $g'=f'/f$ and $g'(x)$ will be absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $g''=(ff''-f'^2)/f^2$.

(iii) For some $1<\alpha<2$, $\beta>0$ and $\gamma>0$

$$f(x)=\alpha\beta x^{\alpha-1}+O(x^{\alpha+\gamma-1}),\ g'(x)=(\alpha-1)x^{-1}+O(x^{\gamma-1}) \ \text{and}$$
$$g''(x)=-(\alpha-1)x^{-2}+O(x^{\gamma-2}) \qquad \text{as } x\to+0\,.$$

(iv) For every $s\geqq 0$

$$\int_0^\infty \{g(x+s)\}^2 f(x)\,dx<\infty\,.$$

(v) For every $a>0$, there exists a $\delta>0$ such that

(a) $\displaystyle\int_a^\infty \sup_{|s|\leqq\delta}\{g'(x+s)\}^2 f(x)\,dx<\infty,$

(b) $\displaystyle\int_a^\infty \sup_{|s|\leqq\delta}\{g''(x+s)\}^2 f(x)\,dx<\infty.$

Condition (i) insures that MLE's of θ for the sample size n exist in the interval $(-\infty, M_n)$ where $M_n=\min(X_1, \cdots, X_n)$. Let $\{\hat\theta_n;\ n\geqq 1\}$ be a sequence of MLE's. In addition to condition (i), if $g(x)$ is continuously differentiable, then $\{\hat\theta_n\}$ will form a sequence of roots of the likelihood equation (see Woodroofe [7]).

Since θ is a translation parameter, we restrict our attention to the case that $\theta=0$. The following Lemma 1 and Lemma 2 may be proved analogously to Lemma 1 and Lemma 2 in [3], respectively.

Lemma 1 (cf. Lemma 2.1 in [7]). *Let conditions* (i)–(iii) *and* (v) (b) *be satisfied. Then for sufficiently small* $\varepsilon>0$, *there are events* A_n, $n\geqq 1$, *for which* $P\{A_n^c\}=O(n^{-1})$ *and* A_n *implies*

$$\sup_{-\varepsilon\leqq t<M_n} n^{-1}\sum_{j=1}^n g''(X_j-t)<-1\,.$$

Proof. Let $a>0$ be so small that $g''(x)\leqq-(\alpha-1)/2x^2$ for $0<x\leqq 2a$. There is a sufficiently small number $\varepsilon>0$ such that

$$(\alpha-1)\int_0^a (x+\varepsilon)^{-2} f(x)\,dx > 2\int_a^\infty \sup_{|t|\leq\varepsilon} |g''(x+t)|\,f(x)\,dx + 3 + \alpha$$

because the left-hand integral diverges to ∞ as $\varepsilon\to 0$. Then the event $M_n \leqq \varepsilon$ implies that

$$\sup_{-\varepsilon\leqq t<M_n} n^{-1}\sum_{j=1}^n g''(X_j-t) \leqq -(\alpha-1)/(2n)\sum\nolimits_0^a (X_j+\varepsilon)^{-2} + n^{-1}\sum\nolimits_a^\infty \sup_{|t|\leqq\varepsilon} |g''(X_j+t)|$$

where $\sum_u^v$ denotes summation over $j\leqq n$ for which $u\leqq X_j<v$. Hence

$$M_n\leqq\varepsilon,\ \left|n^{-1}\sum\nolimits_0^a (X_j+\varepsilon)^{-2} - \int_0^a (x+\varepsilon)^{-2} f(x)\,dx\right| < 1$$

and

$$\left|n^{-1}\sum\nolimits_a^\infty \sup_{|t|\leqq\varepsilon} |g''(X_j+t)| - \int_a^\infty \sup_{|t|\leqq\varepsilon} |g''(x+t)|\,f(x)\,dx\right| < 1$$

imply

$$\sup_{-\varepsilon\leqq t<M_n} n^{-1}\sum_{j=1}^n g''(X_j-t) < -1\,.$$

Since $P\{M_n>\varepsilon\}=o(n^{-1})$, Lemma 1 follows easily from condition (v) (b) and Chebyshev's inequality.

Lemma 2. *Let conditions* (i)–(iv) *be satisfied. Then for every* $\varepsilon>0$

$$P\{|\hat\theta_n|\geqq\varepsilon\} = O(n^{-1})\,.$$

Let $b_n=(\beta n)^{1/\alpha}$ and for $t>0$ define distribution functions $F_{n,0}$, $F_{n,t}$ and $F^*_{n,t}$ by

$$F_{n,0}(x) = P\{b_n^{-1}\sum_{j=1}^n g'(X_j)<x\}\,,$$

$$F_{n,t}(x) = P\{b_n^{-1}t\sum_{j=1}^n g'(X_j+b_n^{-1}t)<x\}\,,$$

$$F^*_{n,t}(x) = P\{b_n^{-1}t\sum_{j=1}^n g'(X_j-b_n^{-1}t)<x\,|\,M_n>b_n^{-1}t\}\,,$$

where $F^*_{n,t}$ denotes conditioned probability given $M_n>b_n^{-1}t$.

Next, for $t>0$ let G_0, G_t and G_t^* be the distribution functions with the following characteristic functions ξ_0, ξ_t and ξ_t^*, respectively:

$$\xi_0(u) = \exp\{-d|u|^\alpha[1-i\,\mathrm{sign}(u)\tan(\pi\alpha/2)]\}^{1)}$$

with

$$d = -\alpha(\alpha-1)^\alpha\cos(\pi\alpha/2)\int_0^\infty (e^{-x}-1+x)x^{-\alpha-1}\,dx>0\,,$$

1) The plus sign of the imaginary part of (2.4) in [7] should be corrected to minus. Consequently, $H(0)$ in [7] (p. 478) must be equal to $1-\alpha^{-1}$, not α^{-1}.

$$\xi_t(u) = \exp\{-ium_\alpha t^\alpha - \alpha(\alpha-1)t^\alpha h(u)\}$$

with

$$m_\alpha = \alpha\Gamma(\alpha)\Gamma(2-\alpha) \text{ and } h(u) = \int_0^{\alpha-1}(1+iux-e^{iux})\,[(\alpha-1)x^{-1}-1]^{\alpha-1}x^{-2}dx\,,$$

$$\xi_t^*(u) = \exp\{ium_\alpha^* t^\alpha - \alpha(\alpha-1)t^\alpha h^*(u)\}$$

with

$$m_\alpha^* = \alpha\int_1^\infty \{\sin[(\alpha-1)/(x-1)]-(\alpha-1)x^{-1}\}x^{\alpha-1}dx-\alpha$$

$$\text{and } h^*(u) = \int_0^\infty (1+iu\sin(x)-e^{iux})\,[(\alpha-1)x^{-1}+1]^{\alpha-1}x^{-2}dx\,.$$

Then $F_{n,0}$, $F_{n,t}$ and $F_{n,t}^*$ converge weakly to G_0, G_t and G_t^*, respectively (see [7], Theorem 2.1–Theorem 2.3). We shall investigate rates of convergence in these cases. Define the distance between two distribution functions F and G by

$$\Delta(F, G) = \sup\{|F(x)-G(x)|\,;\ -\infty<x<\infty\}\,.$$

Theorem 1. *Let conditions* (i)–(iii) *and* (v) (a) *be satisfied for some* $1<\alpha<2$ *and* $\gamma>0$, *then there exists* $C>0$ *such that for all* $n\geqq 1$

$$\Delta(F_{n,0}, G_0) \leqq \begin{cases} C\,n^{-(2-\alpha)/\alpha}\,, & \alpha+\gamma>2\,,\\ C\,n^{-\gamma/\alpha}\log n\,, & \alpha+\gamma=2\,,\\ C\,n^{-\gamma/\alpha}\,, & \alpha+\gamma<2\,.\end{cases}$$

Theorem 2. *Let conditions* (i)–(iii) *and* (v) *be satisfied for some* $1<\alpha<2$ *and* $\gamma>0$.

