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SOME REMARKS ON THE CAUCHY PROBLEM FOR SCHRÖDINGER TYPE EQUATIONS

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Dedicated to the memory of Professor Hitoshi Kumano-go

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0. Introduction

In the present paper we consider the Cauchy problem for the following equation

$$(0.1)\qquad Lu \equiv (i\partial_t+\tau\Delta+\sum_{j=1}^{m} b_j(x)\partial_{x_j}+c(x))u(x, t) = 0$$

with initial data $u_0(x)$ at $t=0$, where τ is a constant such that $0\leqq\tau\leqq 1$, and $b_j(x)$, $c(x)$ belong to $\mathcal{B}^\infty(R^m_x)$. $\mathcal{B}^\infty(R^m_x)$ denotes the set of C^∞-functions whose derivatives of any order are all bounded. If τ is positive, the above equation (0.1) is the typical equation of non-kowalewskian type which is not parabolic. The study of the equation (0.1) is important for the study of equations of general non-kowalewskian type.

For real s let H_s be the Sobolev space with the usual norm $||\cdot||_s$ and let $H_\infty\equiv\bigcap_{s\in R} H_s$ be the Fréchet space with semi-norms $||\cdot||_s$, $s=0, \pm 1, \pm 2, \cdots$. We say that the Cauchy problem for (0.1) is well posed for the future (resp. for the past) in the space H_∞, if there exists a constant $T>0$ (resp. $T<0$) such that for any initial data $u_0(x)\in H_\infty$ a unique solution $u(x, t)\in\mathcal{E}^0_t([0, T]; H_\infty)$ of (0.1), which takes $u_0(x)$ at $t=0$, exists. Here, $f(x, t)\in\mathcal{E}^0_t([0, T]; H_\infty)$ means that the mapping: $[0, T]\ni t\to f(x, t)\in H_\infty$ is continuous in the topology of H_∞.

Our purpose is to prove the following theorem corresponding to the so-called Lax-Mizohata theorem for equations of kowalewskian type (Lax [5], Mizohata [6]).

Theorem. *In order that equation* (0.1) *is well posed for the future or for the past in the space* H_∞, *it is necessary that there exist constants* M *and* N *such that the inequality*

$$(0.2)\qquad \sup_{x\in R^m,\omega\in S^{m-1}}\Big|\sum_{j=1}^{m}\int_0^\rho \mathrm{Re}\, b_j(x+2\tau\theta\omega)\omega_j d\theta\Big| \leqq M\log(1+\rho)+N$$

holds for any $\rho\geqq 0$. S^{m-1} *denotes the unit sphere in* R^m.

REMARK 1. J. Takeuchi in [8] first studied the Cauchy problem for equations of non-kowalewskian type in the frame of L^2 space.

REMARK 2. S. Mizohata in [7] proves the following. It is necessary for (0.1) to be well posed in the space L^2 that the inequality (0.2) with $M=0$ holds for any $\rho\geqq 0$. He proves it by constructing the asymptotic solution based on Birkhoff [1]. In the present paper we use the energy method.

REMARK 3. The author in [3] has given a sufficient condition for (0.1) to be well posed in the space H_∞. In particular, from [3] and the above theorem we can see that in the case $m=1$ the condition (0.2) is necessary and sufficient for (0.1) to be well posed for the future and for the past in the space H_∞.

When constant τ equals zero, equation (0.1) is kowalewskian. Then, we remark that our theorem gives the H_∞ version of the Lax-Mizohata theorem.

Now, a solution $u(x, t)$ of the equation

$$(0.3)\qquad (i\partial_t+\Delta)u(x, t)=0$$

with initial data $u_0(x)$ at $t=0$ is written by

$$(0.4)\qquad u(x, t)=C_0\int e^{i|z|^2}u_0(x+2\sqrt{t}z)dz$$
$$=(2\pi)^{-m}\int e^{-ix\cdot\xi-it|\xi|^2}\hat{u}_0(\xi)d\xi\,,$$

where $C_0^{-1}=\int e^{i|z|^2}dz$ and $\hat{u}_0(\xi)$ is the Fourier transform for $u_0(x)$. (0.4) shows that equation (0.3) is not well posed in the space $\mathcal{E}$, but well posed in the space H_∞. $\mathcal{E}$ is the space of infinitely differentiable functions with the customary topology. In fact, if (0.3) is well posed in the space $\mathcal{E}$, for any compact set K in R_x^m and any $T>0$ there exist a non-negative integer l and a compact set K' in R_x^m such that

$$\sup_{K\times[0,T]}|u(\cdot,\cdot)|\leqq C_{K,K',T}\sum_{|\alpha|\leqq l}\sup_{K'}|\partial_x^\alpha u_0(\cdot)|$$

for a constant $C_{K,K',T}$. So, if the intersection of the support of $u_0(x)$ and K' is empty, $u(x, T)$ equals zero for a point x belonging to K. Hence, it follows from the first equality of (0.4) that for a point $x_0\in K$

$$\int e^{i|z|^2}u_0(x_0+2\sqrt{T}z)dz=0$$

is valid for any $u_0(x)$ whose support does not intersect K'. This is not true. On the other hand we have from the second equality of (0.4) $||u(\cdot, t)||_s=||u_0(\cdot)||_s$ $(s=0, \pm 1, \cdots)$ for any t, which follows that (0.3) is well posed in the space H_∞. Therefore, it is natural to consider the Cauchy problem for (0.1) in the frame of the space H_∞ corresponding to the frame of the space $\mathcal{E}$ for the kowalewskian type.

As is stated in Remark 2, we use the energy method. The technique used in the present paper is based on [6]. But, in particular, localizations in the present paper and [6] are quite different. Roughly speaking, in the present paper we localize the solution of (0.1) in phase space along the classical trajectory for the Hamiltonian $-\tau\Delta$. The symbol $w(x, t; \xi)$ of this localizing (pseudo-differential) operator is defined by the solution of "equation of motion for Hamilton function $-\tau|\xi|^2$"

$$\partial_t w(x, t; \xi) = \{w(x, t; \xi), -\tau|\xi|^2\}, \tag{0.5}$$

where for C^1-functions $f(x, \xi)$ and $g(x, \xi)$ $\{f, g\}(x, \xi)$ implies the Poisson bracket $\sum_{j=1}^{m}(\partial_{x_j}f\partial_{\xi_j}g-\partial_{\xi_j}f\partial_{x_j}g)$.

