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PARALLEL PROJECTIVE MANIFOLDS AND SYMMETRIC BOUNDED DOMAINS

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Introduction. The parallel complex submanifolds, i.e., complex subfolds with parallel second fundamental form, of Fubini-Study spaces were classified by Nakagawa-Takagi [12] and Takeuchi [20]. The first classification [12] was done as an application of the general study of Kähler immersions of locally symmetric Kähler manifolds into Fubini-Study spaces. The second one [20] was done by the determination (Takagi-Takeuchi [18]) of degrees of Kähler immersions of symmetric Kähler manifolds into Fubini-Study spaces.

In this paper we give another way of classification of such submanifolds. Let D be an irreducible symmetric bounded domain and V the holomorphic tangent space of D at a point $p \in D$. Then the isotropy group K at p acts in a natural way on the complex projective space $P(V)$ associated to V. We endow $P(V)$ with a K-invariant Kähler metric with positive constant holomorphic sectional curvature. Take a highest weight vector v of the irreducible K-module V. Then

$$M = K \cdot [v] \subset P(V),$$

where $[v]$ denotes the line $\boldsymbol{C}v$, is a complete full complex submanifold with parallel second fundamental form. This is proved by writing the second fundamental form of M in terms of the Lie algebra of infinitesimal automorphisms of D.

Conversely, any complete full complex submanifold M of a Fubini-Study space $P_N(\boldsymbol{C})$ with parallel second fundamental form is obtained in this way. This is proved by defining a structure of Jordan triple system on $\boldsymbol{C}^{N+1}$ making use of second fundamental form and curvature tensor of M, and then using Koecher's classification theorem for symmetric bounded domains by Jordan triple systems.

As an application, we study the group $\mathrm{Aut}(S)$ of automorphisms of a non-singular hyperplane section S of M. We show that $\mathrm{Aut}(S)$ is reductive if and only if the symmetric bounded domain D corresponding to M is a unit ball or of tube type. This provides a unified construction of compact complex manifolds admitting no Einstein Kähler metric found by Hano [2], Sakane

[14].

1. Projective manifolds

Let U be a finite dimensional real vector space, J a complex structure on U, i.e., a linear endomorphism of U with $J^2=-I_U$, I_U being the identity map on U, and $\langle\ ,\ \rangle$ an inner product on U such that

$$\langle Ju, Jv\rangle = \langle u, v\rangle \quad \text{for} \quad u, v\in U.$$

The triple $(U, J, \langle\ ,\ \rangle)$ is called a *hermitian vector space*. Then U is regarded in a natural way as the underlying real vector space of a complex vector space, which will be denoted by (U, J), and

$$\{u, v\} = \langle u, v\rangle+\sqrt{-1}\langle u, Jv\rangle$$

is a hermitian inner product on (U, J) such that

$$\text{(1.1)} \qquad \mathcal{Re}\{u, v\} = \langle u, v\rangle \quad \text{for} \quad u, v\in U.$$

Denoting the $\boldsymbol{C}$-linear extensions of J and $\langle\ ,\ \rangle$ to the complexification $U^{\boldsymbol{C}}$ of U also by J and $\langle\ ,\ \rangle$, we define

$$U^{\pm} = \{u\in U^{\boldsymbol{C}}; Ju = \pm\sqrt{-1}u\}.$$

Then $U^{\boldsymbol{C}}=U^{+}+U^{-}$ (direct sum), $\bar{U}^{\pm}=U^{\mp}$, $\langle U^{\pm}, U^{\pm}\rangle=\{0\}$ and

$$\langle\!\langle u, v\rangle\!\rangle = \langle u, \bar{v}\rangle \quad \text{for} \quad u, v\in U^{\pm}$$

is a hermitian inner product on $U^{\pm}$. Let $\varpi^{\pm}\colon U\to U^{\pm}$ denote the projections, i.e.,

$$\varpi^{\pm}(u) = \frac{1}{2}(u\mp\sqrt{-1}Ju) \quad \text{for} \quad u\in U.$$

They are then $\boldsymbol{R}$-linear isomorphisms such that

$$\varpi^{\pm}(Ju) = \pm\sqrt{-1}\varpi^{\pm}(u) \quad \text{for} \quad u\in U,$$
$$\text{(1.2)} \qquad 2\langle\!\langle\varpi^{\pm}(u), \varpi^{\pm}(v)\rangle\!\rangle = \{u, v\} \quad \text{for} \quad u, v\in U.$$

In particular, if V is a finite dimensional complex vector space equipped with a hermitian inner product $\{\ ,\ \}$, then the scalar restriction $U=V_{\boldsymbol{R}}$, the natural complex structure J and $\langle u, v\rangle=\mathcal{Re}\{u, v\}$ define a hermitian vector space $(U, J, \langle\ ,\ \rangle)$ in such a way that $(U, J)=V$ and that the hermitian inner product $\{\ ,\ \}$ on (U, J) coincides with the original one $\{\ ,\ \}$. In this case, $(V_{\boldsymbol{R}})^{\pm}$ will be often abbreviated to $V^{\pm}$.

Now let V be a complex vector space with dimension $N+1$ equipped with a hermitian inner product $\{\ ,\ \}$, and $P(V)$ the complex projective space associated to V with dimension N. The complex structure tensor on the tangent bundle $T(P(V))$ is denoted by J. We denote by $\pi\colon V-\{0\}\to P(V)$ the projection $v\mapsto[v]$, where $[v]$ denotes the line $\boldsymbol{C}v$. The unitary group on V with respect to $\{\ ,\ \}$ is denoted by $U(V)$, which acts on $P(V)$ in a natural way. Let $\mathfrak{u}(V)=\mathrm{Lie}\ U(V)$, the Lie algebra of $U(V)$. We fix $c>0$ and let $g(X, Y)=\langle X, Y\rangle$ denote the $U(V)$-invariant Kähler metric on $P(V)$ with constant holomorphic sectional curvature c. It is described as follows.

We define a $U(V)$-invariant symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{gl}(V)=\mathrm{Lie}\ GL(V)$ by

$$\langle X, Y\rangle=-\frac{c}{2}\mathrm{Tr}(XY) \qquad \text{for} \quad X, Y\in\mathfrak{gl}(V)\,.$$

Let $(V_{\boldsymbol{R}}, J, \langle\ ,\ \rangle)$ be the hermitian vector space constructed from V in the above way. Choosing an element $E\in V$ with $\langle E, E\rangle=4/c$, we put $o=[E]\in P(V)$. We define $Q=\{E, JE\}_{\boldsymbol{R}}$ and

$$\mathfrak{k}(V)=\{X\in\mathfrak{u}(V);\ X\cdot E\in Q\}\,,$$
$$\mathfrak{m}(V)=\{X\in\mathfrak{u}(V);\ \langle X, \mathfrak{k}(V)\rangle=\{0\}\}\,.$$

Then $\mathfrak{u}(V)=\mathfrak{k}(V)+\mathfrak{m}(V)$ (direct sum) is a Cartan decomposition of $\mathfrak{u}(V)$, and the linear map $\tilde{j}\colon\mathfrak{m}(V)\to T_o(P(V))$ defined by

$$\tilde{j}(X)=X^*_o \qquad \text{for} \quad X\in\mathfrak{m}(V)\,,$$

where X^* denotes the vector field on $P(V)$ generated by X, is an $\boldsymbol{R}$-linear isomorphism. Then, together with the complex structure J on $\mathfrak{m}(V)$ corresponding to the complex structure tensor J_o on $T_o(P(V))$ by the above $\tilde{j}$, the triple $(\mathfrak{m}(V), J, \langle\ ,\ \rangle)$ becomes a hermitian vector space. Obviously, $(T_o(P(V)), J_o, \langle\ ,\ \rangle)$, $\langle\ ,\ \rangle$ being the Kähler metric of $P(V)$ at o, becomes a hermitian vector space. We put furthermore

$$\tilde{T}=\{v\in V_{\boldsymbol{R}};\ \langle v, Q\rangle=\{0\}\}\,.$$

Then $\tilde{T}$ is a J-invariant subspace of $V_{\boldsymbol{R}}$ (with dimension $2N$), and hence $(\tilde{T}, J, \langle\ ,\ \rangle)$ becomes a hermitian vector space. We may define a linear map $\tilde{\imath}\colon\mathfrak{m}(V)\to\tilde{T}$ by

$$\tilde{\imath}(X)=X\cdot E \qquad \text{for} \quad X\in\mathfrak{m}(V)\,.$$

The differential of π is denoted by π_*. We have then a commutative diagram

$$\begin{array}{ccc} & \mathfrak{m}(V) & \\ {}^{\tilde{\imath}}\swarrow & & \searrow^{\tilde{j}} \\ \tilde{T} & \xrightarrow[\pi_*]{\hspace{2em}} & T_o(P(V))\,, \end{array}$$

and each of $\tilde{i}$, $\tilde{j}$ and π_* is an isomorphism of hermitian vector space, i.e., a linear isomorphism preserving $\langle\,,\rangle$ and commuting with J. The Kähler manifold $(P(V), g)$ and the Kähler metric g are called the *Fubini-Study space* and the *Fubini-Study metric*, respectively.