If $\alpha+\gamma>2$, *then there exists* $C>0$ *such that for all* $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/\alpha}$

$$\Delta(F_{n,t}, G_t)\leqq C[1+t^{(4-\alpha)/2}]\,n^{-(2-\alpha)/\alpha}\,.$$

If $\alpha+\gamma\leqq 2$, *then for every* λ, $0<\lambda<\gamma$, *there exists* $C>0$ *such that for all* $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/\alpha}$

$$\Delta(F_{n,t}, G_t)\leqq C[1+t^{(\alpha+2\lambda)/2}]\,n^{-\lambda/\alpha}\,.$$

Theorem 3. *Let conditions* (i)–(iii) *and* (v) *be satisfied for some* $1<\alpha<2$ *and* $\gamma>0$.

If $\alpha+\gamma>2$, *then there exists* $C>0$ *such that for all* $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/\alpha}$

$$\Delta(F_{n,t}^*, G_t^*)\leqq C(1+t^{2-\alpha})\,n^{-(2-\alpha)/\alpha}\,.$$

If $\alpha+\gamma\leqq 2$, *then for every* λ, $0<\lambda<\gamma$, *there exists* $C>0$ *such that for all* $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/\alpha}$

$$\Delta(F_{n,t}^*, G_t^*)\leqq C(1+t^\lambda)\,n^{-\lambda/\alpha}\,.$$

Theorems 1–3 will be proved in Section 4. As a corollary to them, we shall estimate the distance between the distribution of $b_n\hat{\theta}_n$ and the limiting distribution function (see $H(t)$ below). We shall use ideas related to Woodroofe [7]. It follows easily from Lemma 1 and Lemma 2 that as $n\to\infty$

$$P\{b_n\hat{\theta}_n \leqq -t\} = P\{\sum_{j=1}^n g'(X_j+b_n^{-1}t)\geqq 0\}+O(n^{-1}),$$

where $O(n^{-1})$ is uniform in $t\in[0, b_n\varepsilon)$ with $\varepsilon>0$ as in Lemma 1 and

$$P\{b_n\hat{\theta}_n > t\} = P\{\sum_{j=1}^n g'(X_j-b_n^{-1}t)<0,\ M_n> b_n^{-1}t\}+O(n^{-1}),$$

where $O(n^{-1})$ is uniform in $t>0$. Thus

$$P\{b_n\hat{\theta}_n \leqq -t\} = 1-F_{n,t}(0)+O(n^{-1}), \tag{2.1}$$

$$P\{b_n\hat{\theta}_n \leqq t\} = 1-F_{n,t}^*(0)\, P\{M_n>b_n^{-1}t\}+O(n^{-1}), \tag{2.2}$$

uniformly in $0\leqq t<b_n\varepsilon$ and $t>0$ as $n\to\infty$, respectively.

Let H be the distribution function defined by

$$H(-t) = 1-G_t(0), \qquad t\geqq 0,$$
$$H(t) = 1-G_t^*(0)\exp(-t^\alpha), \qquad t>0.$$

According to Woodroofe [7],

$$H(-t)+[1-H(t)] = o(\exp(-t^\alpha)) \qquad \text{as } t\to\infty,$$

which implies

$$H(-(\log n)^{1/\alpha})+[1-H((\log n)^{1/\alpha})] = o(n^{-1}). \tag{2.3}$$

From (2.1) and Theorem 2 it follows that for $0<t\leqq(\log n)^{1/\alpha}$

$$|P\{b_n\hat{\theta}_n\leqq -t\}-H(-t)| = O((1+t^{(4-\alpha)/2})n^{-(2-\alpha)/\alpha}), \qquad \alpha+\gamma>2, \tag{2.4a}$$

$$|P\{b_n\hat{\theta}_n\leqq -t\}-H(-t)| = O(n^{-\lambda/\alpha}), \qquad \alpha+\gamma\leqq 2,\ 0<\lambda<\gamma. \tag{2.4b}$$

Noting that

$$P\{M_n>b_n^{-1}t\} = \exp(-t^\alpha)\,[1+O(t^\alpha(b_n^{-1}t)^{\min(\alpha,\gamma)})] \tag{2.5}$$

uniformly in $0<t\leqq(\log n)^{1/\alpha}$ as $n\to\infty$, (2.2) together with Theorem 3 implies that

$$|P\{b_n\hat{\theta}_n\leqq t\}-H(t)| = O(n^{-(2-\alpha)/\alpha}), \qquad \alpha+\gamma>2, \tag{2.6a}$$

$$|P\{b_n\hat{\theta}_n\leqq t\}-H(t)| = O(n^{-\lambda/\alpha}), \qquad \alpha+\gamma\leqq 2,\ 0<\lambda<\gamma. \tag{2.6b}$$

Furthermore, from (2.3), (2.4) and (2.6) we see that for $t>(\log n)^{1/\alpha}$

$$|P\{b_n\hat{\theta}_n\leqq -t\}-H(-t)| = O(n^{-(2-\alpha)/\alpha}(\log n)^{(4-\alpha)/2\alpha}), \qquad \alpha+\gamma>2, \tag{2.7a}$$

(2.7b) $\quad |P\{b_n\hat{\theta}_n \leqq -t\} - H(-t)| = O(n^{-\lambda/\alpha})\,, \qquad \alpha+\gamma \leqq 2,\ 0<\lambda<\gamma\,,$

(2.8a) $\quad |P\{b_n\hat{\theta}_n \leqq t\} - H(t)| = O(n^{-(2-\alpha)/\alpha})\,, \qquad \alpha+\gamma>2\,,$

(2.8b) $\quad |P\{b_n\hat{\theta}_n \leqq t\} - H(t)| = O(n^{-\lambda/\alpha})\,, \qquad \alpha+\gamma \leqq 2,\ 0<\lambda<\gamma\,,$

because

$$
\begin{aligned}
&|P\{b_n\hat{\theta}_n \leqq -t\} - H(-t)| \leqq |P\{b_n\hat{\theta}_n \leqq -(\log n)^{1/\alpha}\} - H(-(\log n)^{1/\alpha})| \\
&\qquad\qquad + 2H(-(\log n)^{1/\alpha})\,, \\
&|P\{b_n\hat{\theta}_n \leqq t\} - H(t)| \leqq |P\{b_n\hat{\theta}_n \leqq (\log n)^{1/\alpha}\} - H((\log n)^{1/\alpha})| \\
&\qquad\qquad + 2[1 - H((\log n)^{1/\alpha})]\,.
\end{aligned}
\tag{2.9}
$$

The estimates (2.4), (2.6)–(2.8) and Theorem 1 yield the following theorem.

Theorem 4. *Suppose that conditions* (i)–(v) *hold for some* $1<\alpha<2$ *and* $\gamma>0$.

If $\alpha+\gamma>2$, *then there exists* $C>0$ *such that for all* θ, $t \geqq 0$ *and* $n \geqq 1$

$$
\begin{aligned}
&|P_\theta\{b_n(\hat{\theta}_n-\theta) \leqq -t\} - H(-t)| \leqq C\, n^{-(2-\alpha)/\alpha} (\log n)^{(4-\alpha)/2\alpha}\,, \\
&|P_\theta\{b_n(\hat{\theta}_n-\theta) \leqq t\} - H(t)| \leqq C\, n^{-(2-\alpha)/\alpha}\,.
\end{aligned}
$$

If $\alpha+\gamma \leqq 2$, *then for every* λ, $0<\lambda<\gamma$, *there exists* $C>0$ *such that for all* θ, t *and* $n \geqq 1$

$$|P_\theta\{b_n(\hat{\theta}_n-\theta) \leqq t\} - H(t)| \leqq C\, n^{-\lambda/\alpha}\,.$$

The following corollary is an immediate consequence of Theorem 4 and (2.3).