1. Notations and preliminaries

Let $x=(x_1, \cdots, x_m)$ denote a point of R^m and let $\alpha=(\alpha_1, \cdots, \alpha_m)$ be a multi-index whose components α_j are non-negative integers. We use the usual notation.

$$|\alpha| = \alpha_1+\cdots+\alpha_m,\ x^\alpha = x_1^{\alpha_1}\cdots x_m^{\alpha_m},\ \alpha! = \alpha_1!\cdots\alpha_m!$$
$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_m}^{\alpha_m},\ D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_m}^{\alpha_m},\ \partial_{x_j} = \frac{\partial}{\partial x_j},$$
$$D_{x_j} = -i\frac{\partial}{\partial x_j},\ \langle x\rangle = (1+|x|^2)^{1/2}.$$

Let $\mathcal{S}$ on R^m denote the Schwartz space of rapidly decreasing functions. For $u(x)\in\mathcal{S}$ the Fourier transform $\hat{u}(\xi)$ is defined by

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx,\ x\cdot\xi = x_1\xi_1+\cdots+x_m\xi_m.$$

For real s we define the Sobolev space as the completion of $\mathcal{S}$ in the norm $\|u\|_s=\{\int\langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 d\xi\}^{1/2}$, $d\xi=(2\pi)^{-m}d\xi$.

We first state the definitions and theorems with respect to pseudo-differential operators without proofs. Let $S_{0,0}^0$ be the set of C^∞-functions such that for any α, β we have

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha,\beta},$$

where $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$ and $C_{\alpha,\beta}>0$ are constants independent of $(x, \xi)\in R^{2m}$. $S_{0,0}^0$ is a Fréchet space provided with semi-norms $|p|_{l,l'}^{(0)}=\max_{|\alpha|\leqq l,|\beta|\leqq l'}\sup_{x,\xi}|p_{(\beta)}^{(\alpha)}(x, \xi)|$ $(l, l'=0, 1, \cdots)$. The pseudo-differential operator $P=p(x, D_x)$ with symbol $\sigma(P)(x, \xi)=p(x, \xi)$ is defined by

$$P\phi(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{\phi}(\xi)d\xi$$

for $\phi\in\mathcal{S}$. For $p_j(x,\xi)\in S^0_{0,0}$ $(j=1, 2)$ we define $q_\theta(x,\xi)$ $(0\leqq\theta\leqq 1)$ by

$$\begin{aligned}(1.1)\qquad & q_\theta(x,\xi)\\ &=\mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta}p_1(x,\xi+\theta\eta)p_2(x+y,\xi)dyd\eta\\ &\equiv\lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon y,\varepsilon\eta)p_1(x,\xi+\theta\eta)p_2(x+y,\xi)dyd\eta\,,\end{aligned}$$

where $\chi(y,\eta)$ belongs to $\mathcal{S}(R^{2m})$ such that $\chi(0,0)=1$. Then, it is known that $q_1(x,D_x)=p_1(x,D_x)\circ p_2(x,D_x)$, where "$\circ$" denotes the product of operators (see chap. 2 in [4]). We often write $q_1(x,\xi)=\sigma(P_1\circ P_2)(x,\xi)$.

Theorem A (Theorem 3.1 of chap. 2 in [4]). *Let define $q_1(x,\xi)$ by (1.1). Then, for any positive integer ν we get*

$$q_1(x,\xi)=\sum_{0\leqq|\gamma|\leqq\nu-1}\frac{1}{\gamma!}p_1^{(\gamma)}(x,\xi)p_{2(\gamma)}(x,\xi)+\nu\sum_{|\gamma|=\nu}\int_0^1\frac{(1-\theta)^{\nu-1}}{\gamma!}q_{\theta,\gamma}(x,\xi)d\theta\,,$$

where

$$q_{\theta,\gamma}=\mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta}p_1^{(\gamma)}(x,\xi+\theta\eta)p_{2(\gamma)}(x+y,\xi)dyd\eta\,.$$

Theorem B (Lemma 2.2 of chap. 7 in [4]). *For $q_\theta(x,\xi)$ defined by (1.1) we get*

$$|q_\theta|^{(0)}_{l,l}\leqq C_l|p_1|^{(0)}_{l',l'}|p_2|^{(0)}_{l',l'}\,,$$

where $l'=l+2[m/2+1]$ $(l=0, 1, 2, \cdots)$ and constants C_l are independent of θ $(0\leqq\theta\leqq 1)$, but depend on l. For real r $[r]$ denotes the largest integer not greater than r.

Theorem C (Calderón-Vaillancourt theorem, [2] or Theorem 1.6 of chap. 7 in [4]). *Let $p(x,\xi)$ belong to $S^0_{0,0}$. Then, we get*

$$\|p(x,D_x)\phi\|\leqq C|p|^{(0)}_{l_0,l_0}\|\phi\|$$

for $\phi\in\mathcal{S}$, where $\|\cdot\|=\|\cdot\|_0$, $l_0=2[m/2+1]$ and $C>0$ is a constant independent of $p(x,\xi)$ and ϕ.

Now, we shall prepare two lemmas. At first, we note that when τ is positive,

$$(1.2)\qquad \sum_j\int_0^\rho \mathrm{Re}\ b_j(x+\tau\theta\omega)\omega_j d\theta=\frac{1}{\tau}\sum_j\int_{L_{x,x+\tau\rho\omega}}\mathrm{Re}\ b_j dx_j$$

holds. Here, integral $\int_{L_{x,x+\tau\rho\omega}}(\cdots)dx_j$ means curvilinear integral along the straight line $L_{x,x+\tau\rho\omega}$ from a point $x\in R^m$ to a point $x+\tau\rho\omega\in R^m$.

Lemma 1.1. *The following* (i) *and* (ii) *are equivalent.*

(i) *The inequality* (0.2) *with constants* M *and* N *holds for any* $\rho\geqq 0$.

(ii) *The inequality*

$$(0.2)' \qquad \sup_{x\in R^m,\omega\in S^{m-1}} -\sum_{j=1}^m \int_0^\rho \mathrm{Re}\, b_j(x+2\tau\theta\omega)\omega_j d\theta \leqq M\log(1+\rho)+N$$

holds for any $\rho\geqq 0$.

Proof. We have only to show that (ii) yields (i). When τ equals zero, the proof is easy. We shall prove in the case $\tau>0$. By (1.2) and (ii) we have

$$\begin{aligned}&\sum_j \int_0^\rho \mathrm{Re}\, b_j(x+2\tau\theta\omega)\omega_j d\theta\\ &= -\sum_j \int_0^\rho \mathrm{Re}\, b_j(x+2\tau\rho\omega+2\tau\theta(-\omega))(-\omega_j)d\theta\\ &\leqq M\log(1+\rho)+N,\end{aligned}$$

which completes the proof. Q.E.D.