Next we recall some basic identities for projective manifolds, i.e., those for complex submanifolds of complex projective spaces. Let (M, g) be a Kähler manifold and $f: (M, g) \rightarrow (P(V), g)$ a holomorphic isometric immersion. Let TM and NM denote the tangent bundle of M and the normal bundle for f, respectively. The curvature tensor of (M, g), the second fundamental form, the shape operator and the normal curvature tensor of f (and also their extensions to complexified bundles $(TM)^C$ etc.) are denote by R, α, A and $R^\perp$, respectively. The complex structure tensors J on TM and NM give rise Whitney sum decompositions

$$(TM)^C = (TM)^+ \oplus (TM)^- ,$$
$$(NM)^C = (NM)^+ \oplus (NM)^- ,$$

and thus we get direct sum decompositions

$$C^\infty((TM)^C) = C^\infty((TM)^+) + C^\infty((TM)^-) ,$$
$$C^\infty((NM)^C) = C^\infty((NM)^+) + C^\infty((NM)^-)$$

for the spaces of smooth sections. Then (cf. Kobayashi-Nomizu [8])

(1.3) $\alpha(X, \bar{Y}) = 0$ for $X, Y \in C^\infty((TM)^\pm)$,

(1.4) $\alpha(X, Y) \in C^\infty((NM)^\pm)$ for $X, Y \in C^\infty((TM)^\pm)$,

(1.5) $A_\xi X = 0$ for $\xi \in C^\infty((NM)^\pm)$, $X \in C^\infty((TM)^\pm)$,

(1.6) $A_\xi \bar{X} \in C^\infty((TM)^\pm)$ for $\xi \in C^\infty((NM)^\pm)$, $X \in C^\infty((TM)^\pm)$,

(1.7) (Bianchi) $R(X, \bar{Y})Z = R(Z, \bar{Y})X$ for $X, Y, Z \in C^\infty((TM)^\pm)$,

(1.8) (Gauss) $A_{\alpha(X,Y)}\bar{Z} = \frac{c}{2}\{\langle X, \bar{Z}\rangle Y + \langle Y, \bar{Z}\rangle X\} - R(X, \bar{Z})Y$ for $X, Y, Z \in C^\infty((TM)^\pm)$,

(1.9) (Ricci) $\alpha(X, A_\xi \bar{Y}) = R^\perp(X, \bar{Y})\xi - \frac{c}{2}\langle X, \bar{Y}\rangle \xi$ for $X, Y \in C^\infty((TM)^\pm)$, $\xi \in C^\infty((NM)^\pm)$.

Here the $\boldsymbol{C}$-linear extension of the Fubini-Study metric is also denoted by $\langle\,,\rangle$. The equations (1.8) and (1.9) follow from Gauss-Ricci equations for a general isometric immersion and (1.3)$\sim$(1.6).

2. Projective manifolds associated to hermitian symmetric Lie algebras

Let $(\mathfrak{g}, \sigma, J)$ be an effective hermitian symmetric Lie algebra of compact type, i.e., $(\mathfrak{g}, \sigma)$ is an effective orthogonal symmetric Lie algebra of compact type and J is a $\mathfrak{k}$-invariant complex structure on $\mathfrak{p}$, where

$$\mathfrak{k} = \{X \in \mathfrak{g};\ \sigma X = X\},$$
$$\mathfrak{p} = \{X \in \mathfrak{g};\ \sigma X = -X\}.$$

Let $\mathfrak{g}^C$, $\mathfrak{k}^C$ and $\mathfrak{p}^C$ denote the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$, respectively, and $\mathfrak{p}^C = \mathfrak{p}^+ + \mathfrak{p}^-$ the decomposition by J as in §1. Let $\mathfrak{t}$ be an arbitrary maximal abelian subalgebra of $\mathfrak{k}$, so that the complexification $\mathfrak{h}$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^C$. The root system Σ of $\mathfrak{g}^C$ relative to $\mathfrak{h}$ will be identified with a subset of the real part $\mathfrak{h}_R = \sqrt{-1}\,\mathfrak{t}$ by means of the Killing form $(\ ,\ )$ of $\mathfrak{g}^C$. The root space for $\alpha \in \Sigma$ is denoted by $\mathfrak{g}^C_\alpha$. Let $\Sigma_{\mathfrak{k}} \subset \Sigma$ denote the set of roots of $\mathfrak{k}^C$ and put $\Sigma_{\mathfrak{p}} = \Sigma - \Sigma_{\mathfrak{k}}$. For an arbitrary (lexicographic) order on $\mathfrak{h}_R$, the set of positive roots and the fundamental root system are denoted by Σ^+ and Π, respectively, and put $\Sigma^+_{\mathfrak{k}} = \Sigma^+ \cap \Sigma_{\mathfrak{k}}$ and $\Sigma^+_{\mathfrak{p}} = \Sigma^+ \cap \Sigma_{\mathfrak{p}}$.

Lemma 2.1. *Let $E^+ \in \mathfrak{p}^+ - \{0\}$. Let $H_0 = -[E^+, \bar{E}^+] \in \sqrt{-1}\,\mathfrak{k}$, $\mathfrak{g}^C_\lambda$ the λ-eigenspace of $ad(H_0)$ on $\mathfrak{g}^C$ and $\mathfrak{p}^C_0 = \mathfrak{g}^C_0 \cap \mathfrak{p}^C$. Suppose*

(i) $\mathfrak{g}^C = \mathfrak{g}^C_0 + \mathfrak{g}^C_1 + \mathfrak{g}^C_{-1} + \mathfrak{g}^C_2 + \mathfrak{g}^C_{-2}$, *with* $\mathfrak{g}^C_2 = CE^+$;

(ii) $\mathfrak{p}^C_0 \subset [\mathfrak{g}^C_1, \mathfrak{g}^C_{-1}]$.

Then $(\mathfrak{g}, \sigma, J)$ is irreducible, and therefore $\mathfrak{p}^+$ is an irreducible $\mathfrak{k}$-module.

Proof. Put $E^- = \bar{E}^+$ and $\mathfrak{s} = \{H_0, E^+, E^-\}_C$. Then we have $[H_0, E^\pm] = \pm 2E^\pm$, $[E^+, E^-] = -H_0$, and so $\mathfrak{s}$ is a 3-dimensional simple subalgebra of $\mathfrak{g}^C$. Suppose that $(\mathfrak{g}, \sigma, J)$ is not irreducible, i.e., $(\mathfrak{g}, \sigma, J)$ has a non-trivial direct sum decomposition

$$(\mathfrak{g}, \sigma, J) = (\mathfrak{g}_{(1)}, \sigma_{(1)}, J_{(1)}) \oplus (\mathfrak{g}_{(2)}, \sigma_{(2)}, J_{(2)}).$$

Let π_i: $\mathfrak{g}^C \to \mathfrak{g}^C_{(i)}$ be projections and $\mathfrak{s}_i = \pi_i(\mathfrak{s})$ for $i = 1, 2$.

Suppose first that both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are not $\{0\}$. Then π_i: $\mathfrak{s} \to \mathfrak{s}_i$ is an isomorphism and hence $E^+_i = \pi_i(E^+) \neq 0$ for each i. Denoting by

$$\mathfrak{g}^C_{(i)} = \sum_\lambda (\mathfrak{g}^C_{(i)})_\lambda \qquad (i = 1, 2)$$

the same decomposition of $\mathfrak{g}^C_{(i)}$ with respect to E^+_i, we get

$$\mathfrak{g}^C_2 = (\mathfrak{g}^C_{(1)})_2 \oplus (\mathfrak{g}^C_{(2)})_2 \quad \text{with} \quad E^+_i \in (\mathfrak{g}^C_{(i)})_2.$$

This is a contradiction to (i): $\dim \mathfrak{g}^C_2 = 1$.

Suppose next that say $\mathfrak{s}_1 \neq \{0\}$ and $\mathfrak{s}_2 = \{0\}$. Then $\mathfrak{s} \subset \mathfrak{g}^C_{(1)}$, and hence $\mathfrak{g}^C_{\pm 1} = (\mathfrak{g}^C_{(1)})_{\pm 1}$, $\mathfrak{p}^C_0 = (\mathfrak{p}^C_{(1)})_0 \oplus \mathfrak{p}^C_{(2)}$ under the obvious notation, and thus $[\mathfrak{g}^C_1, \mathfrak{g}^C_{-1}] \subset \mathfrak{g}^C_{(1)}$. This is a contradiction to (ii). q.e.d.