Corollary 1. *Suppose that conditions* (i)–(v) *hold for some* $1<\alpha<2$ *and* $\gamma>0$.

If $\alpha+\gamma>2$, *then there exists* $C>0$ *such that for all* θ *and* $n \geqq 1$

$$P_\theta\{b_n|\hat{\theta}_n-\theta| \geqq (\log n)^{1/\alpha}\} \leqq C\, n^{-(2-\alpha)/\alpha} (\log n)^{(4-\alpha)/2\alpha}\,.$$

If $\alpha+\gamma \leqq 2$, *then for every* λ, $0<\lambda<\gamma$, *there exists* $C>0$ *such that for all* θ *and* $n \geqq 1$

$$P_\theta\{b_n|\hat{\theta}_n-\theta| \geqq (\log n)^{1/\alpha}\} \leqq C\, n^{-\lambda/\alpha}\,.$$

In the case $\alpha+\gamma \leqq 2$, the above bound $n^{-\lambda/\alpha}$ can be improved in Corollary 3 of Section 5.

EXAMPLES ([7]). (1) Let

$$f(x) = \alpha\beta x^{\alpha-1} \exp(-x^\gamma)\,, \quad x>0\,,$$

where $1<\alpha<2$, $\beta>0$ and $\gamma>0$, then the conditions are all satisfied.

(2) Let

$$f(x) = \Gamma(\alpha+\tau)\,[\Gamma(\alpha)\Gamma(\tau)]^{-1}\,x^{\alpha-1}(1+x)^{-\alpha-\tau}\,, \quad x>0\,,$$

where $1<\alpha<2$ and $\tau>0$, then the conditions are all satisfied with $\gamma=1$.

3. Auxiliary results. Let $\phi(u)$, $\phi(u, s)$ and $\phi^*(u, s)$ be the characteristic functions as follows

$$\phi(u) = E\{\exp[iu\, g'(X_1)]\}\,,$$
$$\phi(u, s) = E\{\exp[ius\, g'(X_1+s)]\}\,,$$
$$\phi^*(u, s) = E^*\{\exp[ius\, g'(X_1-s)]\}\,,$$

where $s>0$ and E^* denotes conditional expectation given $M_n>s$. It is remarked that the conditional distribution of $X_1, \cdots, X_n$, given $M_n>s$, is that of independent random variables with common density

$$f^*(x) = l(s)f(x)\,, \qquad x>s\,,$$
$$= 0\,, \qquad \text{otherwise},$$

where

$$l(s) = \left(\int_s^\infty f(x)\,dx\right)^{-1} = 1+\beta s^\alpha+o(s^\alpha) \quad \text{as } s\to 0\,. \tag{3.1}$$

For real v let

$$Q(v, \gamma) = \begin{cases} v^2\,, & \alpha+\gamma>2\,, \\ v^2|\log|v||\,, & \alpha+\gamma=2\,, \\ |v|^{\alpha+\gamma}\,, & \alpha+\gamma<2\,. \end{cases}$$

To simplify our notations we shall use c as a generic constant to denote factors occurring in the bounds, which do not depend on n, s or u.

Lemma 3. *If the conditions of Theorem* 1 *are satisfied, then as* $u\to 0$,

$$\phi(u) = 1-d\beta|u|^\alpha[1-i\,\mathrm{sign}(u)\,\tan(\pi\alpha/2)]+O(Q(u, \gamma))\,.$$

Proof. Suppose $u>0$. Since $E\{g'(X_1)\}=0$, we may write

$$\phi(u) = 1+E\{\exp[iu\,g'(X_1)]-1-iu\,g'(X_1)\}$$
$$= 1+\left(\int_0^a+\int_a^\infty\right)\{\exp[iu\,g'(x)]-1-iu\,g'(x)\}f(x)\,dx$$
$$= 1+I_1+I_2, \text{ say,}$$

with $a>0$.

To evaluate the integral I_1, we express it in the form $I_{11}+I_{12}+I_{13}$, where

$$I_{11} = \int_0^a \alpha\beta\{\exp[iu(\alpha-1)/x]-1-iu[(\alpha-1)/x]\}x^{\alpha-1}\,dx\,,$$

$$I_{12} = \int_0^a \{\exp[iu(\alpha-1)/x] - 1 - iu[(\alpha-1)/x]\}\,(f(x)-\alpha\beta x^{\alpha-1})\,dx\,,$$

$$I_{13} = \int_0^a \{\exp[iug'(x)] - \exp[iu(\alpha-1)/x] - iu(g'(x)-[(\alpha-1)/x])\}\,f(x)\,dx\,.$$

We shall make use of the following inequality:

$$(3.2) \qquad |e^{ix}-1-ix| \leqq |x|^\rho/[\rho(\rho-1)] \qquad \text{for } -\infty<x<\infty \text{ and } 1<\rho\leqq 2\,.$$

For the estimate of I_{11} use (3.2) with $\rho=2$, then

$$\begin{aligned} I_{11} &= \alpha\beta(\alpha-1)^\alpha u^\alpha \int_{u(\alpha-1)/a}^{\infty} (e^{ix}-1-ix)x^{-\alpha-1}\,dx \\ &= -d\beta u^\alpha[1-i\tan(\pi\alpha/2)] + O(u^2)\,. \end{aligned}$$

From (3.2) and condition (iii) it follows that

$$|I_{12}| \leqq c\{\int_0^u u^\rho x^{\alpha+\gamma-\rho-1}\,dx + \int_u^a u^2 x^{\alpha+\gamma-3}\,dx\}$$

for $0<u<a$ with sufficiently small a. Choosing $\rho=2$ if $\alpha+\gamma>2$ and $1<\rho<\alpha+\gamma$ if $\alpha+\gamma\leqq 2$, it is easily seen that $I_{12}=O(Q(u,\gamma))$. Taking account of the inequality

$$\begin{aligned} (3.3) \qquad |e^{ix}-e^{iy}-i(x-y)| &= |\int_y^x i(e^{iz}-1)\,dz| \leqq |\int_y^x |z|^{\rho-1}/(\rho-1)\,dz| \\ &\leqq |x^\rho - y^\rho|/[\rho(\rho-1)] \end{aligned}$$

for $x>0$, $y>0$ and $1<\rho\leqq 2$, condition (iii) implies that

$$|\exp[iug'(x)] - \exp[iu(\alpha-1)/x] - iu(g'(x)-[(\alpha-1)/x])| \leqq cu^\rho x^{\gamma-\rho}$$

for $0<x<a$ with sufficiently small a. Then the estimate of I_{13} is the same as I_{12}. Thus we have

$$I_1 = -d\beta u^\alpha[1-i\tan(\pi\alpha/2)] + O(Q(u,\gamma))\,.$$

Because of (3.2) and condition (v) (a), it is easy to see that

$$|I_2| \leqq \frac{u^2}{2}\int_a^\infty \{g'(x)\}^2 f(x)\,dx = O(u^2)\,.$$

To complete the proof there remains only to note that for $u<0$, $\phi(u)=\overline{\phi(-u)}$.