We set

$$(1.3) \qquad b(x;\xi) = -\sum_{j=1}^m \mathrm{Re}\, b_j(x)\xi_j\,.$$

Then, we get

Lemma 1.2. *Assume that for any large constants* M *and* N *the inequality* (0.2) *does not hold. Then, for any large constant* M *there exist sequences* $x^{(k)}\in R^m$, $\omega^{(k)}\in S^{m-1}$, $\rho_k\geqq 0$, $k=1, 2, \cdots$ *such that*

$$(1.4) \qquad \rho_k\to\infty \text{ as } k\to\infty\,,$$

$$(1.5) \qquad \int_0^{\rho_k} b(x^{(k)}+2\tau\theta\omega^{(k)};\omega^{(k)})d\theta \geqq M\log(1+\rho_k)+k$$

and for any $t\in[0,\rho_k]$

$$(1.6) \qquad \int_0^t b(x^{(k)}+2\tau\theta\omega^{(k)};\omega^{(k)})d\theta\geqq 0\,.$$

Proof. Noting Lemma 1.1 and the assumption in this lemma, for any large constant M we can find sequences $y^{(k)}\in R^m$, $\sigma^{(k)}\in S^{m-1}$, $\delta_k\geqq 0$, $k=1, 2, \cdots$ such that

$$(1.7) \qquad \int_0^{\delta_k} b(y^{(k)}+2\tau\theta\sigma^{(k)};\sigma^{(k)})d\theta\geqq M\log(1+\delta_k)+k\,.$$

Set $F_k(t)=\int_0^t b(y^{(k)}+2\tau\theta\sigma^{(k)};\sigma^{(k)})d\theta$ and let t_k be the point at which $F_k(t)$

has the minimal value on $[0, \delta_k]$. Then, we shall prove that (1.4), (1.5) and (1.6) hold, if we determine $x^{(k)}$, $\omega^{(k)}$ and ρ_k $(k=1, 2, \cdots)$ by

$$x^{(k)} = y^{(k)}+2\tau t_k\sigma^{(k)},\ \omega^{(k)} = \sigma^{(k)},\ \rho_k=\delta_k-t_k\,. \tag{1.8}$$

We can see that for $t\in[0, \rho_k]$

$$\begin{aligned}&\int_0^t b(x^{(k)}+2\tau\theta\omega^{(k)};\, \omega^{(k)})d\theta \\ &= \int_0^t b(y^{(k)}+2\tau(t_k+\theta)\sigma^{(k)};\, \sigma^{(k)})d\theta \\ &= \int_{t_k}^{t+t_k} b(y^{(k)}+2\tau\theta\sigma^{(k)};\, \sigma^{(k)})d\theta \\ &= F_k(t+t_k)-F_k(t_k)\,.\end{aligned} \tag{1.9}$$

So, the choice of t_k shows that (1.6) holds for $t\in[0, \rho_k]$. By (1.7)–(1.9) and $F_k(t_k)\leqq F_k(0)=0$, we have

$$\begin{aligned}&\int_0^{\rho_k} b(x^{(k)}+2\tau\theta\omega^{(k)};\, \omega^{(k)})d\theta \\ &= F_k(\delta_k)-F_k(t_k) \\ &\geqq F_k(\delta_k) \\ &\geqq M\log(1+\delta_k)+k \\ &\geqq M\log(1+\rho_k)+k\,,\end{aligned} \tag{1.10}$$

which implies that (1.4) and (1.5) hold. Q.E.D.

2. Localization in phase space and proof of Theorem

We prove our theorem by contradiction. That is, we assume the following:

(A.1) Equation (0.1) is well posed for the future or for the past in the space H_∞.

(A.2) Inequality (0.2) does not hold for any large constants M and N.

Here, we may assume without loss of generality in place of (A.1)

(A.1)′ Equation (0.1) is well posed for the future in the space H_∞.

Then, by the assumption (A.1)′ there exists a $T>0$ such that for any initial data $u_0(x)\in H_\infty$ a unique solution $u(x, t)\in\mathcal{E}_t^0([0, T]; H_\infty)$ of (0.1) exists. Since the space $\mathcal{E}_t^0([0, T]; H_\infty)$ is a Fréchet space with semi-norms $\max_{0\leqq t\leqq T}||f(\cdot, t)||_s$, $s=0, \pm1, \pm2, \cdots$, we see by the closed graph theorem that the mapping: $H_\infty \ni u_0(x)\rightarrow u(x, t)\in\mathcal{E}_t^0([0, T]; H_\infty)$ is continuous. Consequently, there exist a non-negative integer q and a constant $C(T)>0$ such that

$$||u(\cdot, t)||\leqq C(T)||u_0(\cdot)||_q \tag{2.1}$$

holds for $t\in[0, T]$.

For the above q we take a constant M such that

$$M>\frac{m}{2}+2\left[\frac{m}{2}+1\right]+3q \tag{2.2}$$

and fix it through sections 2 and 3. Then, since Lemma 1.2 holds from the assumption (A.2), for this M we can take the sequences $x^{(k)}\in R^m$, $\omega^{(k)}\in S^{m-1}$, $\rho_k\geqq 0$ ($k=1, 2, \cdots$) satisfying (1.4), (1.5) and (1.6). Moreover, we take a positive constant δ such that

$$M>\frac{m}{2}+2\left[\frac{m}{2}+1\right]+(3+\delta)q\,. \tag{2.3}$$

We can assume from (1.4) that

$$\rho_k\geqq 1,\ \rho_k^{-(2+\delta)}\leqq T \tag{2.4}$$

for any k. We also fix these sequences and δ hereafter.

Let $h(x)$ be the C^∞-function such that

$$\begin{cases} h(x)=1 & \text{on } \{x;\ |x|\leqq 1/4\}\,, \\ \operatorname{supp} h(\cdot)\subset\{x;\ |x|\leqq 1/2\}\,, \end{cases}$$

where supp $h(\cdot)$ implies the support of the function $h(x)$. Let $w_{n,k}(x, t;\ \xi)$ be the solution of (0.5) with initial data $\rho_k^{m/2}\, h(\rho_k(x-x^{(k)}))h(\rho_k^2(\xi-n\omega^{(k)})/n)$ at $t=0$. Then, we can easily get

$$\begin{aligned} &w_{n,k}(x, t;\ \xi) \\ &=\rho_k^{m/2}\, h(\rho_k(x-x^{(k)}-2\tau t\xi))h(\rho_k^2(\xi-n\omega^{(k)})/n)\,. \end{aligned} \tag{2.5}$$

For the solution $u(x, t)$ of (0.1) we call $W_{n,k}\, u(x, t)=w_{n,k}(x, t;\ D_x)u(x, t)$ the localized solution (in phase space along the solution $(x^{(k)}+2n\tau t\omega^{(k)},\ n\omega^{(k)})\in R^{2n}_{x,\xi}$ of the canonical equation with initial value $(x^{(k)}, n\omega^{(k)})$ at $t=0$ for the Hamilton function $\tau|\xi|^2$) (see Lemma 2.3). We note

$$\begin{aligned} &\sigma([i\partial_t+\tau\Delta,\ W_{n,k}])\,(x, t;\ \xi) \\ &=i\partial_t w_{n,k}(x, t;\ \xi)-i\{w_{n,k},\ -\tau|\xi|^2\}+\tau\Delta w_{n,k} \\ &=\tau(\Delta w_{n,k})\,(x, t;\ \xi)\,, \end{aligned} \tag{2.6}$$

where $[\cdot, \cdot]$ indicates the commutator of operators and $\Delta w_{n,k}(x, t;\ \xi)=\sum_j (\partial^2_{x_j} w_{n,k})(x, t;\ \xi)$. Equality (2.6) is essential for the proof of Theorem. For any multi-indices α and β we set

$$w^{\alpha,\beta}_{n,k}(x, t;\ \xi)=\rho_k^{m/2}(\partial_x^\alpha h)\,(x)\,(\partial_\xi^\beta h)\,(\xi)\Big|_{\substack{x\,=\,\rho_k(x-x^{(k)}-2\tau t\xi)\\ \xi\,=\,\rho_k^2(\xi-n\omega^{(k)})/n}}\,. \tag{2.7}$$

We note that $w_{n,k}^{0,0}(x, t; \xi)=w_{n,k}(x, t; \xi)$.