Lemma 2.2. *Let $(\mathfrak{g}, \sigma, J)$ be irreducible and $E^+ \in \mathfrak{p}^+ - \{0\}$. Then the* *following three conditions for* E^+ *are mutually equivalent.*

1) E^+ *defines the decomposition of* $\mathfrak{g}^C$ *with properties* (i) *and* (ii) *in Lemma* 2.1;

2) E^+ *is a highest weight vector* $(\neq 0)$ *of the irreducible* $\mathfrak{k}$*-module* $\mathfrak{p}^+$ *with respect to an order on* $\mathfrak{h}_R=\sqrt{-1}\mathfrak{t}$ *for a maximal abelian subalgebra* $\mathfrak{t}$ *of* $\mathfrak{k}$, *normalized as* $-[[E^+, \bar{E}^+], E^+]=2E^+$;

3) E^+ *is a highest root vector* $(\neq 0)$ *of* $\mathfrak{g}^C$ *with respect to an order on* $\mathfrak{h}_R=\sqrt{-1}\mathfrak{t}$ *with the following properties* (a) *and* (b) *for a maximal abelian subalgebra* $\mathfrak{t}$ *of* $\mathfrak{k}$, *normalized as* $-[[E^+, \bar{E}^+], E^+]=2E^+$:

(a) $\mathfrak{p}^+ = \sum_{\alpha\in\Sigma_{\mathfrak{p}}^+} \mathfrak{g}_\alpha^C$,

(b) $\Sigma_{\mathfrak{k}} = \Sigma \cap \{\Pi_{\mathfrak{k}}\}_Z$, *where* $\Pi_{\mathfrak{k}} = \Pi \cap \Sigma_{\mathfrak{k}}$.

These vectors E^+ *are conjugate to each other under the action of the connected subgroup* K^+ *of* $GL(\mathfrak{p}^+)$ *generated by* $ad_{\mathfrak{p}^+}(\mathfrak{k})$.

Proof. 1)⇒2): We define

$$\mathfrak{k}_\lambda^C = \mathfrak{k}^C \cap \mathfrak{g}_\lambda^C, \quad \mathfrak{p}_\lambda^C = \mathfrak{p}^C \cap \mathfrak{g}_\lambda^C,$$
$$\mathfrak{k}_0^{C\prime} = \{X\in\mathfrak{k}_0^C;\ (X, H_0) = 0\}.$$

Then $\mathfrak{k}_0^C=\mathfrak{k}_0^{C\prime}\oplus CH_0$ (direct sum of ideals), and by (i) we have decompositions

$$\mathfrak{k}^C = \mathfrak{k}_0^C+\mathfrak{k}_1^C+\mathfrak{k}_{-1}^C, \tag{2.1}$$

$$\mathfrak{p}^C = \mathfrak{p}_0^C+\mathfrak{p}_1^C+\mathfrak{p}_{-1}^C+\mathfrak{p}_2^C+\mathfrak{p}_{-2}^C, \quad \text{with} \quad \mathfrak{p}_2^C = CE^+. \tag{2.2}$$

Now by applying the representation theory of 3-dimensional complex simple Lie algebras (cf. Serre [17]) to the $\mathfrak{s}$-module $\mathfrak{g}^C$, we know that the kernel Ker $(ad(E^+))$ of $ad(E^+)$ is given by

$$\mathrm{Ker}\,(ad(E^+)) = \mathfrak{k}_0^{C\prime}+\mathfrak{p}_0^C+\mathfrak{g}_1^C+\mathfrak{g}_2^C.$$

We take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ containing $\sqrt{-1}H_0$, and put

$$\mathcal{B} = \{\alpha\in\Sigma_{\mathfrak{k}};\ (\alpha, H_0)\geqq 0\},$$

under the notation in the beginning of this section. Then $[\mathfrak{g}_\alpha^C, E^+]=\{0\}$ for each $\alpha\in\mathcal{B}$, since $\mathfrak{g}_\alpha^C\subset\mathfrak{k}_0^{C\prime}+\mathfrak{k}_1^C\subset\mathrm{Ker}\,(ad(E^+))$. Moreover $[H', E^+]=0$ for each $H'\in\mathfrak{h}_R$ with $(H', H_0)=0$, since $H'\in\mathfrak{k}_0^{C\prime}\subset\mathrm{Ker}\,(ad(E^+))$. Thus the element $\gamma_1\in\mathfrak{h}_R$ determined by

$$(\gamma_1, aH_0+H') = 2a \quad \text{for} \quad a\in R,\ H'\in\mathfrak{h}_R \quad \text{with} \quad (H', H_0) = 0,$$

is a weight of the $\mathfrak{k}$-module $\mathfrak{p}^+$ with a weight vector E^+. On the other hand, by a characterization of positive roots by Borel-Hirzebruch [1] we may find an order on $\mathfrak{h}_R$ such that $\Sigma_{\mathfrak{k}}^+\subset\mathcal{B}$. Thus E^+ is a highest weight vector with respect to this order. The normalization condition follows by $E^+\in\mathfrak{p}_2^C$.

2)⇒3): Let E^+ be a normalized weight vector in $\mathfrak{p}^+$ belonging to the highest weight γ_1 with respect to an order $>''$ on $\mathfrak{h}_R=\sqrt{-1}\mathfrak{t}$. Let $\Pi_{\mathfrak{k}}''$ denote the fundamental root system of $\Sigma_{\mathfrak{k}}$. By a theorem of Borel-Hirzebruch [1] there is an order $>'$ on $\mathfrak{h}_R$ such that

$$\text{(a)}' \quad \mathfrak{p}^+ = \sum_{\alpha\in\Sigma_{\mathfrak{p}}'^+} \mathfrak{g}_\alpha^C ,$$

$$\text{(b)}' \quad \Sigma_{\mathfrak{k}} = \Sigma\cap\{\Pi_{\mathfrak{k}}'\}_Z , \quad \text{where} \quad \Pi_{\mathfrak{k}}' = \Pi'\cap\Sigma_{\mathfrak{k}} .$$

We choose an element s of the Weyl group of $\mathfrak{k}$ such that $s\Pi_{\mathfrak{k}}'=\Pi_{\mathfrak{k}}''$. Let $>$ be an order on $\mathfrak{h}_R$ such that its fundamental root system Π of Σ coincides with $s\Pi'$. We put $\Pi_{\mathfrak{k}}=\Pi\cap\Sigma_{\mathfrak{k}}$. Then we have $s\Sigma_{\mathfrak{p}}'^+=\Sigma_{\mathfrak{p}}^+$. On the other hand, we have $s\Sigma_{\mathfrak{p}}'^+=\Sigma_{\mathfrak{p}}'^+$ in virtue of (a)′. Thus we have

$$\text{(a)} \quad \mathfrak{p}^+ = \sum_{\alpha\in\Sigma_{\mathfrak{p}}^+} \mathfrak{g}_\alpha^C .$$

Moreover we have $\Pi_{\mathfrak{k}}''=s(\Pi'\cap\Sigma_{\mathfrak{k}})=\Pi\cap\Sigma_{\mathfrak{k}}=\Pi_{\mathfrak{k}}$, and hence $\Sigma\cap\{\Pi_{\mathfrak{k}}\}_Z=\Sigma\cap\{\Pi_{\mathfrak{k}}''\}_Z=s\Sigma_{\mathfrak{k}}$ by (b)′. Thus we get

$$\text{(b)} \quad \Sigma_{\mathfrak{k}} = \Sigma\cap\{\Pi_{\mathfrak{k}}\}_Z , \quad \text{where} \quad \Pi_{\mathfrak{k}} = \Pi\cap\Sigma_{\mathfrak{k}} .$$

Now $\Pi_{\mathfrak{k}}''=\Pi_{\mathfrak{k}}$ implies $\Sigma_{\mathfrak{k}}''^+=\Sigma_{\mathfrak{k}}^+$. Therefore γ_1 is highest in $\Sigma_{\mathfrak{p}}^+$, and thus it is highest in Σ^+.

3)⇒1): Let E^+ be a normalized root vector of $\mathfrak{g}^C$ belonging to the highest root γ_1 with respect to an order on $\mathfrak{h}_R=\sqrt{-1}\mathfrak{t}$ with (a), (b). Let $\Delta=\{\gamma_1,\cdots,\gamma_r\}$ be the maximal system of strongly orthogonal roots in $\Sigma_{\mathfrak{p}}^+$ in the sense of Takeuchi [19], and $\varpi\colon\mathfrak{h}_R\to\{\Delta\}_R$ the orthogonal projection with respect to (,). Then (Takeuchi [19])

$$\text{(2.3)} \quad \begin{cases} \varpi\Sigma_{\mathfrak{p}}^+ = \left\{\dfrac{1}{2}(\gamma_i+\gamma_j) \quad (1\leqq i\leqq j\leqq r)\right\}, \\ \varpi\Sigma_{\mathfrak{k}}^+ = \left\{\dfrac{1}{2}(\gamma_i-\gamma_j) \quad (1\leqq i\leqq j\leqq r)\right\}, \end{cases}$$

or

$$\text{(2.4)} \quad \begin{cases} \varpi\Sigma_{\mathfrak{p}}^+ = \left\{\dfrac{1}{2}(\gamma_i+\gamma_j) \quad (1\leqq i\leqq j\leqq r),\ \dfrac{1}{2}\gamma_i(1\leqq i\leqq r)\right\}, \\ \varpi\Sigma_{\mathfrak{k}}^+ = \left\{\dfrac{1}{2}(\gamma_i-\gamma_j) \quad (1\leqq i\leqq j\leqq r),\ \dfrac{1}{2}\gamma_i(1\leqq i\leqq r)\right\}, \end{cases}$$

and a root $\alpha\in\Sigma_{\mathfrak{p}}^+$ such that $\varpi(\alpha)=\gamma_i$ is the only $\alpha=\gamma_i$ $(1\leqq i\leqq r)$. Moreover the normalization condition implies that $H_0=-[E^+,\bar{E}^+]$ is given by

$$\text{(2.5)} \qquad H_0 = \frac{2}{(\gamma_1,\gamma_1)}\gamma_1 .$$

Since $(\gamma_i, H_0)=2\delta_{i1}\,(1\leqq i\leqq r)$, the eigenspace decomposition of $ad(H_0)$ is as (2.1) and (2.2). Thus we get the condition (i).