It is convenient to use $\sigma=\sigma(\gamma,\lambda)$ defined by

$$\sigma = \begin{cases} 2\,, & \text{if } \alpha+\gamma>2\,, \\ \alpha+\lambda\,, & \text{if } \alpha+\gamma\leqq 2\,, \end{cases}$$

for $0<\lambda\leqq\gamma$.

Lemma 4. *If the conditions of Theorem 2 are satisfied, then for every*

$0<\lambda<\gamma$

$$\phi(u, s) = 1-ium_\alpha\beta s^\alpha-\alpha(\alpha-1)\beta s^\alpha h(u)+O(|u|Q(s, \gamma)+|us|^\sigma)$$

uniformly in u, $-\infty<u<\infty$, *as* $s\to 0$.

Proof. We write

$$\phi(u, s) = 1+ius\,E\{g'(X_1+s)\}+E\{\exp[iusg'(X_1+s)]-1-iusg'(X_1+s)\}\,. \tag{3.4}$$

For $a>0$ let

$$s\,E\{g'(X_1+s)\} = \left(\int_0^a+\int_a^\infty\right)s\{g'(x+s)-g'(x)\}f(x)\,dx$$
$$=J_1+J_2, \text{ say.}$$

To evaluate the integral J_1, we express it in the form $J_{11}+J_{12}+J_{13}$, where

$$J_{11} = \int_0^a s\alpha\beta(\alpha-1)\,[(x+s)^{-1}-x^{-1}]x^{\alpha-1}\,dx\,,$$
$$J_{12} = \int_0^a s(\alpha-1)\,[(x+s)^{-1}-x^{-1}]\,(f(x)-\alpha\beta x^{\alpha-1})\,dx\,,$$
$$J_{13} = \int_0^a s\{g'(x+s)-g'(x)-(\alpha-1)\,[(x+s)^{-1}-x^{-1}]\}f(x)\,dx\,.$$

If a is chosen small enough, then easy computations show that

$$J_{11} = -\alpha\beta s^\alpha\Gamma(\alpha)\Gamma(2-\alpha)+O(s^2)\,,$$
$$J_{12} = O(Q(s, \gamma))\,.$$

Since

$$|g'(x+s)-g'(x)-(\alpha-1)\,[(x+s)^{-1}-x^{-1}]| = \left|\int_x^{x+s}[g''(y)+(\alpha-1)y^{-2}]\,dy\right|$$
$$\leqq \int_x^{x+s}cy^{\gamma-2}\,dy$$

because of condition (iii), from Fubini's theorem we see that

$$J_{13} = O(Q(s, \gamma))\,.$$

It follows from condition (v) (b) that $J_2=O(s^2)$, and consequently

$$ius\,E\{g'(X_1+s)\} = -ium_\alpha\beta s^\alpha+O(|u|Q(s, \gamma))\,. \tag{3.5}$$

Next, we go on to evaluate the third term on the right side of (3.4), expressing it in the form J_3+J_4, where J_3 and J_4 are the integrals over the intervals $(0, a)$ and $[a, \infty)$, respectively. Let divide J_3 into J_{31}–J_{33} as follows:

$$J_{31} = \int_0^a\alpha\beta\{\exp[ius(\alpha-1)/(x+s)]-1-ius[(\alpha-1)/(x+s)]\}x^{\alpha-1}\,dx\,,$$

$$J_{32} = \int_0^a \{\exp[ius(\alpha-1)/(x+s)]-1-ius[(\alpha-1)/(x+s)]\}(f(x)-\alpha\beta x^{\alpha-1})dx\,,$$

$$J_{33} = \int_0^a \{\exp[iusg'(x+s)]-\exp[ius(\alpha-1)/(x+s)] -ius(g'(x+s)-[(\alpha-1)/(x+s)])\}f(x)dx\,.$$

Simple calculations give

$$J_{31} = \alpha(\alpha-1)\beta s^{\alpha}\int_{s(\alpha-1)/(a+s)}^{\alpha-1}(e^{iux}-1-iux)\,[(\alpha-1)x^{-1}-1]^{\alpha-1}x^{-2}dx\,,$$

so that (3.2) with $\rho=\sigma$ implies that

$$J_{31} = -\alpha(\alpha-1)\beta s^{\alpha}h(u)+O(|us|^{\sigma})\,.$$

Similarly,

$$\begin{aligned}|J_{32}| &\leq \int_0^a c|us|^{\sigma}(x+s)^{-\sigma}x^{\alpha+\gamma-1}dx \\ &= c|u|^{\sigma}s^{\alpha+\gamma}\int_0^{a/s}(1+x)^{-\sigma}x^{\alpha+\gamma-1}dx \\ &= O(|us|^{\sigma})\,.\end{aligned}$$

The estimate of J_{33} is the same as J_{32} except that the inequality (3.3) is used instead of (3.2). Thus we have

$$(3.6)\qquad J_3 = -\alpha(\alpha-1)\beta s^{\alpha}h(u)+O(|us|^{\sigma})\,.$$

It is easily seen that $J_4=O(|us|^{\sigma})$ because of (3.2) and condition (v) (a). This together with (3.4)–(3.6) implies the desired assertion.

Lemma 5. *If the conditions of Theorem* 3 *are satisfied, then for every* $0<\lambda<\gamma$

$$\phi^*(u,s) = 1+ium_{\alpha}{}^*\beta s^{\alpha}-\alpha(\alpha-1)\beta s^{\alpha}h^*(u) +O(|u|^{1/2}s^{\min(2\alpha,\alpha+\gamma)}+|u|Q(s,\gamma)+|us|^{\sigma})$$

uniformly in u, $-\infty<u<\infty$, *as* $s\to 0$.

Proof. We may write

$$(3.7)\qquad \phi^*(u,s) = 1+iu\,E^*\{\sin[sg'(X_1-s)]\} +E^*\{\exp[iusg'(X_1-s)]-1-iu\sin[sg'(X_1-s)]\}\,.$$

To estimate the second term on the right side of (3.7), we express it in the form $iul(s)\,(K_1+K_2+K_3)$, where

$$K_1 = \int_s^a \{\sin[sg'(x-s)]-sg'(x)\}f(x)dx\,,$$

$$K_2 = \int_a^\infty \{\sin[sg'(x-s)] - sg'(x)\} f(x)\,dx\,,$$
$$K_3 = \int_s^\infty sg'(x)f(x)\,dx$$

with sufficiently small $a>0$. Moreover, let divide K_1 into K_{11}–K_{13} as follows:

$$K_{11} = \int_s^a \alpha\beta\{\sin[s(\alpha-1)/(x-s)] - [s(\alpha-1)/x]\} x^{\alpha-1}\,dx\,,$$
$$K_{12} = \int_s^a \{\sin[s(\alpha-1)/(x-s)] - [s(\alpha-1)/x]\}(f(x) - \alpha\beta x^{\alpha-1})\,dx\,,$$
$$K_{13} = \int_s^a \{\sin[sg'(x-s)] - \sin[s(\alpha-1)/(x-s)] - s[g'(x) - (\alpha-1)x^{-1}]\} f(x)\,dx\,.$$

Using the inequality

$$|\sin[(\alpha-1)/(x-1)] - [(\alpha-1)/x]| \leqq (\alpha-1)^2/[2(x-1)^2] + (\alpha-1)/[x(x-1)]\,,$$

we obtain

$$K_{11} = \alpha\beta s^\alpha \int_1^\infty \{\sin[(\alpha-1)/(x-1)] - [(\alpha-1)/x]\} x^{\alpha-1}\,dx + O(s^2)\,,$$
$$K_{12} = O(Q(s,\,\gamma))\,.$$