Now, we define a series of solutions of (0.1) as in [6] by using $x^{(k)}$, $\omega^{(k)}$ and ρ_k determined above. Namely, we define their initial values. We set hereafter throughout sections 2 and 3

$$(2.8)\qquad n=n(k)=\rho_k^{3+\delta}\,.$$

Let $\psi(x)\in\mathcal{S}$ be a function such that $\psi(0)=2$ and

$$(2.9)\qquad \text{supp}\ \hat{\psi}(\cdot)\subset\{\xi;\ h(\xi)=1\}\,,$$

and then, we define

$$(2.10)\qquad \hat{\psi}_k(\xi)=e^{-ix^{(k)}\cdot\xi}\hat{\psi}(\xi-n\omega^{(k)})\qquad (n=\rho_k^{3+\delta})\,,$$

that is,

$$(2.10)'\qquad \psi_k(x)=e^{i(x-x^{(k)})\cdot n\omega^{(k)}}\psi(x-x^{(k)})\,.$$

Let $u_k(x, t)\in\mathcal{E}_t^0([0, T]; H_\infty)$ be the solution of (0.1) with initial data $\psi_k(x)$ at $t=0$. Then, we can easily get by (2.1) and the definition of $\psi_k(x)$

$$(2.11)\qquad ||u_k(\cdot, t)||\leqq C_1(T)n^q$$

with a constant $C_1(T)>0$ for $t\in[0, T]$. We set

$$(2.12)\qquad v_k^{\alpha,\beta}(x, t)=W_{n,k}^{\alpha,\beta}u_k(x, t)\,,$$

where $W_{n,k}^{\alpha,\beta}=w_{n,k}^{\alpha,\beta}(x, t; D_x)$. We often write $v_k(x, t)=v_k^{0,0}(x, t)$. Since supp $\hat{\psi}_k(\cdot)\subset\{\xi;\ h(\rho_k^2(\xi-n\omega^{(k)})/n)=1\}$ is valid from (2.9), (2.10) and $\rho_k\geqq 1$, we get

$$\begin{aligned}||v_k(\cdot, 0)||^2&=||w_{n,k}(x, 0; D_x)u_k(\cdot, 0)||^2\\&=\int\rho_k^m\,|h(\rho_k(x-x^{(k)}))\psi(x-x^{(k)})\,|^2dx\,,\end{aligned}$$

which follows from $\psi(0)=2$ and (1.4) that for large k

$$(2.13)\qquad ||v_k(\cdot, 0)||\geqq||h(\cdot)||>0\,.$$

Now, take a positive integer s such that

$$(2.14)\qquad \delta\left[\frac{s+2}{2}\right]\geqq\frac{m-2}{2}+2\left[\frac{m}{2}+1\right]+(3+\delta)\,(q+1)$$

and set by the localized solution $v_k(x, t)$

$$(2.15)\qquad \sigma_k(t)=\sum_{0\leqq|\alpha+\beta|\leqq s}(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}||v_k^{\alpha,\beta}(\cdot, t)||$$

for $t\in[0, T]$, where for real r $[r]$ denotes the largest integer not greater than r. We remark that since $\rho_k/n=\rho_k^{-(2+\delta)}$ is not greater than T for any k, $\sigma_k(t)$ has been defined on the interval $[0, \rho_k/n]$. Then, we obtain

Lemma 2.1. *We have*

$$\sigma_k(t) \leqq C_0 \rho_k^{m/2+2[m/2+1]+(3+\delta)q} \tag{2.16}$$

for any $t \in [0, \rho_k/n]$ $(n=\rho_k^{3+\delta})$, *where* C_0 *is a constant independent of* k.

Proposition 2.2. *For large k we get*

$$\sigma_k(\rho_k/n) \geqq C_1(1+\rho_k)^M \qquad (n=\rho_k^{3+\delta}) \tag{2.17}$$

with a positive constant C_1 *independent of* k.

Lemma 2.1 will be proved after the proof of Theorem and Proposition 2.2 will be proved in section 3.

Proof of Theorem. Since we have determined constant $\delta>0$ so that (2.3) holds, (2.16) and (2.17) is not compatible for large k. Thus, we can prove Theorem. Q.E.D.

Proof of Lemma 2.1. By Theorem C we get

$$\begin{aligned} \|v_k(\cdot, t)\| &\leqq C\rho_k^{m/2} |h(\rho_k(x-x^{(k)}-2\tau t\xi))h(\rho_k^2(\xi-n\omega^{(k)})/n)|_{l_0,l_0}^{(0)} \|u_k(\cdot, t)\| \\ &\leqq C'\rho_k^{m/2+l_0}\|u_k(\cdot, t)\|, \end{aligned}$$

where $l_0=2[m/2+1]$. Here, we used $0 \leqq \rho_k t \leqq \rho_k^2/n = \rho_k^{-(1+\delta)}$ for $t \in [0, \rho_k/n]$. Consequently, we obtain from (2.11) for $t \in [0, \rho_k/n]$

$$\|v_k(\cdot, t)\| \leqq C\, \rho_k^{m/2+l_0}\, n^q$$

with another constant C independent of k. In the same way we obtain for $t \in [0, \rho_k/n]$

$$\|v_k^{\alpha,\beta}(\cdot, t)\| \leqq C_{\alpha,\beta}\, \rho_k^{m/2+l_0}\, n^q \tag{2.18}$$

with constant $C_{\alpha,\beta}$ independent of k. Hence, we get (2.16) by $n=\rho_k^{3+\delta}$. Q.E.D.