In order to prove the condition (ii) we consider the involutive automorphism $\theta=\exp ad(\pi\sqrt{-1}H_0)$ of $\mathfrak{g}$. We put

$$\mathfrak{g}_\pm=\{X\in\mathfrak{g};\ \theta X=\pm X\}\,.$$

Their complexifications $\mathfrak{g}_\pm^C$ are given by

$$\mathfrak{g}_+^C=\mathfrak{k}_0^C+\mathfrak{p}_0^C+\mathfrak{p}_2^C+\mathfrak{p}_{-2}^C\,,$$
$$\mathfrak{g}_-^C=\mathfrak{g}_1^C+\mathfrak{g}_{-1}^C\,.$$

If $\mathfrak{g}_-=\{0\}$, then $\mathfrak{p}_0^C=\{0\}$ in virtue of (2.3), (2.4). So we may assume $\mathfrak{g}_-\neq\{0\}$. Now the theory of symmetric Lie algebras implies $[\mathfrak{g}_-, \mathfrak{g}_-]=\mathfrak{g}_+$, because $\mathfrak{g}$ is simple and thus $(\mathfrak{g}, \theta)$ is effective. Since $[\mathfrak{g}_-^C, \mathfrak{g}_-^C]=[\mathfrak{g}_1^C, \mathfrak{g}_{-1}^C]$, we obtain the condition (ii).

The second statement of the lemma follows from the fact that the vectors with 2) are conjugate to each other under the action of K^+. Note here that K^+ contains the circle group $\{\varepsilon I_{\mathfrak{p}^+};\ \varepsilon\in\boldsymbol{C},\ |\varepsilon|=1\}$. q.e.d.

In the following in this section we assume that $(\mathfrak{g}, \sigma, J)$ is irreducible. Take an element $E^+\in\mathfrak{p}^+$ as in Lemma 2.2 and fix it. Let $E^-=\bar{E}^+\in\mathfrak{p}^-$, $E=E^++E^-\in\mathfrak{p}$, $H_0=-[E^+, E^-]\in\sqrt{-1}\mathfrak{k}$ and (2.1), (2.2) the decompositions by $ad(H_0)$. By (2.3), (2.4) and (2.5) we have $\mathfrak{p}_{\pm\lambda}^C\subset\mathfrak{p}^\pm$ for $\lambda=2, 1$. So we write

$$\mathfrak{p}_\lambda^\pm=\mathfrak{p}_{\pm\lambda}^C\qquad(\lambda=2, 1)\,.$$

We put furthermore

$$\mathfrak{p}_0^\pm=\mathfrak{p}_0^C\cap\mathfrak{p}^\pm\,,$$
$$\mathfrak{m}^\pm=\mathfrak{k}_{\mp1}^C\,.$$

Thus (2.1) and (2.2) are written as

$$\mathfrak{k}^C=\mathfrak{k}_0^C+\mathfrak{m}^++\mathfrak{m}^-\,,$$
$$\mathfrak{p}^\pm=\mathfrak{p}_2^\pm+\mathfrak{p}_1^\pm+\mathfrak{p}_0^\pm,\quad\text{with}\quad\mathfrak{p}_2^\pm=\boldsymbol{C}E^\pm\,.$$

We define

$$\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{k}_0^C,$$
$$\mathfrak{m}=\mathfrak{k}\cap(\mathfrak{m}^++\mathfrak{m}^-)\,,$$
$$\mathfrak{p}_\lambda=\mathfrak{p}\cap(\mathfrak{p}_\lambda^++\mathfrak{p}_\lambda^-)\qquad(\lambda=2, 1, 0)\,.$$

We have then direct sum decompositions

(2.6) $$\mathfrak{k}=\mathfrak{k}_0+\mathfrak{m}\,,$$

(2.7) $$\mathfrak{p}=\mathfrak{p}_2+\mathfrak{p}_1+\mathfrak{p}_0\,.$$

The complexifications $\mathfrak{m}^C$ and $(\mathfrak{p}_\lambda)^C$ of $\mathfrak{m}$ and $\mathfrak{p}_\lambda$, respectively, have direct sum decompositions

$$\mathfrak{m}^C = \mathfrak{m}^+ + \mathfrak{m}^- , \tag{2.8}$$

$$(\mathfrak{p}_\lambda)^C = \mathfrak{p}_\lambda^+ + \mathfrak{p}_\lambda^- \qquad (\lambda = 2, 1, 0) . \tag{2.9}$$

Considering that $[\mathfrak{k}^C, \mathfrak{k}^C] \subset \mathfrak{k}^C$, $[\mathfrak{k}^C, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$, $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] = \{0\}$, $[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}^C$ and $[\mathfrak{g}_\lambda^C, \mathfrak{g}_\mu^C] \subset \mathfrak{g}_{\lambda+\mu}^C$, we get the following relations.

$$\begin{aligned}
&[\mathfrak{m}^\pm, \mathfrak{m}^\pm] = \{0\} , \quad [\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{k}_0^C, \quad [\mathfrak{k}_0^C, \mathfrak{m}^\pm] \subset \mathfrak{m}^\pm ,\\
&[\mathfrak{p}_\lambda^\pm, \mathfrak{p}_\lambda^\pm] = \{0\} , \quad [\mathfrak{p}_\lambda^+, \mathfrak{p}_\lambda^-] \subset \mathfrak{k}_0^C \qquad (\lambda = 2, 1, 0) ,\\
&[\mathfrak{p}_0^\pm, \mathfrak{p}_1^+] \subset \mathfrak{m}^- , \quad [\mathfrak{p}_0^\pm, \mathfrak{p}_1^-] \subset \mathfrak{m}^+ ,\\
&[\mathfrak{p}_0^\pm, \mathfrak{p}_2^+] = [\mathfrak{p}_0^\pm, \mathfrak{p}_2^-] = [\mathfrak{p}_1^\pm, \mathfrak{p}_2^\pm] = \{0\} ,\\
&[\mathfrak{p}_1^\pm, \mathfrak{p}_2^\mp] \subset \mathfrak{m}^\pm, \quad [\mathfrak{k}_0^C, \mathfrak{p}_\lambda^\pm] \subset \mathfrak{p}_\lambda^\pm \quad (\lambda = 2, 1, 0) ,\\
&[\mathfrak{m}^\pm, \mathfrak{p}_0^\pm] = \{0\} , \quad [\mathfrak{m}^\pm, \mathfrak{p}_0^\mp] \subset \mathfrak{p}_1^\mp ,\\
&[\mathfrak{m}^\pm, \mathfrak{p}_1^\pm] \subset \mathfrak{p}_0^\pm , \quad [\mathfrak{m}^\pm, \mathfrak{p}_1^\mp] \subset \mathfrak{p}_2^\mp ,\\
&[\mathfrak{m}^\pm, \mathfrak{p}_2^\pm] \subset \mathfrak{p}_1^\pm , \quad [\mathfrak{m}^\pm, \mathfrak{p}_2^\mp] = \{0\} .
\end{aligned}$$

From these we have the following relations.

$$\begin{aligned}
&[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}_0 , \quad [\mathfrak{k}_0, \mathfrak{m}] \subset \mathfrak{m} ,\\
&[\mathfrak{p}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{k}_0 \qquad (\lambda = 2, 1, 0) ,\\
&[\mathfrak{p}_0, \mathfrak{p}_1] \subset \mathfrak{m} , \quad [\mathfrak{p}_0, \mathfrak{p}_2] = \{0\} , \quad [\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{m} ,\\
&[\mathfrak{k}_0, \mathfrak{p}_\lambda] \subset \mathfrak{p}_\lambda \qquad (\lambda = 2, 1, 0) ,\\
&[\mathfrak{m}, \mathfrak{p}_0] \subset \mathfrak{p}_1 , \quad [\mathfrak{m}, \mathfrak{p}_1] \subset \mathfrak{p}_0 + \mathfrak{p}_2 , \quad [\mathfrak{m}, \mathfrak{p}_2] \subset \mathfrak{p}_1 .
\end{aligned}$$

These relations will be constantly used in the sequel.