We write $K_{13} = K_{131} + K_{132}$, where

$$K_{131} = \int_s^a \{\sin[sg'(x-s)] - \sin[s(\alpha-1)/(x-s)] - s[g'(x-s) - (\alpha-1)(x-s)^{-1}]\} f(x)\,dx\,,$$
$$K_{132} = \int_s^a s\{g'(x-s) - g'(x) + (\alpha-1)[x^{-1} - (x-s)^{-1}]\} f(x)\,dx\,.$$

Suppose $\alpha+\gamma>2$ and put $\zeta = (\alpha+\gamma-1)^{-1} < 1$. Since

(3.8) $\qquad |\sin(x) - \sin(y) - x + y| \leqq \min(2|x-y|,\ |x^2-y^2|/2), \quad \text{for } x>0,\ y>0\,,$

it follows that

$$|K_{131}| \leqq \int_s^{s^\zeta} cs(x-s)^{\gamma-1} x^{\alpha-1}\,dx + \int_{s^\zeta}^a cs^2(x-s)^{\gamma-2} x^{\alpha-1}\,dx = O(s^2)\,.$$

If $\alpha+\gamma \leqq 2$, then

$$|K_{131}| \leqq \int_s^{2s} cs(x-s)^{\gamma-1} x^{\alpha-1}\,dx + \int_{2s}^a cs^2(x-s)^{\gamma-2} x^{\alpha-1}\,dx = O(Q(s,\,\gamma))\,.$$

Furthermore, the estimate of K_{132} is similar to J_{13}, so that

$$K_{13} = O(Q(s,\,\gamma))\,.$$

It is easy to see that $K_2 = O(s^2)$ because of condition (v), and condition (iii) implies

$$K_3 = -s\int_0^s g'(x)f(x)\,dx = -\alpha\beta s^{\alpha}+O(s^{\alpha+\gamma})\,.$$

Thus, taking account of (3.1) we find that

$$iu\,E^*\{\sin[sg'(X_1-s)]\} = ium_{\alpha}{}^*\beta s^{\alpha}+O(|u|Q(s,\gamma))\,. \tag{3.9}$$

We express the third term on the right side of (3.7) in the form $l(s)(K_4+K_5)$, where

$$K_4 = \int_s^a \{\exp[iusg'(x-s)]-1-iu\sin[sg'(x-s)]\}f(x)\,dx\,,$$

$$K_5 = \int_a^\infty \{\exp[iusg'(x-s)]-1-iu\sin[sg'(x-s)]\}f(x)\,dx\,,$$

with sufficiently small $a>0$. To evaluate K_4 divide it into $K_{41}-K_{43}$ as follows:

$$K_{41} = \int_s^a \alpha\beta\{\exp[ius(\alpha-1)/(x-s)]-1-iu\sin[s(\alpha-1)/(x-s)]\}x^{\alpha-1}dx\,,$$

$$K_{42} = \int_s^a \{\exp[ius(\alpha-1)/(x-s)]-1-iu\sin[s(\alpha-1)/(x-s)]\}(f(x)-\alpha\beta x^{\alpha-1})\,dx\,,$$

$$K_{43} = \int_s^a \{\exp[iusg'(x-s)-\exp[ius(\alpha-1)/(x-s)]$$
$$-iu(\sin[sg'(x-s)]-\sin[s(\alpha-1)/(x-s)])\}f(x)\,dx\,.$$

Since for real x, u and $1<\rho\leqq 2$

$$|e^{iux}-1-iu\sin(x)|\leqq|ux|^{\rho}/[\rho(\rho-1)]+|u|x^2/2\,,$$

if we choose $\rho=\sigma$, then

$$K_{41} = -\alpha(\alpha-1)\beta s^{\alpha}h^*(u)+O(|u|s^2+|us|^{\sigma}) \qquad (\text{see } J_{31}).$$

Also, the above inequality and the inequality $|e^{iux}-1|\leqq 2|ux|^{1/2}$ imply that

$$|K_{42}|\leqq cs^{\alpha+\gamma}\int_{s(\alpha-1)/(a-s)}^{\infty}|e^{iux}-1-iu\sin(x)|[(\alpha-1)x^{-1}+1]^{\alpha+\gamma-1}x^{-2}dx$$
$$\leqq cs^{\alpha+\gamma}\{\int_{s(\alpha-1)/(a-s)}^{1}(|u|x^{1-\alpha-\gamma}+|u|^{\sigma}x^{\sigma-\alpha-\gamma-1})\,dx+\int_1^\infty(|ux|^{1/2}+|u|)x^{-2}dx\}$$
$$\leqq c(|u|^{1/2}s^{\alpha+\gamma}+|u|Q(s,\gamma)+|us|^{\sigma})\,.$$

By (3.3) and (3.8), the integrand of K_{43} does not exceed

$$c\{|u|s^2(x-s)^{\gamma-2}+|us|^{\sigma}(x-s)^{\gamma-\sigma}\}$$

in absolute value. On the other hand, it does not exceed $c|us|(x-s)^{\gamma-1}$ in absolute value, since $|e^{ix}-e^{iy}|\leqq|x-y|$ and $|\sin(x)-\sin(y)|\leqq|x-y|$. As in the estimate of K_{13} choose $\zeta=(\alpha+\gamma-1)^{-1}$ for $\alpha+\gamma>2$, then

$$|K_{43}| \leqq \int_s^{s\zeta} c|us|(x-s)^{\gamma-1}x^{\alpha-1}dx + \int_{s\zeta}^a c(|u|+u^2)s^2(x-s)^{\gamma-2}x^{\alpha-1}dx$$
$$= O((|u|+u^2)s^2).$$

If $\alpha+\gamma \leqq 2$, then

$$|K_{43}| \leqq \int_s^{2s} c|us|(x-s)^{\gamma-1}x^{\alpha-1}dx + \int_{2s}^a c\{|u|s^2(x-s)^{\gamma-2} + |us|^\sigma(x-s)^{\gamma-\sigma}\}x^{\alpha-1}dx$$
$$= O(|u|Q(s, \gamma) + |us|^\sigma),$$

so that

$$K_4 = -\alpha(\alpha-1)\beta s^\alpha h^*(u) + O(|u|^{1/2}s^{\alpha+\gamma} + |u|Q(s, \gamma) + |us|^\sigma).$$

From condition (v) (a) and (3.2) with $\rho=\sigma$ it follows easily that

$$K_5 = O(|u|s^2 + |us|^\sigma).$$

Thus these estimates together with (3.1) yield

$$\text{(3.10)}\quad E^*\{\exp[iusg'(X_1-s)] - 1 - iu\sin[sg'(X_1-s)]\}$$
$$= -\alpha(\alpha-1)\beta s^\alpha h^*(u) + O(|u|^{1/2}s^{\min(2\alpha,\alpha+\gamma)} + |u|Q(s, \gamma) + |us|^\sigma),$$

because $|h^*(u)| \leqq c(|u|^{1/2} + |u|^\sigma)$ for every u (cf. K_{42}). Now Lemma 5 follows from (3.7), (3.9) and (3.10).

Let $k(u)$ and $k^*(u)$ be the real parts of $h(u)$ and $h^*(u)$, respectively, and for $p>-1$ and $t>0$ define

$$\Lambda_p(t) = \int_{-\infty}^{\infty} |u|^p \exp\{-\frac{1}{2}\alpha(\alpha-1)t^\alpha k(u)\}\,du,$$
$$\Lambda_p^*(t) = \int_{-\infty}^{\infty} |u|^p \exp\{-\frac{1}{2}\alpha(\alpha-1)t^\alpha k^*(u)\}\,du.$$

$\Lambda_p(t)$ and $\Lambda_p^*(t)$ are continuous in $t>0$. Moreover, we have the following

Lemma 6.