Lemma 2.3. *If* $t \in [0, \rho_k/n]$ $(n=\rho_k^{3+\delta})$, *then we have*

$$\begin{aligned} &\operatorname{supp} w_{n,k}^{\alpha,\beta}(\cdot, t; \cdot) \\ &\subset \{(x, \xi);\ |x-(x^{(k)}+2n\tau t\omega^{(k)})| \leqq 2/\rho_k,\ |\xi/n-\omega^{(k)}| \leqq 1/(2\rho_k^2)\}. \end{aligned} \tag{2.19}$$

Proof. If $(x, \xi) \in \operatorname{supp} w_{n,k}^{\alpha,\beta}(\cdot, t; \cdot)$, we have from the definition (2.7) of $w_{n,k}^{\alpha,\beta}$

$$|x-(x^{(k)}+2\tau t\xi)| \leqq 1/(2\rho_k),\ |\xi/n-\omega^{(k)}| \leqq 1/(2\rho_k^2).$$

So, noting that $0 \leqq \tau \leqq 1$, it follows that

$$\begin{aligned}&|x-(x^{(k)}+2n\tau t\omega^{(k)})|\\&\leqq |x-(x^{(k)}+2\tau t\xi)|+2n\tau t|\xi/n-\omega^{(k)}|\\&\leqq 2/\rho_k\end{aligned}$$

for any $t\in[0, \rho_k/n]$. This completes the proof. Q.E.D.

Now, if we use the equality (2.6), we can easily get for the localized solution $v_k(x, t)=W_{n,k}u_k(x, t)$

$$\begin{aligned}(2.20)\quad & Lv_k(x, t)\\&=f_k(x, t)\\&\equiv \{[\sum_j b_j(x)\partial_{x_j}+c(x), W_{n,k}]+\tau(\Delta w_{n,k})(x, t; D_x)\}u_k.\end{aligned}$$

Then, we obtain

Lemma 2.4. *Let $t\in[0, \rho_k/n]$ $(n=\rho_k^{3+\delta})$. Then, for any $p=1, 2, \cdots$ we get*

$$\begin{aligned}(2.21)\quad & \|f_k(\cdot, t)\|\\&\leqq \rho_k^2\sum_{|\alpha+\beta|=2}\|v_k^{\alpha,\beta}(\cdot, t)\|+C_p n\sum_{1\leq|\alpha+\beta|\leq p+1}(\rho_k^2/n)^{|\alpha+\beta|}\|v_k^{\alpha,\beta}(\cdot, t)\|\\&\qquad +C_p\, n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{p+1},\end{aligned}$$

where $\lambda=m/2+4[m/2+1]$ and C_p is a positive constant independent of k.

Proof. We can easily see from (2.20)

$$\begin{aligned}(2.22)\quad & \|f_k(\cdot, t)\|\\&\leqq \sum_j\|[b_j\partial_{x_j}, W_{n,k}]u_k(\cdot, t)\|+\|[c(x), W_{n,k}]u_k(\cdot, t)\|+\rho_k^2\sum_{|\alpha+\beta|=2}\|v_k^{\alpha,\beta}(\cdot, t)\|.\end{aligned}$$

We first consider the term $[b_j\partial_{x_j}, W_{n,k}]u_k(x, t)$. If we use the notation (2.7), we can write

$$\begin{aligned}(2.23)\quad & [b_j\partial_{x_j}, W_{n,k}]u_k(x, t)\\&=\rho_k\, b_j(x)W_{n,k}^{e_j,0}u_k(x, t)+[b_j, W_{n,k}]\partial_{x_j}u_k(x, t),\end{aligned}$$

where e_j is the multi-index whose j-th component is one and other components are all zero. Then, for the first term of the right hand side of (2.23) its L^2 norm is estimated by the second term of the right hand side of (2.21).

We consider the second term in (2.23). By Theorem A in section 1 we obtain

$$\begin{aligned}(2.24)\quad & \frac{1}{i}\sigma([b_j(x), W_{n,k}]\partial_{x_j})\\&=-\{\sum_{1\leq|\gamma|\leq p}\frac{1}{\gamma!}D_x^\gamma b_j(x)\,\partial_\xi^\gamma w_{n,k}(x, t; \xi)\}\xi_j+r_{p,k}(x, t; \xi),\end{aligned}$$

where $r_{p,k}(x, t; \xi)$ consists of the sum of

$$-(p+1)\frac{1}{\gamma!}\int_0^1(1-\theta)^p d\theta\ \mathrm{O_s}\text{-}\iint e^{-iy\cdot\eta}(D_x^\gamma b_j)(x+y)$$
$$(\partial_\xi^\gamma w_{n,k})(x, t; \xi+\theta\eta)\xi_j\, dyd\eta$$

over γ such that $|\gamma|=p+1$. Using

$$\mathrm{O_s}\text{-}\iint e^{-iy\cdot\eta}(D_x^\gamma b_j)(x+y)(\partial_\xi^\gamma w_{n,k})(x, t; \xi+\theta\eta)\xi_j\, dyd\eta$$
$$=\mathrm{O_s}\text{-}\iint e^{-iy\cdot\eta}(D_x^\gamma b_j)(x+y)(\partial_\xi^\gamma w_{n,k})(x, t; \xi+\theta\eta)(\xi_j+\theta\eta_j)\, dyd\eta$$
$$-\theta\ \mathrm{O_s}\text{-}\iint e^{-iy\cdot\eta} D_{y_j}(D_x^\gamma b_j)(x+y)(\partial_\xi^\gamma w_{n,k})(x, t; \xi+\theta\eta)\, dyd\eta$$

and then applying Theorem B, we get the estimates from (2.5) and Lemma 2.3

$$(2.25)\qquad \begin{aligned}&|r_{p,k}(\cdot, t; \cdot)|_{l_0,l_0}^{(0)}\\ &\leqq C_{p,1}\, n\rho_k^\lambda \sum_{|\gamma_1+\gamma_2|=p+1}(\rho_k t)^{|\gamma_1|}(\rho_k^2/n)^{|\gamma_2|}\\ &\leqq C_{p,2}\, n\rho_k^\lambda(\rho_k^2/n)^{p+1}\end{aligned}$$

for $t\in[0, \rho_k/n]$, where $l_0=2[m/2+1]$ and $C_{p,1}$, $C_{p,2}$ are positive constants depending only on p. Here, we used $\rho_k t\leqq\rho_k^2/n$ for $t\in[0, \rho_k/n]$. Consequently, applying Theorem C, we get

$$(2.26)\qquad \|r_{p,k}(x, t; D_x)u_k(\cdot, t)\|\leqq C_{p,3}\, n^{q+1}\rho_k^\lambda(\rho_k^2/n)^{p+1}$$

by (2.11).