We define linear maps i: $\mathfrak{m} \to \mathfrak{p}_1$ and h: $\mathfrak{p}_1 \to \mathfrak{m}$ by

$$\begin{aligned}
i(X) &= [X, E] \qquad \text{for} \quad X \in \mathfrak{m} ,\\
h(X) &= [E, X] \qquad \text{for} \quad X \in \mathfrak{p}_1 .
\end{aligned}$$

Note here that $E^\pm \in \mathfrak{p}_2^\pm$, $E \in \mathfrak{p}_2$, and so i and h are well defined.

Lemma 2.3. *Both i and h are isomorphisms. More precisely, we have $h \circ i = I_{\mathfrak{m}}$ and $i \circ h = I_{\mathfrak{p}_1}$.*

Proof. Let $\mathfrak{s}$ be the 3-dimensional simple subalgebra of $\mathfrak{g}^C$ defined in the proof of Lemma 2.1. Then by applying to $\mathfrak{s}$ the representation theory of 3-dimensional complex simple Lie algebras, we know that $ad(E^\pm)$ induces an isomorphism $\mathfrak{g}_{\mp1}^C \to \mathfrak{g}_{\pm1}^C$. In particular, $ad(E^\pm)$ induces isomorphisms $\mathfrak{p}_{\mp1}^C \to \mathfrak{k}_{\pm1}^C$ and $\mathfrak{k}_{\mp1}^C \to \mathfrak{p}_{\pm1}^C$. Moreover $ad(E^\mp)ad(E^\pm)$ induces $-$(identity) on $\mathfrak{g}_{\mp1}^C$. In fact, since $ad(E^\mp)ad(E^\pm) = ad(E^\pm)ad(E^\mp) \pm ad(H_0)$ we have

$$ad(E^{\mp})ad(E^{\pm})X = \pm ad(H_0)X = -X \quad \text{for} \quad X \in \mathfrak{g}^{C}_{\pm 1}.$$

In particular, $ad(E^{\mp})ad(E^{\pm})$ induces $-$(identity) on $\mathfrak{k}^{C}_{\mp 1}$ and on $\mathfrak{p}^{C}_{\mp 1}$.

Now the $\boldsymbol{C}$-linear extensions of i and h, denoted also by i and h, are given by

$$(2.10) \qquad i(X) = [X, E^{\pm}] \in \mathfrak{p}^{\pm}_{1} \quad \text{for} \quad X \in \mathfrak{m}^{\pm} = \mathfrak{k}^{C}_{\mp 1},$$

$$(2.11) \qquad h(X) = [E^{\mp}, X] \in \mathfrak{m}^{\pm} \quad \text{for} \quad X \in \mathfrak{p}^{\pm}_{1}.$$

Thus $h \circ i = -ad(E^{\mp})ad(E^{\pm})$ on $\mathfrak{m}^{\pm}$ and $i \circ h = -ad(E^{\pm})ad(E^{\mp})$ on $\mathfrak{p}^{\pm}_{1}$. Therefore by the facts proved above we get $h \circ i = I_{\mathfrak{m}}$ and $i \circ h = I_{\mathfrak{p}_1}$. q.e.d.

Now we shall construct a complex submanifold M of the projective space $P(\mathfrak{p}, J)$ associated to $(\mathfrak{p}, J)$ with parallel second fundamental form.

Fix $c>0$ and take a (unique) $\mathfrak{k}$-invariant inner product $\langle\, ,\rangle$ on $\mathfrak{p}$ such that $\langle E, E\rangle = 4/c$ and

$$\langle JX, JY\rangle = \langle X, Y\rangle \quad \text{for} \quad X, Y \in \mathfrak{p}.$$

Then $\langle\, ,\rangle$ is uniquely extended to a $\mathfrak{g}$-invariant inner product on $\mathfrak{g}$, which will be also denoted by $\langle\, ,\rangle$, and (2.6) and (2.7) are orthogonal sums with respect to $\langle\, ,\rangle$. The $\boldsymbol{C}$-linear extension of $\langle\, ,\rangle$ to $\mathfrak{g}^{C}$ is also denoted by $\langle\, ,\rangle$. Let $G \subset GL(\mathfrak{g})$ (resp. $G^{C} \subset GL(\mathfrak{g}^{C})$) be the adjoint group of $\mathfrak{g}$ (resp. of $\mathfrak{g}^{C}$) and K (resp. K^{C}) the connected subgroup of G (resp. of G^{C}) generated by $\mathfrak{k}$ (resp. by $\mathfrak{k}^{C}$). We may regard as $G \subset G^{C}$ and $K \subset K^{C}$. Then K is a maximal compact subgroup of K^{C}. Let $\rho: K \to GL(\mathfrak{p}, J)$ and $\rho^{+}: K \to GL(\mathfrak{p}^{+})$ be the natural (faithful unitary) representations. (Note that $\rho^{+}(K) = K^{+}$.) They are equivalent by the $\boldsymbol{C}$-linear isomorphism $\varpi^{+}: (\mathfrak{p}, J) \to \mathfrak{p}^{+}$. Their complexifications are also denoted by $\rho: K^{C} \to GL(\mathfrak{p}, J)$ and $\rho^{+}: K^{C} \to GL(\mathfrak{p}^{+})$. Through these representations the groups K and K^{C} act on complex projective spaces $P(\mathfrak{p}, J)$ and $P(\mathfrak{p}^{+})$. Putting $o = [E] \in P(\mathfrak{p}, J)$ we define

$$M = K \cdot o \subset P(\mathfrak{p}, J).$$

We know the following properties of M (Takeuchi [20]): We have $M = K^{C} \cdot o$, and thus it is holomorphically isomorphic to $K^{C} \cdot [E^{+}] \subset P(\mathfrak{p}^{+})$ by the holomorphic isomorphism $P(\mathfrak{p}, J) \to P(\mathfrak{p}^{+})$ induced by ϖ^{+}. Since ρ^{+} is irreducible and E^{+} is a highest weight vector in $\mathfrak{p}^{+}$, M is a kählerian C-space and it is a full compact connected complex submanifold of $P(\mathfrak{p}, J)$. The center of K^{C} acts on M trivially. The derived group $K'^{C} = [K^{C}, K^{C}]$ acts almost effectively on M and it is locally isomorphic to the identity component $\mathrm{Aut}^{0}(M)$ of the group of holomorphic automorphisms of M. The coset space structure of M is described as follows. Let

$$\mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \Sigma_{\mathfrak{k}}, (\gamma_1, \alpha) \geq 0} \mathfrak{g}^{C}_{\alpha},$$

under the notation in Lemma 2.2, L the connected complex subgroup of K^C generated by $\mathfrak{l}$ and $K_0 = K \cap L$. Then Lie $K_0 = \mathfrak{k}_0$ and

$$M = K^C/L = K/K_0 .$$

Thus $T_o(M)$ is identified with $\mathfrak{m}$ in virtue of (2.6). The complex submanifold $M \subset P(\mathfrak{p}, J)$ is said to be *associated to* $(\mathfrak{g}, \sigma, J)$. It is independent of the choice of E^+ by the last statement of Lemma 2.2.

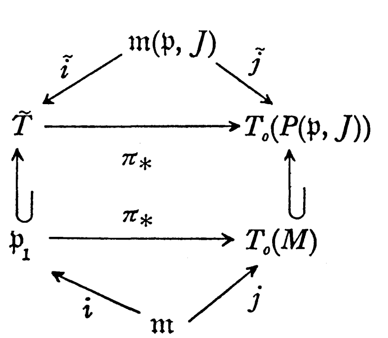
Now we introduce a K-invariant Kähler metric g on $P(\mathfrak{p}, J)$ with constant holomorphic sectional curvature c as in §1 by our inner product $\langle\, ,\rangle$ on $\mathfrak{p}$. The induced Kähler metric on M is also denoted by g. We use the same notation as in §1 for $(M, g) \subset (P(\mathfrak{p}, J), g)$ (f=the inclusion). Note that each $\mathfrak{p}_\lambda$ ($\lambda=2, 1, 0$) is stable under J, thus becomes a hermitian vector space, and (2.9) is the decomposition by J. Moreover, if we denote by J the complex structure on $\mathfrak{m}$ corresponding to the complex structure tensor J_o on $T_o(M)$, then (2.8) is also the decomposition by J. This follows from (2.10). The inner product $\langle\, ,\rangle$ restricted to $\mathfrak{m} \times \mathfrak{m}$ and J define a hermitian vector space structure on $\mathfrak{m}$. The subspace $\tilde{T} \subset \mathfrak{p}$ in §1 is identified as

$$\tilde{T} = \mathfrak{p}_1 + \mathfrak{p}_0 ,$$

since we have $Q = \mathfrak{p}_2$ in our case. We define a linear map j: $\mathfrak{m} \to T_o(M)$ by

$$j(X) = X_o^* \qquad \text{for} \quad X \in \mathfrak{m} .$$

Lemma 2.4. *The diagram*



is commutative, and each of i, j and π_* (on $\mathfrak{p}_1$) is an isomorphism of hermitian vector space. Therefore, g coincides with $\langle\, ,\rangle$ restricted to $\mathfrak{m} \times \mathfrak{m}$ at o, and h: $\mathfrak{p}_1 \to \mathfrak{m}$ is also an isomorphism of hermitian vector space.