(1) $\lim_{t\to 0} t^{p+1}\Lambda_p(t) = \lim_{t\to 0} t^{p+1}\Lambda_p^*(t) = \int_{-\infty}^{\infty} |v|^p \exp\{-\frac{1}{2}\alpha(\alpha-1)\bar{k}|v|^\alpha\}dv,$

where $\bar{k} = (\alpha-1)^{\alpha-1}\int_0^\infty [1-\cos(x)]x^{-\alpha-1}dx > 0.$

(2) $\Lambda_p(t) = O(t^{-\alpha(p+1)/2})$ as $t \to \infty$,

and $\lim_{t\to\infty} t^{\alpha(p+1)}\Lambda_p^*(t) = \int_{-\infty}^{\infty} |v|^p \exp\{-\frac{1}{4}\pi\alpha(\alpha-1)|v|\}dv.$

Proof. (1) Since

$$\text{(3.11)}\quad \lim_{|u|\to\infty} |u|^{-\alpha}k(u) = (\alpha-1)^{\alpha-1}\int_0^\infty [1-\cos(x)]x^{-\alpha-1}dx = \bar{k}$$

(see [7], 478) and $t^{\alpha}k(t^{-1}v)\geqq k(v)$ for all v and $0<t\leqq 1$, it follows from the dominated convergence theorem that as $t\to 0$

$$t^{p+1}\Lambda_p(t)=\int_{-\infty}^{\infty}|v|^p\exp\{-\frac{1}{2}\alpha(\alpha-1)t^{\alpha}k(t^{-1}v)\}dv$$
$$\to\int_{-\infty}^{\infty}|v|^p\exp\{-\frac{1}{2}\alpha(\alpha-1)\bar{k}|v|^{\alpha}\}dv\,.$$

The same result is valid for $\Lambda_p{}^*$ because $\lim_{|u|\to\infty}|u|^{-\alpha}k^*(u)=\bar{k}$ and $t^{\alpha}k^*(t^{-1}v)\geqq\bar{k}|v|^{\alpha}$ for all v and $t>0$.

(2) For $0<\eta<1$, there exists $\delta>0$ such that

$$u^2x^2(1-\eta)/2\leqq 1-\cos(ux)\leqq u^2x^2/2$$

for $0<x<\alpha-1$ and $|u|\leqq\delta$. Then we have

$$\frac{1-\eta}{2}\int_0^{\alpha-1}[(\alpha-1)x^{-1}-1]^{\alpha-1}dx\leqq u^{-2}k(u)\leqq\frac{1}{2}\int_0^{\alpha-1}[(\alpha-1)x^{-1}-1]^{\alpha-1}dx\,.$$

Since $\eta>0$ was arbitrary, it follows easily that

$$\lim_{u\to 0}u^{-2}k(u)=\frac{1}{2}\int_0^{\alpha-1}[(\alpha-1)x^{-1}-1]^{\alpha-1}dx=\frac{1}{2}(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)\,.$$

This together with (3.11) implies that $\Lambda_p(t)=O(t^{-\alpha(p+1)/2})$ as $t\to\infty$.

Finally, we have that

$$\lim_{u\to 0}|u|^{-1}k^*(u)=\int_0^{\infty}[1-\cos(x)]x^{-2}dx=\pi/2$$

and $t^{\alpha}k^*(t^{-\alpha}v)\geqq\pi|v|/2$ for all v and $t>0$. The remaining part of the proof is identical with that of the assertion (1).

Lemma 7. *There exists $C>0$ such that for every $t>0$*

(1) $$\sup_{-\infty<x<\infty}|dG_t(x)/dx|\leqq C(1+t^{-1})\,,$$

(2) $$\sup_{-\infty<x<\infty}|dG_t{}^*(x)/dx|\leqq C(1+t^{-1})\,.$$

Proof. Applying the well-known theorem (Kawata [1], Theorem 11.6.2) we see that

$$\sup_{-\infty<x<\infty}|dG_t(x)/dx|\leqq(2\pi)^{-1}\int_{-\infty}^{\infty}|\xi_t(u)|du$$
$$=(2\pi)^{-1}\int_{-\infty}^{\infty}\exp\{-\alpha(\alpha-1)t^{\alpha}k(u)\}du\,.$$

This implies the assertion (1) because of Lemma 6 (with an obvious modification). In the same way we can show the assertion (2).

4. Proofs of Theorems 1–3. For $0<\lambda\leqq\gamma$ let define $T_{n,\gamma,\lambda}$ as follows

$$T_{n,\gamma,\lambda}=n^{(\sigma-\alpha)/\alpha}$$

with $\sigma=\sigma(\gamma, \lambda)$ as in the statement of Lemma 4.

Proof of Theorem 1. Let $\psi_{n,0}$ denote the characteristic function of $F_{n,0}$, then

$$\begin{aligned}(4.1)\qquad \log\psi_{n,0}(u) &= n\log\phi(b_n^{-1}u)\\ &= n[\phi(b_n^{-1}u)-1]+n\tau_n(u)\,[\phi(b_n^{-1}u)-1]^2,\end{aligned}$$

where

$$\tau_n(u)=-\frac{1}{2}+\frac{1}{3}[\phi(b_n^{-1}u)-1]-\frac{1}{4}[\phi(b_n^{-1}u)-1]^2+\cdots.$$

It is easily checked that if $|\phi(b_n^{-1}u)-1|\leqq\frac{1}{3}$, then $|\tau_n(u)|\leqq\frac{2}{3}$. Suppose $|u|\leqq T_{n,\gamma,\gamma}$. Since $b_n^{-1}u\to 0$ uniformly in $|u|\leqq T_{n,\gamma,\gamma}$ as $n\to\infty$, it follows from Lemma 3 that as $n\to\infty$, uniformly in $|u|\leqq T_{n,\gamma,\gamma}$,

$$(4.2)\qquad n[\phi(b_n^{-1}u)-1]=-d|u|^{\alpha}[1-i\,\mathrm{sign}(u)\tan(\pi\alpha/2)]+O(nQ(b_n^{-1}u, \gamma)),$$

where

$$nQ(b_n^{-1}u, \gamma)=\begin{cases}\beta^{-2/\alpha}u^2(T_{n,\gamma,\gamma})^{-1}, & \alpha+\gamma>2,\\ \beta^{-2/\alpha}u^2(T_{n,\gamma,\gamma})^{-1}|\log|b_n^{-1}u||, & \alpha+\gamma=2,\\ \beta^{-(\alpha+\gamma)/\alpha}|u|^{\alpha+\gamma}(T_{n,\gamma,\gamma})^{-1}, & \alpha+\gamma<2.\end{cases}$$

Also, we have, uniformly in $|u|\leqq T_{n,\gamma,\gamma}$,

$$|\phi(b_n^{-1}u)-1|\leqq c\{|u|^{\alpha}n^{-1}+Q(b_n^{-1}u, \gamma)\}=o(1)$$

and

$$n|\phi(b_n^{-1}u)-1|^2=o(nQ(b_n^{-1}u, \gamma)).$$

Taking account of (4.1) and (4.2) it follows that for $|u|\leqq T_{n,\gamma,\gamma}$

$$\log\psi_{n,0}(u)=-d|u|^{\alpha}[1-i\,\mathrm{sign}(u)\tan(\pi\alpha/2)]+S_n(u, \gamma),$$

where $S_n(u, \gamma)=O(nQ(b_n^{-1}u, \gamma))$ uniformly in $|u|\leqq T_{n,\gamma,\gamma}$ as $n\to\infty$.