Next, we consider the first term in (2.24). We remark

$$(2.27)\qquad \begin{aligned}&(\partial_\xi^\gamma w_{n,k})(x, t; \xi)\xi_j\\ &=\sum_{\alpha+\beta=\gamma}\frac{\gamma!}{\alpha!\beta!}(-2\tau t\rho_k)^{|\alpha|}(\rho_k^2/n)^{|\beta|}w_{n,k}^{\alpha,\beta}(x, t; \xi)\xi_j\,.\end{aligned}$$

We can easily see

$$(2.28)\qquad \begin{aligned}&\|W_{n,k}^{\alpha,\beta} D_{x_j}u_k(\cdot, t)\|\\ &\leqq \rho_k\|W_{n,k}^{\alpha+e_j,\beta}u_k(\cdot, t)\|+\|D_{x_j}\circ W_{n,k}^{\alpha,\beta}\, u_k(\cdot, t)\|\,,\end{aligned}$$

in which the second term is estimated by

$$K\, n\|W_{n,k}^{\alpha,\beta}\, u_k(\cdot, t)\|+C_{p,\alpha,\beta}\, n^{q+1}\rho_k^\lambda(\rho_k^2/n)^{p+1}\,,$$

where $K=3\max_{x\in R^m}|h(x)|$ and $C_{p,\alpha,\beta}$ are constants independent of k, but depend on α and β.

In fact, if we set

$$(2.29)\qquad \chi_{1,k}(\xi)=h(\rho_k(\xi-n\omega^{(k)})/3n),$$

we have

$$\begin{aligned}&||D_{x_j}\circ W_{n,k}^{\alpha,\beta}\,u_k(\cdot,t)||\\&\leqq||\chi_{1,k}(D_x)D_{x_j}\circ W_{n,k}^{\alpha,\beta}\,u_k(\cdot,t)||+\rho_k||(I-\chi_{1,k}(D_x))\circ W_{n,k}^{\alpha+e_j,\beta}\,u_k(\cdot,t)||\\&+||(I-\chi_{1,k}(D_x))\circ W_{n,k}^{\alpha,\beta}\,D_{x_j}\,u_k(\cdot,t)||\,.\end{aligned}$$

Since supp $\chi_{1,k}(\cdot)\subset\{\xi;\,|\xi|\leqq 3n\}$ is valid, the term $||\chi_{1,k}(D_x)D_{x_j}\circ W_{n,k}^{\alpha,\beta}\,u_k(\cdot,t)||$ is estimated by $Kn||W_{n,k}^{\alpha,\beta}\,u_k(\cdot,t)||$. Apply Theorems A and B to the symbol $\sigma((I-\chi_{1,k}(D_x))\circ W_{n,k}^{\alpha,\beta}\,D_{x_j})\,(x,t;\xi)$. Then, if we note from Lemma 2.3 that supp $(1-\chi_{1,k}(\cdot))\cap$ supp $w_{n,k}^{\alpha,\beta}(\cdot,t;\cdot)=\phi$ for $t\in[0,\rho_k/n]$, we can easily have

$$\begin{aligned}&|\sigma((I-\chi_{1,k}(D_x))\circ W_{n,k}^{\alpha,\beta}\,D_{x_j})\,(\cdot,t;\cdot)|_{l_0,l_0}^{(0)}\\&\leqq C'_{p,\alpha,\beta}\,n\rho_k^{\lambda}(\rho_k^2/n)^{p+1}\end{aligned}$$

as in the proof of (2.25) for $t\in[0,\rho_k/n]$ with a constant $C'_{p,\alpha,\beta}$. So, we get

$$(2.30)\qquad \begin{aligned}&||(I-\chi_{1,k}(D_x))\circ W_{n,k}^{\alpha,\beta}\,D_{x_j}\,u_k(\cdot,t)||\\&\leqq C'_{p,\alpha,\beta}\,n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{p+1}\end{aligned}$$

with another constant $C'_{p,\alpha,\beta}$. In the same way we can also estimate $\rho_k||(I-\chi_{1,k}(D_x))\circ W_{n,k}^{\alpha+e_j,\beta}u_k(\cdot,t)||$.

Hence, noting that $\rho_k t\leqq\rho_k^2/n$ for $t\in[0,\rho_k/n]$, we obtain from (2.27)

$$(2.31)\qquad \begin{aligned}&||(\partial_\xi^\gamma\,w_{n,k})\,(x,t;D_x)D_{x_j}\,u_k(\cdot,t)||\\&\leqq C_\gamma(\rho_k^2/n)^{|\gamma|}\sum_{\alpha+\beta=\gamma}(\rho_k||v_k^{\alpha+e_j,\beta}(\cdot,t)||+n||v_k^{\alpha,\beta}(\cdot,t)||)+C_{p,\gamma}\,n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{p+1}\end{aligned}$$

for constants C_γ and $C_{p,\gamma}$, which shows from (2.24) together with (2.26) that

$$(2.32)\qquad \begin{aligned}&||[b_j(x),W_{n,k}]\partial_{x_j}\,u_k(\cdot,t)||\\&\leqq C'_p\,n\sum_{1\leqq|\alpha+\beta|\leqq p+1}(\rho_k^2/n)^{|\alpha+\beta|}||v_k^{\alpha,\beta}(\cdot,t)||+C'_p\,n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{p+1}\end{aligned}$$

for constants C'_p independent of k. Since we can also estimate $||[c(x),W_{n,k}]u_k(\cdot,t)||$ in the same way, we can complete the proof. Q.E.D.

3. Proof of Proposition 2.2

We first prove for $v_k(x,t)=v_k^{0,0}(x,t)$ defined by (2.12)

Lemma 3.1. *Let* $t\in[0,\rho_k/n]$ $(n=\rho_k^{3+\delta})$. *Then, for any* $\nu=1,2,\cdots$

$$(3.1)\qquad \frac{1}{2}\frac{d}{dt}||v_k(\cdot,t)||^2$$

$$\geqq \{b(x^{(k)}+2n\tau t\omega^{(k)};\, n\omega^{(k)})-A(1+\frac{n}{\rho_k})\}\|v_k(\cdot, t)\|^2$$
$$-\|f_k(\cdot, t)\|\times\|v_k(\cdot, t)\|-\tilde{C}_\nu\, n^{q+1}\rho_k^\lambda(\rho_k^2/n)^\nu\|v_k(\cdot, t)\|$$

is valid, where λ *is the same constant in Lemma* 2.4, A *is a constant independent of* ν *and* k, *and* $\tilde{C}_\nu$ *are constants independent of* k *but depend on* ν. *As set in section* 1, $b(x;\xi)$ *denotes* $-\sum_{j=1}^{m}\mathrm{Re}\, b_j(x)\xi_j$.