Proof. The commutativity $\pi_* \circ i = j$ follows from the definitions, and hence the whole diagram is commutative. Since π_*: $\tilde{T} \to T_o(P(\mathfrak{p}, J))$ is an isomorphism of hermitian vector space and i is a linear isomorphism by Lemma 2.3, it remains to show that i preserves the inner products. For each $X, Y \in \mathfrak{m}$

we have

$$\langle i(X), i(Y)\rangle = \langle [X, E], [Y, E]\rangle = \langle X, [E, [Y, E]]\rangle$$
$$= \langle X, hi(Y)\rangle = \langle X, Y\rangle$$

by Lemma 2.3. This proves the assertion. q.e.d.

Now we identify $\tilde{T}$ with $T_o(P(\mathfrak{p}, J))$ through the isomorphism π_*. Then by the above lemma we have the following identifications.

$$\mathfrak{p}_1 = T_o(M), \quad \mathfrak{p}_1^{\pm} = T_o(M)^{\pm},$$
$$\mathfrak{p}_0 = N_o(M), \quad \mathfrak{p}_0^{\pm} = N_o(M)^{\pm}.$$

Recall also that

$$\mathfrak{p}_2 = Q, \quad \mathfrak{p}_2^{\pm} = Q^{\pm}.$$

Lemma 2.5. 1) *For $X \in \mathfrak{p}$, let X_0 denote the $\mathfrak{p}_0$-component of X with respect to the decomposition* (2.7). *Then*

$$\alpha(X, Y) = [h(X), Y]_0 = [h(Y), X]_0 \quad \textit{for} \quad X, Y \in \mathfrak{p}_1.$$

2) $\alpha(X, Y) = [h(X), Y] = [h(Y), X]$ *for* $X, Y \in \mathfrak{p}_1^+$.

Proof. 1) Let $X \in \mathfrak{p}_1$ and $X' \in \mathfrak{m}(\mathfrak{p}, J)$ with $\tilde{i}(X') = X$. For $Z \in \mathfrak{u}(\mathfrak{p}, J)$, $Z_{\mathfrak{k}(\mathfrak{p})}$ denotes the $\mathfrak{k}(\mathfrak{p}, J)$-component of Z with respect to the Cartan decomposition $\mathfrak{u}(\mathfrak{p}, J) = \mathfrak{k}(\mathfrak{p}, J) + \mathfrak{m}(\mathfrak{p}, J)$. Then (cf. Helgason [3]) the covariant derivative in $(P(\mathfrak{p}, J), g)$ of Z^* by $x = \pi_*(X) \in T_o(M)$ is given by

$$\tilde{\nabla}_x Z^* = \tilde{j}([Z_{\mathfrak{k}(\mathfrak{p})}, X']).$$

Let $y \in T_o(M)$, $Y \in \mathfrak{p}_1$ with $\pi_*(Y) = y$, $\rho_* = ad_{\mathfrak{p}}: \mathfrak{k} \to \mathfrak{u}(\mathfrak{p}, J)$ be the differential of ρ, and put $\tilde{Y} = \rho_*(h(Y)) \in \mathfrak{u}(\mathfrak{p}, J)$. We shall compute $\tilde{\nabla}_x Z^*$ for $Z = \tilde{Y}$. We define a linear map $\phi: \mathfrak{p}_1 \to \mathfrak{p}_0$ by

$$\phi(W) = [h(Y), W]_0 \qquad \text{for} \quad W \in \mathfrak{p}_1.$$

Then $\tilde{Y} \cdot \mathfrak{p}_2 \subset \mathfrak{p}_1$, $\tilde{Y} \cdot \mathfrak{p}_1 \subset \mathfrak{p}_2 + \mathfrak{p}_0$, $\tilde{Y} \cdot \mathfrak{p}_0 \subset \mathfrak{p}_1$ and the $\mathfrak{p}_0$-component of $\tilde{Y} \cdot W$ equals $\phi(W)$ for each $W \in \mathfrak{p}_1$. Thus $\tilde{Y}_{\mathfrak{k}(\mathfrak{p})} \cdot \mathfrak{p}_2 = \{0\}$, $\tilde{Y}_{\mathfrak{k}(\mathfrak{p})} \cdot W = \phi(W)$ for each $W \in \mathfrak{p}_1$. Therefore

$$\tilde{i}([\tilde{Y}_{\mathfrak{k}(\mathfrak{p})}, X']) = (\tilde{Y}_{\mathfrak{k}(\mathfrak{p})} X' - X' \tilde{Y}_{\mathfrak{k}(\mathfrak{p})}) \cdot E$$
$$= \tilde{Y}_{\mathfrak{k}(\mathfrak{p})} \cdot (X' \cdot E) = \tilde{Y}_{\mathfrak{k}(\mathfrak{p})} \cdot X = \phi(X),$$

and hence

$$\tilde{\nabla}_x \tilde{Y}^* = \tilde{j}([\tilde{Y}_{\mathfrak{k}(\mathfrak{p})}, X']) = \pi_*(\tilde{i}([\tilde{Y}_{\mathfrak{k}(\mathfrak{p})}, X']))$$
$$= \pi_*(\phi(X)) \in N_o(M).$$

On the other hand, by Lemma 2.3 we have

$$(\tilde{Y}^*)_o = \pi_*(\tilde{Y}\cdot E) = \pi_*([h(Y), E]) = \pi_* ih(Y) \\ = \pi_*(Y) = y .$$

Thus we get

$$\alpha(x, y) = \pi_*(\phi(X)) = \pi_*([h(Y), X]_0) ,$$

which proves the assertion 1).

2) This follows from 1), $[\mathfrak{m}^+, \mathfrak{p}_1^+] \subset \mathfrak{p}_0^+$ and $h(\mathfrak{p}_1^+) = \mathfrak{m}^+$. q.e.d.

Lemma 2.6. $A_\xi X = -[h(X), \xi] = [\xi, h(X)]$ *for* $\xi \in \mathfrak{p}_0$, $X \in \mathfrak{p}_1$.

Proof. For each $Y \in \mathfrak{p}_1$, by Lemma 2.4 we have

$$\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle = \langle [h(X), Y], \xi\rangle \\ = -\langle [h(X), \xi], Y\rangle .$$

q.e.d.

Theorem 2.7. *Let $M \subset P(\mathfrak{p}, J)$ be the complex submanifold associated to an irreducible hermitian symmetric Lie algebra $(\mathfrak{g}, \sigma, J)$ of compact type. Then the second fundamental form α of M is parallel.*

Proof. Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}_0$, $[\mathfrak{k}_0, \mathfrak{m}] \subset \mathfrak{m}$, our (M, g) is a locally symmetric Kähler manifold. Thus by Takeuchi [20] the parallelness of α is equivalent to $\alpha(\mathfrak{p}_1^+ \otimes \mathfrak{p}_1^+) = \mathfrak{p}_0^+$. With the notation in the proof of Lemma 2.2, we may assume that $\mathfrak{g}_- \neq \{0\}$ and hence $[\mathfrak{g}_-^C, \mathfrak{g}_-^C] = \mathfrak{g}_+^C$. Then by the condition (ii) in Lemma 2.1 we have $[\mathfrak{k}_{-1}^C, \mathfrak{p}_1^+] = \mathfrak{p}_0^+$. Now $\alpha(\mathfrak{p}_1^+ \otimes \mathfrak{p}_1^+) = \mathfrak{p}_0^+$ follows from Lemmas 2.3 and 2.5. q.e.d.

3. Jordan triple products and second fundamental forms

Let $(\mathfrak{g}, \sigma, J)$ be an irreducible hermitian symmetric Lie algebra of compact type, and retain the notation in §2. We define a triple product on $\mathfrak{p}^+$ by

$$\{x, y, z\} = -[[x, \bar{y}], z] \qquad \text{for} \quad x, y, z \in \mathfrak{p}^+ ,$$

and then an endomorphism $x \square y$ of $\mathfrak{p}^+$ for $x, y \in \mathfrak{p}^+$ by

$$(x \square y)z = \{x, y, z\} \qquad \text{for} \quad z \in \mathfrak{p}^+ .$$

Then (Koecher [9], cf. also Satake [16]) the triple system $(\mathfrak{p}^+, \square)$ becomes a *positive definite hermitian Jordan triple system*, i.e., $\mathfrak{p}^+$ is a finite dimensional complex vector space and

(i) $\{x, y, z\}$ is $\boldsymbol{C}$-linear in x, z and conjugate linear in y;

(ii) $(x \square y)z = (z \square y)x$;

(iii) $[l \square m, x \square y] = ((l \square m)x) \square y - x \square ((m \square l)y)$;

(iv) $\tau(x, y) = \operatorname{Tr}(x \square y)$ is a hermitian inner product on $\mathfrak{p}^+$.