Using the inequality $|1-e^x|\leqq|x|e^{|x|}$ for complex number x, it follows that for $|u|\leqq T_{n,\gamma,\gamma}$

$$\begin{aligned}|\psi_{n,0}(u)-\xi_0(u)| &\leqq |S_n(u, \gamma)|\exp\{-d|u|^{\alpha}+|S_n(u, \gamma)|\}\\ &\leqq |S_n(u, \gamma)|\exp\{-d|u|^{\alpha}/2\}.\end{aligned}$$

Here we used the fact that $|S_n(u, \gamma)|\leqq d|u|^{\alpha}/2$ for $|u|\leqq T_{n,\gamma,\gamma}$ with all sufficiently large n. Since $|\xi_0(u)|$ is integrable, G_0 has a bounded continuous derivative. Thus, from Esseen's inequality (see Petrov [4], p. 109, Theorem 2) we obtain

$$\Delta(F_{n,0}, G_0) \leqq \int_{|u| \leqq T_{n,\gamma,\gamma}} \left| \frac{\psi_{n,0}(u) - \xi_0(u)}{u} \right| du + O((T_{n,\gamma,\gamma})^{-1}),$$

which implies the desired assertion.

Proof of Theorem 2. Let $\psi_{n,t}$ denote the characteristic function of $F_{n,t}$, then

$$\log \psi_{n,t}(u) = n \log \phi(u, b_n^{-1}t).$$

Noting that $b_n^{-1}t \to 0$ uniformly in $0 < t \leqq (\log n)^{1/\alpha}$ as $n \to \infty$, it follows from Lemma 4 that as $n \to \infty$, uniformly in u and $0 < t \leqq (\log n)^{1/\alpha}$,

$$n[\phi(u, b_n^{-1}t) - 1] = -ium_\alpha t^\alpha - \alpha(\alpha-1)t^\alpha h(u) + O(t^\sigma[|u| + |u|^\sigma](T_{n,\gamma,\lambda})^{-1}).$$

As in the proof of Theorem 1 it remains to estimate $[\phi(u, b_n^{-1}t) - 1]$ and $n[\phi(u, b_n^{-1}t) - 1]^2$. Since $k(u) = O(|u|^\alpha)$ as $|u| \to \infty$ and

$$\lim_{|u| \to \infty} |u|^{-\alpha} \int_0^{\alpha-1} |ux - \sin(ux)| ((\alpha-1)x^{-1} - 1)^{\alpha-1} x^{-2} dx$$
$$= (\alpha-1)^{\alpha-1} \int_0^\infty [x - \sin(x)] x^{-\alpha-1} dx,$$

there exists $L > 1$ such that $|h(u)| \leqq c|u|^\alpha$ for every $|u| > L$. Suppose $L < |u| \leqq (1+t^{-1})T_{n,\gamma,\lambda}$ for $0 < t \leqq (\log n)^{1/\alpha}$. Then Lemma 4 implies that

$$|\phi(u, b_n^{-1}t) - 1| \leqq c[|tu|^\alpha n^{-1} + |tu|^\sigma (nT_{n,\gamma,\lambda})^{-1}].$$

From the inequalities

$$|tu|^\alpha n^{-1} \leqq (1+t)^\alpha n^{-1} (T_{n,\gamma,\lambda})^\alpha,$$
$$|tu|^\sigma (nT_{n,\gamma,\lambda})^{-1} \leqq (1+t)^\sigma n^{-1} (T_{n,\gamma,\lambda})^{\sigma-1},$$

it follows that $|\phi(u, b_n^{-1}t) - 1| = o(1)$. Similarly, for $L < |u| \leqq (1+t^{-1})T_{n,\gamma,\lambda}$ we have

$$n|\phi(u, b_n^{-1}t) - 1|^2 \leqq c[|tu|^{2\alpha} n^{-1} + |tu|^{2\sigma} n^{-1} (T_{n,\gamma,\lambda})^{-2}],$$

which implies $n|\phi(u, b_n^{-1}t) - 1|^2 = o(|tu|^\sigma (T_{n,\gamma,\lambda})^{-1})$ because

$$|tu|^{2\alpha} n^{-1} = |tu|^\sigma (T_{n,\gamma,\lambda})^{-1} [|tu|^{2\alpha-\sigma} n^{-1} T_{n,\gamma,\lambda}] = o(|tu|^\sigma (T_{n,\gamma,\lambda})^{-1}),$$
$$|tu|^{2\sigma} n^{-1} (T_{n,\gamma,\lambda})^{-2} = |tu|^\sigma (T_{n,\gamma,\lambda})^{-1} [|tu|^\sigma (nT_{n,\gamma,\lambda})^{-1}] = o(|tu|^\sigma (T_{n,\gamma,\lambda})^{-1}).$$

Next, suppose $|u| \leqq L$. Since $|h(u)| \leqq cu^2$ for all u, a similar argument will show that $|\phi(u, b_n^{-1}t) - 1| = o(1)$ and $n|\phi(u, b_n^{-1}t) - 1|^2 = o(|tu|^\sigma (T_{n,\gamma,\lambda})^{-1})$. Therefore, we have for $0<t\leqq(\log n)^{1/\alpha}$ and $|u|\leqq(1+t^{-1})T_{n,\gamma,\lambda}$

$$\log \psi_{n,t}(u)=-ium_{\alpha}t^{\alpha}-\alpha(\alpha-1)t^{\alpha}h(u)+S_n(u,t,\gamma,\lambda),$$

where $S_n(u,t,\gamma,\lambda)=O(t^{\sigma}[|u|+|u|^{\sigma}](T_{n,\gamma,\lambda})^{-1})$ uniformly in t and u as $n\to\infty$.

Let $L>1$. Since for $0<t\leqq(\log n)^{1/\alpha}$ and $L<|u|\leqq(1+t^{-1})T_{n,\gamma,\lambda}$

$$\begin{aligned}|S_n(u,t,\gamma,\lambda)|&\leqq c|tu|^{\alpha}[|tu|^{\sigma-\alpha}(T_{n,\gamma,\lambda})^{-1}]\\&\leqq c|tu|^{\alpha}[(1+t)^{\sigma-\alpha}(T_{n,\gamma,\lambda})^{\sigma-\alpha-1}],\end{aligned}$$

it follows that $|S_n(u,t,\gamma,\lambda)|\leqq\alpha(\alpha-1)t^{\alpha}k(u)/2$ for sufficiently large L and all sufficiently large n. This implies that

$$|\psi_{n,t}(u)-\xi_t(u)|\leqq\begin{cases}2|S_n(u,t,\gamma,\lambda)|\exp\{-\alpha(\alpha-1)t^{\alpha}k(u)\}, & |u|\leqq L,\\ |S_n(u,t,\gamma,\lambda)|\exp\{-\frac{1}{2}\alpha(\alpha-1)t^{\alpha}k(u)\}, & \\ & L<|u|\leqq(1+t^{-1})T_{n,\gamma,\lambda},\end{cases}$$

so that

$$|\psi_{n,t}(u)-\xi_t(u)|\leqq2|S_n(u,t,\gamma,\lambda)|\exp\{-\frac{1}{2}\alpha(\alpha-1)t^{\alpha}k(u)\},\ |u|\leqq(1+t^{-1})T_{n,\gamma,\lambda}.$$

(To show the first inequality, use the obvious inequality $|1-e^x|\leqq2|x|$ for $|x|<1/2$.) Applying the Esseen's inequality we have for $0<t\leqq(\log n)^{1/\alpha}$

$$\begin{aligned}\Delta(F_{n,t},G_t)&\leqq ct^{\sigma}(T_{n,\gamma,\lambda})^{-1}\int_{|u|\leqq(1+t^{-1})T_{n,\gamma,\lambda}}(1+|u|^{\sigma-1})\exp\{-\frac{1}{2}\alpha(\alpha-1)t^{\alpha}k(u)\}du\\&\quad+c[\sup_{-\infty<x<\infty}|dG_t(x)/dt|]\,[(1+t^{-1})T_{n,\gamma,\lambda}]^{-1}\\&\leqq c(1+t^{(2\sigma-\alpha)/2})(T_{n,\gamma,\lambda})^{-1}.\end{aligned}$$

Here Lemma 6 and Lemma 7 were used to lead the last inequality.