Proof. From (2.20) we can see that

$$\begin{aligned}(3.2)\quad &\frac{d}{dt}\|v_k(\cdot, t)\|^2\\ &= 2\mathrm{Re}\,(\partial_t v_k(\cdot, t),\, v_k(\cdot, t))\\ &= 2\mathrm{Re}\; i((\tau\Delta+\sum_j b_j\partial_{x_j}+c)v_k(\cdot, t),\, v_k(\cdot, t))-2\mathrm{Re}\; i(f_k(\cdot, t),\, v_k(\cdot, t))\\ &\geqq -2\mathrm{Re}\,(\sum_j(\mathrm{Re}\, b_j)(x)D_{x_j}v_k(\cdot, t),\, v_k(\cdot, t))\\ &\quad -A_1\|v_k(\cdot, t)\|^2-2\|f_k(\cdot, t)\|\times\|v_k(\cdot, t)\|\end{aligned}$$

for a constant A_1 independent of k. We shall estimate

$$-(\sum_j(\mathrm{Re}\, b_j)(x)D_{x_j}v_k(\cdot, t),\, v_k(\cdot, t))=(b(x;D_x)v_k(\cdot, t),\, v_k(\cdot, t))\,.$$

We write

$$\begin{aligned}(3.3)\quad &-(\mathrm{Re}\, b_j)(x)D_{x_j}\\ &= -(\mathrm{Re}\, b_j)(x^{(k)}+2n\tau t\omega^{(k)})n\omega_j^{(k)}\\ &\quad +(\mathrm{Re}\, b_j)(x^{(k)}+2n\tau t\omega^{(k)})\,(n\omega_j^{(k)}-D_{x_j})\\ &\quad +\{(\mathrm{Re}\, b_j)(x^{(k)}+2n\tau t\omega^{(k)})-(\mathrm{Re}\, b_j)(x)\}D_{x_j}\\ &\equiv \sum_{j=1}^{3}I_j\,.\end{aligned}$$

We first estimate $I_2v_k(x, t)$. Since $\mathrm{supp}\; \chi_{1,k}(\cdot)\subset\{\xi;\; |\xi-n\omega^{(k)}|\leqq 3n/(2\rho_k)\}$ holds for $\chi_{1,k}(\xi)$ defined by (2.29), we see that

$$\begin{aligned}(3.4)\quad &\|\chi_{1,k}(D_x)\circ(n\omega_j^{(k)}-D_{x_j})v_k(\cdot, t)\|\\ &\leqq A_2\frac{n}{\rho_k}\|v_k(\cdot, t)\|\end{aligned}$$

for a constant A_2 independent of k. Hereafter, in this proof, if there is no confusion, we do not indicate that constants are independent of k. Next, we write by $v_k(x, t)=W_{n,k}u_k(x, t)$

$$\begin{aligned}(3.5)\quad J&\equiv (I-\chi_{1,k}(D_x))\circ(n\omega_j^{(k)}-D_{x_j})v_k\\ &= (I-\chi_{1,k}(D_x))\circ w_{n,k}(x, t;\, D_x)\,(n\omega_j^{(k)}-D_{x_j})u_k\end{aligned}$$

$$-\frac{1}{i}\rho_k(I-\chi_{1,k}(D_x))\circ w_{n,k}^{e_j,0}(x,t;D_x)u_k.$$

Apply Theorems A and B in section 1 to the term $p(x,t;\xi)=\sigma((I-\chi_{1,k}(D_x))\circ w_{n,k}(x,t;D_x)(n\omega_j^{(k)}-D_{x_j}))(x,t;\xi)$. Then, we can show in the similar way to the proof of (2.30) that for any ν

$$|p(\cdot,t;\cdot)|_{l_0,l_0}^{(0)}\leqq C_{\nu,1}\,n\rho_k^{\lambda}(\rho_k^2/n)^{\nu}$$

is valid for $t\in[0,\rho_k/n]$ and so we get

$$\begin{aligned}&\|(I-\chi_{1,k}(D_x))\circ w_{n,k}(x,t;D_x)(n\omega_j^{(k)}-D_{x_j})u_k(\cdot,t)\|\\&\leqq C_{\nu,2}\,n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{\nu}\end{aligned}$$

for $t\in[0,\rho_k/n]$, where constants $C_{\nu,1}$ and $C_{\nu,2}$ depend only on ν. In the same way we can also estimate $\rho_k\|(I-\chi_{1,k}(D_x))\circ w_{n,k}^{e_j,0}(x,t;D_x)u_k(\cdot,t)\|$. Namely, we obtain

$$\begin{aligned}(3.6)\qquad&\|J\|\\&\leqq C_{\nu,3}\,n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{\nu},\end{aligned}$$

which shows together with (3.4) that

$$\begin{aligned}(3.7)\qquad&\|I_2v_k(\cdot,t)\|\\&\leqq A_3\frac{n}{\rho_k}\|v_k(\cdot,t)\|+C_{\nu,4}\,n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{\nu}\end{aligned}$$

for $t\in[0,\rho_k/n]$.

Next, we shall estimate $I_3v_k(x,t)$. If we set

$$\chi_{2,k}(x)=h(\rho_k(x-x^{(k)}-2n\tau t\omega^{(k)})/9),$$

supp $(1-\chi_{2,k}(\cdot))\cap$ supp $w_{n,k}^{\alpha,\beta}(\cdot,t;\cdot)=\phi$ holds for $t\in[0,\rho_k/n]$ from Lemma 2.3. So,

$$\begin{aligned}&(I-\chi_{2,k}(x))D_{x_j}v_k(x,t)\\&=(I-\chi_{2,k}(x))\{w_{n,k}(x,t;D_x)D_{x_j}u_k+\frac{1}{i}\rho_k w_{n,k}^{e_j,0}(x,t;D_x)u_k\}\\&=0.\end{aligned}$$

That is,

$$\begin{aligned}(3.8)\qquad&I_3v_k(x,t)\\&=\chi_{2,k}(x)\{(\mathrm{Re}\,b_j)(x^{(k)}+2n\tau t\omega^{(k)})-(\mathrm{Re}\,b_j)(x)\}D_{x_j}v_k,\end{aligned}$$

which follows that

$$(3.9)\qquad\|I_3v_k(\cdot,t)\|\leqq(A_4/\rho_k)\|D_{x_j}v_k(\cdot,t)\|$$

for $t\in[0, \rho_k/n]$. Now, as in the proof of the estimate for the second term of the right hand side of (2.28) we get

$$\begin{aligned}&||D_{x_j}v_k(\cdot, t)||\\ &\leqq A_5\, n||v_k(\cdot, t)||+C_{\nu,5}\, n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{\nu}\,,\end{aligned}$$

which follows

$$\begin{aligned}(3.10)\qquad &||I_3 v_k(\cdot, t)||\\ &\leqq A_6\frac{n}{\rho_k}\,||v_k(\cdot, t)||+C_{\nu,6}\, n^{q+1}\rho_k^{\lambda-1}(\rho_k^2/n)^{\nu}\,.\end{aligned}$$

Using (3.2), (3.3), (3.7) and (3.10), we can complete the proof. Q.E.D.