Lemma 3.1. *For an endomorphism ϕ of $\mathfrak{p}^+$ we denote by ϕ^* the adjoint endomorphism of ϕ with respect to the hermitian inner product $\langle\langle x, y\rangle\rangle = \langle x, \bar{y}\rangle$. Then*

$$(x\Box y)^* = y\Box x \qquad \text{for} \quad x, y\in\mathfrak{p}^+ .$$

Proof. For each $z, w\in\mathfrak{p}^+$ we have

$$\begin{aligned}\langle\langle(x\Box y)z, w\rangle\rangle &= \langle(x\Box y)z, \bar{w}\rangle = -\langle[[x, \bar{y}], z], \bar{w}\rangle \\ &= \langle z, [[x, \bar{y}], \bar{w}]\rangle = \langle z, \overline{[[\bar{x}, y], w]}\rangle = -\langle z, \overline{[[y, \bar{x}], w]}\rangle \\ &= \langle\langle z, (y\Box x)w\rangle\rangle .\end{aligned}$$

q.e.d.

In the following in this section, let $X, Y, Z, U, \cdots, \xi, \eta, \zeta, \cdots$ and $A, B, C, \cdots$ denote general elements of $\mathfrak{p}_1^+ = T_o(M)^+$, $\mathfrak{p}_0^+ = N_o(M)^+$ and $\mathfrak{p}_2^+ = Q^+$, respectively. The purpose of this section is to prove the following table of $(x\Box y)z$ for our Jordan triple system $(\mathfrak{p}^+, \Box)$.

$x\Box y$ \ z	Z	ζ	C
$X\Box Y$	$R(X, \bar{Y})Z$	$\alpha(X, A_\zeta \bar{Y})$	$\frac{c}{2}\langle X, \bar{Y}\rangle C$
$\xi\Box\eta$	$A_\xi A_{\bar{\eta}} Z$	$2\alpha(A_\xi A_{\bar{\eta}} U, U)$ for $\zeta = \alpha(U, U)$	0
$A\Box B$	$\frac{c}{2}\langle A, \bar{B}\rangle Z$	0	$c\langle A, \bar{B}\rangle C$
$X\Box\xi$	$\frac{c}{2}\langle A_{\bar{\xi}} X, Z\rangle E^+$	$A_\zeta A_{\bar{\xi}} X$	0
$\xi\Box X$	$\alpha(A_\xi \bar{X}, Z)$	0	$\frac{c}{2}\langle \bar{E}^+, C\rangle A_\xi \bar{X}$
$X\Box A$	$\frac{c}{2}\langle \bar{A}, E^+\rangle\alpha(X, Z)$	0	$\frac{c}{2}\langle \bar{A}, C\rangle X$
$A\Box X$	$\frac{c}{2}\langle \bar{X}, Z\rangle A$	$\frac{c}{2}\langle A, \bar{E}^+\rangle A_\zeta \bar{X}$	0
$\xi\Box A$	0	0	0
$A\Box\xi$	0	0	0

Lemma 3.2.

(3.1) $[[X, \bar{Y}], E^+] = -\frac{c}{2}\langle X, \bar{Y}\rangle E^+ .$

(3.2) $[[h(X), h(\bar{Y})], E^+] = \frac{c}{2}\langle X, \bar{Y}\rangle E^+ .$

(3.3) $[X, \bar{Y}] = [h(X), h(\bar{Y})] - \frac{c}{2}\langle X, \bar{Y}\rangle H_0$.

Proof. Since $[X, \bar{Y}]$, $[h(X), h(\bar{Y})] \in \mathfrak{k}_0^C$ we have

$$[[X, \bar{Y}], E^+] = aE^+, \quad [[h(X), h(\bar{Y})], E^+] = bE^+,$$

where

$$a = \langle [[X, \bar{Y}], E^+], \bar{E}^+\rangle / \langle E^+, \bar{E}^+\rangle,$$
$$b = \langle [[h(X), h(\bar{Y})], E^+], \bar{E}^+\rangle / \langle E^+, \bar{E}^+\rangle.$$

Here we have $\langle E^+, \bar{E}^+\rangle = \frac{c}{2}$ by (1.1) and (1.2). On the other hand

$$\langle [[X, \bar{Y}], E^+], \bar{E}^+\rangle = \langle [X, \bar{Y}], [E^+, \bar{E}^+]\rangle = -\langle [X, \bar{Y}], H_0\rangle$$
$$= \langle X, [H_0, \bar{Y}]\rangle = -\langle X, \bar{Y}\rangle.$$

Thus we get $a = -\frac{c}{2}\langle X, \bar{Y}\rangle$. This proves (3.1). In the same way we have

$$\langle [[h(X), h(\bar{Y})], E^+], \bar{E}^+\rangle = \langle h(X), [H_0, h(\bar{Y})]\rangle = \langle h(X), h(\bar{Y})\rangle = \langle X, \bar{Y}\rangle$$

by Lemma 2.4. Thus we get $b = \frac{c}{2}\langle X, \bar{Y}\rangle$. This proves (3.2). By (3.1) we have

$$[[[X, \bar{Y}], E^+], E^-] = \frac{c}{2}\langle X, \bar{Y}\rangle H_0.$$

The left hand side is

$$\begin{aligned} &[[[X, E^+], \bar{Y}], E^-] + [[X, [\bar{Y}, E^+]], E^-] \\ &= [[X, E^-], [\bar{Y}, E^+]] + [X, [[\bar{Y}, E^+], E^-]] && \text{by } [X, E^+] = 0 \\ &= [h(X), h(\bar{Y})] - [X, ih(\bar{Y})] && \text{by (2.10), (2.11)} \\ &= [h(X), h(\bar{Y})] - [X, \bar{Y}], && \text{by Lemma 2.3.} \end{aligned}$$

Thus the equality (3.3) holds. q.e.d.

(I) $X \square Y$

Since (M, g) is symmetric, we have (cf. Helgason [3])

$$R(X, Y)Z = -i([[X', Y'], Z']) \qquad \text{for} \quad X, Y, Z \in \mathfrak{p}_1,$$

where $X', Y', Z' \in \mathfrak{m}$ with $i(X') = X$, $i(Y') = Y$, $i(Z') = Z$. Now let $X, Y, Z \in \mathfrak{p}_1^+$. Then, by Lemma 2.4 $ih(X) = X$, $ih(Y) = Y$, $ih(Z) = Z$, and hence

$$-R(X, \bar{Y})Z = i([[h(X), h(\bar{Y})], h(Z)]) = [[[h(X), h(\bar{Y})], h(Z)], E^+]$$

$$= [[[h(X), h(\bar{Y})], E^+], h(Z)] + [[h(X), h(\bar{Y})], [h(Z), E^+]]$$

$$= \frac{c}{2}\langle X, \bar{Y}\rangle [E^+, h(Z)] + [[h(X), h(\bar{Y})], ih(Z)] \qquad \text{by (3.2)}$$

$$= -\frac{c}{2}\langle X, \bar{Y}\rangle Z + [[h(X), h(\bar{Y})], Z] \qquad \text{by Lemma 2.4}$$

$$= [[X, \bar{Y}], Z], \qquad \text{by (3.3).}$$

Thus we get $(X\Box Y) = R(X, \bar{Y})Z$. We have

$$\langle [[X, \bar{Y}], \zeta], \bar{\xi}\rangle = \langle [[h(X), h(\bar{Y})], \zeta], \bar{\xi}\rangle \qquad \text{by (3.3)}$$

$$= \langle [[h(X), \zeta], h(\bar{Y})], \bar{\xi}\rangle + \langle [h(X), [h(\bar{Y}), \zeta]], \bar{\xi}\rangle$$

$$= -\langle [h(\bar{Y}), \zeta], [h(X), \bar{\xi}]\rangle \qquad \text{by } [h(X), \zeta] = 0$$

$$= -\langle A_\zeta \bar{Y}, A_{\bar{\xi}} X\rangle \qquad \text{by Lemma 2.6}$$

$$= -\langle \alpha(X, A_\zeta \bar{Y}), \bar{\xi}\rangle .$$

Thus we get $[[X, \bar{Y}], \zeta] = -\alpha(X, A_\zeta \bar{Y})$, and hence $(X\Box Y)\zeta = \alpha(X, A_\zeta \bar{Y})$. By (3.1) we have $[[X, \bar{Y}], C] = -\frac{c}{2}\langle X, \bar{Y}\rangle C$, and hence $(X\Box Y)C = \frac{c}{2}\langle X, \bar{Y}\rangle C$.