Proof of Theorem 3. Let $\psi^*_{n,t}(u)$ be the characteristic function of $F^*_{n,t}$. Since $k^*(u)=O(|u|^{\alpha})$ as $|u|\to\infty$ and for $|u|>1$

$$\begin{aligned}&\int_0^{\infty}|u\sin(x)-\sin(ux)|[(\alpha-1)x^{-1}+1]^{\alpha-1}x^{-2}dx\\&=|u|^{\alpha}\int_0^{\infty}||u|\sin(x/|u|)-\sin(x)|[(\alpha-1)x^{-1}+|u|^{-1}]^{\alpha-1}x^{-2}dx\\&\leqq|u|^{\alpha}\{\int_0^1(\alpha/x)^{\alpha-1}dx+\int_1^{|u|}(\alpha/x)^{\alpha-1}(1+x)x^{-2}dx+\int_{|u|}^{\infty}(\alpha/|u|)^{\alpha-1}(1+|u|)x^{-2}dx\}\\&\leqq c|u|^{\alpha},\end{aligned}$$

it follows that $h^*(u)=O(|u|^{\alpha})$ as $|u|\to\infty$. Moreover, it has already shown that $|h^*(u)|\leqq c(|u|^{1/2}+|u|^{\sigma})$. Thus, using the same method employed for the derivation of $\log\psi_{n,t}(u)$, we see that for $0<t\leqq(\log n)^{1/\alpha}$ and $|u|\leqq(1+t^{-1})T_{n,\gamma,\lambda}$

$$\log \psi_{n,t}^*(u) = ium_\alpha{}^* t^\alpha - \alpha(\alpha-1)t^\alpha h^*(u) + S_n{}^*(u, t, \gamma, \lambda),$$

where

$$S_n{}^*(u, t, \gamma, \lambda) = O(t^{\min(2\alpha, \alpha+\gamma)} n^{-\min(1, \gamma/\alpha)} |u|^{1/2} + t^\sigma[|u| + |u|^\sigma] (T_{n,\gamma,\lambda})^{-1})$$

uniformly in t and u as $n \to \infty$. And consequently, we have

$$|\psi_{n,t}^*(u) - \xi_t{}^*(u)| \leqq 2|S_n{}^*(u, t, \gamma, \lambda)| \exp\{-\frac{1}{2}\alpha(\alpha-1)t^\alpha k^*(u)\}$$

for $0 < t \leqq (\log n)^{1/\alpha}$ and $|u| \leqq (1+t^{-1})T_{n,\gamma,\lambda}$. Now, Theorem 3 follows easily from the Esseen's inequality, Lemma 6 and Lemma 7.

5. Remarks. For real t let

$$D(t) = |P_\theta\{b_n(\hat{\theta}_n - \theta) \leqq t\} - H(t)|.$$

(It is obvious that $D(t)$ does not depend on θ.) Theorem 4 gives an upper bound for $\sup\{D(t); -\infty < t < \infty\}$. In this section we investigate an upper bound for $D(t)$ which depends on t. At first, it follows from Theorem 1 and (2.1) that

$$D(0) = \begin{cases} O(n^{-(2-\alpha)/\alpha}), & \alpha+\gamma > 2, \\ O(n^{-\gamma/\alpha} \log n), & \alpha+\gamma = 2, \\ O(n^{-\gamma/\alpha}), & \alpha+\gamma < 2. \end{cases}$$

For $0 \neq |t| \leqq (\log n)^{1/\alpha}$, (2.4) and (2.6) give the bounds for $D(t)$. However, in the case $\alpha+\gamma \leqq 2$, we can improve (2.4b) and (2.6b). For this purpose we need the following lemmas instead of Lemma 4 and Lemma 5. We shall omit the proofs because the lemmas may be proved analogously to Lemma 4 and Lemma 5, respectively. Hereafter, suppose that $1 < \alpha < 2$, $\gamma > 0$ and $\alpha+\gamma \leqq 2$.

Lemma 4′. *If the conditions* (i)–(iii) *and* (v) *be satisfied, then*

$$\phi(u, s) = 1 - ium_\alpha \beta s^\alpha - \alpha(\alpha-1)\beta s^\alpha h(u) + O(|u| Q(s, \gamma) + |us|^{\alpha+\gamma} |\log s|)$$

uniformly in u as $s \to 0$.

Lemma 5′. *If the conditions* (i)–(iii) *and* (v) *be satisfied, then*

$$\phi^*(u, s) = 1 + ium_\alpha{}^* \beta s^\alpha - \alpha(\alpha-1)\beta s^\alpha h^*(u) + O(|u|^{1/2} s^{\min(2\alpha, \alpha+\gamma)} + |u| Q(s, \gamma) + |us|^{\alpha+\gamma} |\log s|)$$

uniformly in u as $s \to 0$.

Analogously to the proofs of Theorem 2 and Theorem 3, the following Theorems 2′ and Theorem 3′ are derived from Lemma 4′ and Lemma 5′, respectively, if $(1+t^{-1})T_{n,\gamma,\lambda}$ is replaced by $(1+t^{-1})|\log[\min(t, e^{-1})]|^{-1/\gamma}n^{\gamma/\alpha}$.

Theorem 2′. *Let conditions* (i)–(iii) *and* (v) *be satisfied, then there exists* $C>0$ *such that for all* $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/\alpha}$

$$\Delta(F_{n,t}, G_t)\leqq C\, n^{-\gamma/\alpha}[(1+t^{(\alpha+2\gamma)/2})\log n+|\log t|^{1/\gamma}]\,.$$

Theorem 3′. *Let conditions* (i)–(iii) *and* (v) *be satisfied, then there exists* $C>0$ *such that for all* $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/\alpha}$

$$\Delta(F_{n,t}^*, G_t^*)\leqq C\, n^{-\gamma/\alpha}[(1+t^{\gamma})\log n+|\log t|^{1/\gamma}]\,.$$

The following corollary is an immediate consequence of (2.1), (2.2), (2.5), Theorem 2′ and Theorem 3′.

Corollary 2. *Suppose that conditions* (i)–(v) *hold. Then there exists* $C>0$ *such that for all* $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/\alpha}$

$$D(-t)\leqq C\, n^{-\gamma/\alpha}[(1+t^{(\alpha+2\gamma)/2})\log n+|\log t|^{1/\gamma}]\,,$$
$$D(t)\leqq C\, n^{-\gamma/\alpha}(\log n+|\log t|^{1/\gamma})\,.$$

Corollary 2 together with (2.3) and (2.9) yields the following result which is a refinement of Corollary 1 in the case $\alpha+\gamma\leqq 2$.

Corollary 3. *Suppose that conditions* (i)–(v) *hold. Then there exists* $C>0$ *such that for all* θ *and* $n\geqq 1$

$$P_\theta\{b_n|\hat{\theta}_n-\theta|\geqq(\log n)^{1/\alpha}\}\leqq Cn^{-\gamma/\alpha}(\log n)^{(3\alpha+2\gamma)/2\alpha}\,.$$

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