Proof of Proposition 2.2. We can take a positive integer p such that

$$(3.11)\qquad \sup_k n^{q+1}\rho_k^{\lambda}(\rho_k^2/n)^{p+1}<\infty\,,$$

noting $n=\rho_k^{3+\delta}$ and fix it. Then, it is easily seen from Lemma 2.4 and Lemma 3.1 that

$$\begin{aligned}(3.12)\qquad &\frac{d}{dt}\,||v_k(\cdot, t)||\geqq B(t;k)||v_k(\cdot, t)||-\text{const.}\,\frac{n}{\rho_k}\,\{(\rho_k^3/n)\sum_{|\alpha+\beta|=2}||v_k^{\alpha,\beta}(\cdot, t)||\\ &+\sum_{1\leqq|\alpha+\beta|\leqq p+1}\rho_k(\rho_k^2/n)^{|\alpha+\beta|}||v_k^{\alpha,\beta}(\cdot, t)||\}-\text{const.},\end{aligned}$$

where

$$(3.13)\qquad B(t;k)=b(x^{(k)}+2n\tau t\omega^{(k)};\, n\omega^{(k)})-A\Big(1+\frac{n}{\rho_k}\Big)$$

with the same constant A in (3.1). Since the inequality $\rho_k(\rho_k^2/n)^{|\gamma|}\leqq(\rho_k^3/n)^{[(|\gamma|+1)/2]}$ ($|\gamma|\geqq 1$) is valid, we obtain from (3.12)

$$\begin{aligned}(3.14)\qquad &\frac{d}{dt}\,||v_k(\cdot, t)||\\ &\geqq B(t;k)||v_k(\cdot, t)||-\text{const.}\,\frac{n}{\rho_k}\sum_{1\leqq|\alpha+\beta|\leqq p+1}(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}||v_k^{\alpha,\beta}(\cdot, t)||\\ &-\text{const.}.\end{aligned}$$

If we make the same process for $v_k^{\alpha,\beta}(x, t)=W_{n,k}^{\alpha,\beta}u(x, t)$ (($|\alpha+\beta|\geqq 1$) as for $v_k(x, t)=W_{n,k}u(x, t)$, corresponding to (3.14) we have

$$\begin{aligned}&\frac{d}{dt}\,||v_k^{\alpha,\beta}(\cdot, t)||\geqq B(t;k)||v_k^{\alpha,\beta}(\cdot, t)||\\ &-C_{\alpha,\beta}\frac{n}{\rho_k}\sum_{1\leqq|\tilde\alpha+\tilde\beta|\leqq p+1}(\rho_k^3/n)^{[(|\tilde\alpha+\tilde\beta|+1)/2]}||v_k^{\alpha+\tilde\alpha,\beta+\tilde\beta}(\cdot, t)||-C_{\alpha,\beta}\end{aligned}$$

for constants $C_{\alpha,\beta}$ independent of k. So, we obtain

$$(3.15)\qquad \frac{d}{dt}(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}\|v_k^{\alpha,\beta}(\cdot,t)\|$$
$$\geqq B(t;k)\,(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}\|v_k^{\alpha,\beta}(\cdot,t)\|$$
$$-C_{\alpha,\beta}\frac{n}{\rho_k}\sum_{1\leqq|\tilde\alpha+\tilde\beta|\leqq p+1}(\rho_k^3/n)^{[(|\alpha+\tilde\alpha+\beta+\tilde\beta|+1)/2]}\|v_k^{\alpha+\tilde\alpha,\beta+\tilde\beta}(\cdot,t)\|-C_{\alpha,\beta}\,.$$

Here, we used

$$(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]+[(|\tilde\alpha+\tilde\beta|+1)/2]}$$
$$\leqq(\rho_k^3/n)^{[(|\alpha+\tilde\alpha+\beta+\tilde\beta|+1)/2]}$$

for $|\tilde\alpha+\tilde\beta|\geqq 1$.

Now, we already determined s so that (2.14) holds. Hence, if $|\alpha+\beta|\geqq s+1$, we have by (2.18)

$$\frac{n}{\rho_k}(\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]}\|v_k^{\alpha,\beta}(\cdot,t)\|$$
$$\leqq C'_{\alpha,\beta}\frac{n}{\rho_k}(\rho_k^3/n)^{[(s+2)/2]}\,\rho_k^{m/2+2[m/2+1]}\,n^q$$
$$\leqq C''_{\alpha,\beta}<\infty$$

for any k and for any $t\in[0,\rho_k/n]$. Therefore, for $\sigma_k(t)$ defined by (2.15) we obtain from (3.14) and (3.15)

$$(3.16)\qquad \frac{d}{dt}\sigma_k(t)\geqq\Big(B(t;k)-C\frac{n}{\rho_k}\Big)\sigma_k(t)-O(1)$$

for any k and $t\in[0,\rho_k/n]$, where C is a constant independent of k.

The integration of (3.16) gives

$$(3.17)\qquad \sigma_k(\rho_k/n)$$
$$\geqq\Big(\exp\int_0^{\rho_k/n}B(\theta;k)-C\frac{n}{\rho_k}\,d\theta\Big)$$
$$\times\Big\{\sigma_k(0)-O(1)\int_0^{\rho_k/n}\Big(\exp-\int_0^t B(\theta;k)-C\frac{n}{\rho_k}\,d\theta\Big)\,dt\Big\}\,.$$

Here, we note from (3.13) that

$$B(\theta;k)=b(x^{(k)}+2n\tau\theta\omega^{(k)};\,n\omega^{(k)})-A\Big(1+\frac{n}{\rho_k}\Big)\,.$$

Also, from the choice of $x^{(k)}$, $\omega^{(k)}$, ρ_k we know that

$$\int_0^{\rho_k/n}b(x^{(k)}+2n\tau\theta\omega^{(k)};\,n\omega^{(k)})\,d\theta$$
$$=\int_0^{\rho_k}b(x^{(k)}+2\tau\theta\omega^{(k)};\,\omega^{(k)})\,d\theta$$
$$\geqq M\log(1+\rho_k)+k$$

and for $t\in[0, \rho_k/n]$

$$\begin{aligned}&\int_0^t b(x^{(k)}+2n\tau\theta\omega^{(k)};\, n\omega^{(k)})\, d\theta\\ &=\int_0^{nt} b(x^{(k)}+2\tau\theta\omega^{(k)};\, \omega^{(k)})\, d\theta\\ &\geqq 0\,.\end{aligned}$$

Moreover, $\sigma_k(0)\geqq||v_k(\cdot, 0)||\geqq||h(\cdot)||$ holds for large k by (2.13). Hence, if k is large enough, we obtain from (3.17)

$$\sigma_k(\rho_k/n)\geqq C_1(1+\rho_k)^M$$

for a positive constant C_1, which shows Proposition 2.2. Q.E.D.

REMARK 4. In more detail we can see from the proof of Theorem the following is necessary in order that there exists a constant $T>0$ such that for any initial data $u_0(x)\in H_\infty$ a unique solution $u(x, t)\in\mathcal{E}_t^0([0, T]; H_\infty)$ of (0.1) exists and the inequality (2.1) holds for some q. For any M greater than $m/2+2[m/2+1]+3q$ there exists a constant N such that the inequality (0.2) holds.

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