(II) $\xi\Box\bar{\eta}$

$$\langle [[\xi, \bar{\eta}], Z], \bar{X}\rangle = \langle [\xi, \bar{\eta}], [Z, \bar{X}]\rangle$$

$$= \langle [[Z, \bar{X}], \xi], \bar{\eta}\rangle = -\langle \alpha(Z, A_\xi \bar{X}), \bar{\eta}\rangle \qquad \text{by (I)}$$

$$= -\langle A_\xi \bar{X}, A_{\bar{\eta}} Z\rangle = -\langle A_\xi A_{\bar{\eta}} Z, \bar{X}\rangle .$$

Thus we get $[[\xi, \bar{\eta}], Z] = -A_\xi A_{\bar{\eta}} Z$, and hence $(\xi\Box\eta)Z = A_\xi A_{\bar{\eta}} Z$. For $\zeta = \alpha(U, U)$ we have

$$[[\xi, \bar{\eta}], \zeta] = [[\xi, \bar{\eta}], \alpha(U, U)]$$

$$= [[\xi, \bar{\eta}], [h(U), U]] \qquad \text{by Lemma 2.5}$$

$$= [[[\xi, \bar{\eta}], h(U)], U] + [h(U), [[\xi, \bar{\eta}], U]]$$

$$= [[[\xi, \bar{\eta}], [E^-, U]], U] - [h(U), A_\xi A_{\bar{\eta}} U]$$

$$= [[[E^-, [[\xi, \bar{\eta}], U]], U] - \alpha(U, A_\xi A_{\bar{\eta}} U)$$

$$\text{by } [\mathfrak{p}_0, \mathfrak{p}_2] = \{0\} \text{ and Lemma 2.5}$$

$$= [h([[\xi, \bar{\eta}], U]), U] - \alpha(U, A_\xi A_{\bar{\eta}} U)$$

$$= -2\alpha(A_\xi A_{\bar{\eta}} U, U), \qquad \text{by Lemma 2.5.}$$

Thus we have $(\xi\Box\eta)\zeta = 2\alpha(A_\xi A_{\bar{\eta}} U, U)$. Since $[\mathfrak{p}_0, \mathfrak{p}_2] = \{0\}$ we have $[[\xi, \bar{\eta}], C] = 0$, and hence $(\xi\Box\eta)C = 0$.

(III) $A\Box B$

We show first

$$[A, B] = -\frac{c}{2}\langle A, \bar{B}\rangle H_0 .$$

In fact, let $A=aE^+$, $B=bE^+$ $(a, b\in \boldsymbol{C})$. Then $\langle A, \bar{B}\rangle=a\bar{b}\langle E^+, \bar{E}^+\rangle=\frac{c}{2}a\bar{b}$, and so $a\bar{b}=\frac{c}{2}\langle A, \bar{B}\rangle$. Thus $[A, \bar{B}]=a\bar{b}[E^+, E^-]=-\frac{c}{2}\langle A, \bar{B}\rangle H_0$. Therefore $[[A, \bar{B}], Z]=-\frac{c}{2}\langle A, \bar{B}\rangle Z$, $[[A, \bar{B}], \zeta]=0$ and $[[A, \bar{B}], C]=-c\langle A, \bar{B}\rangle C$. Hence $(A\square B)Z=\frac{c}{2}\langle A, \bar{B}\rangle Z$, $(A\square B)\zeta=0$ and $(A\square B)C=c\langle A, \bar{B}\rangle C$.

(IV) $X\square\xi$

Since $[[X, \bar{\xi}], Z]\in\mathfrak{p}_2^+$, we may write $[[X, \bar{\xi}], Z]=aE^+(a\in\boldsymbol{C})$. Then

$$\langle[[X, \bar{\xi}], Z], \bar{E}^+\rangle = a\langle E^+, \bar{E}^+\rangle = \frac{2}{c}a .$$

The left hand side is

$$\langle[X, \bar{\xi}], [Z,\bar{E}^+]\rangle = -\langle X, [\bar{\xi}, h(Z)]\rangle = -\langle X, A_{\bar{\xi}}Z\rangle$$

by Lemma 2.6. Thus $a = -\frac{c}{2}\langle X, A_{\bar{\xi}}Z\rangle = -\frac{c}{2}\langle A_{\bar{\xi}}X, Z\rangle$. Therefore $[[X, \bar{\xi}], Z]=-\frac{c}{2}\langle A_{\bar{\xi}}X, Z\rangle E^+$, and hence $(X\square\xi)Z=\frac{c}{2}\langle A_{\bar{\xi}}X, Z\rangle E^+$. By (ii) and (II) we have

$$(X\square\xi)\zeta = (\zeta\square\xi)X = A_\zeta A_{\bar{\xi}}X .$$

Since $[\mathfrak{m}^-, \mathfrak{p}_2^+]=\{0\}$ we have $[[X, \bar{\xi}], C]=0$, and hence $(X\square\xi)C=0$.

(V) $\xi\square X$

By Lemma 3.1 we have $\xi\square X=(X\square\xi)^*$. Recalling that $(\xi\square X)\mathfrak{p}_1^+\subset\mathfrak{p}_0^+$, $(\xi\square X)\mathfrak{p}_0^+=\{0\}$ and $(\xi\square X)\mathfrak{p}_2^+\subset\mathfrak{p}_1^+$, we have by (IV) $(\xi\square X)Z=\alpha(A_\xi\bar{X}, Z)$, $(\xi\square X)\zeta=0$ and $(\xi\square X)C=\frac{c}{2}\langle\bar{E}^+, C\rangle A_\xi\bar{X}$.

(VI) $X\square A$

$$[[X, \bar{E}^+], Z] = -[[E^-, X], Z] = -[h(X), Z] = -\alpha(X, Z)$$

by Lemma 2.5. Thus $[[X, \bar{A}], Z]=-\frac{c}{2}\langle\bar{A}, E^+\rangle\alpha(X, Z)$, and hence $(X\square A)Z =\frac{c}{2}\langle\bar{A}, E^+\rangle\alpha(X, Z)$. Since $[\mathfrak{m}^+, \mathfrak{p}_0^+]=\{0\}$ we have $[[X, \bar{A}], \zeta]=0$, and hence $(X\square A)\zeta=0$. By (ii) and (III) we have

$$(X\square A)C = (C\square A)X = \frac{c}{2}\langle\bar{A}, C\rangle X .$$

(VII) $A\square X$

In the same way as in (V), we get $(A\square X)Z=\frac{c}{2}\langle\bar{X}, Z\rangle A$, $(A\square X)\zeta=\frac{c}{2}\langle A, \bar{E}^{+}\rangle A_{\zeta}\bar{X}$ and $(A\square X)C=0$.

(VIII): $\xi\square A=0$ and (IX): $A\square\xi=0$ follow from $[\mathfrak{p}_0, \mathfrak{p}_2]=\{0\}$.

4. Projective manifolds with parallel second fundamental form

Let $V=\boldsymbol{C}^{N+1}$ equipped with the standard hermitian inner product $\{\ ,\ \}$, and $P_N(\boldsymbol{C})=P(V)$ the associated complex projective space with the Kähler metric g with constant holomorphic sectional curvature $c>0$ as in §1. The Fubini-Study space $(P_N(\boldsymbol{C}), g)$ will be abbreviated to $P_N(c)$. Suppose that a full complex submanifold M of $P_N(c)$ with parallel second fundamental form is given. Let g be the induced Kähler metric on M. We use the same notation as in §1 for $(M, g)\subset P_N(c)$. Then (1.8) and (1.9) imply that both R and $R^{\perp}$ are parallel. Fix a point $o\in M$ and choose $E\in V-\{0\}$ such that $[E]=o$ and $\langle E, E\rangle=4/c$. Put $E^{+}=\varpi^{+}(E)\in Q^{+}$. Let T (resp. N) be the subspace of $\tilde{T}$ such that $\pi_*(T)=T_o(M)$ (resp. $\pi_*(N)=N_o(M)$). Then $\tilde{T}=T+N$ (direct sum) and

$$V_{\boldsymbol{R}}=Q+T+N \quad \text{(orthogonal sum with respect to } \langle\ ,\ \rangle),$$
$$V^{+}=Q^{+}+T^{+}+N^{+} \quad \text{(orthogonal sum with respect to } \langle\!\langle\ ,\ \rangle\!\rangle).$$

Through the map π_* we identify T (resp. N) with $T_o(M)$ (resp. with $N_o(M)$), and thus $T^{\pm}$ (resp. $N^{\pm}$) with $T_o(M)^{\pm}$ (resp. with $N_o(M)^{\pm}$). Note that then α: $T^{+}\otimes T^{+}\to N^{+}$ is surjective, as mentioned in the proof of Theorem 2.7. In the following in this and next section, $x, y, z, \cdots, X, Y, Z, W, U, V, L, M, \cdots, \xi, \eta, \zeta, \lambda, \mu, \cdots$ and $A, B, C, D, R, S, \cdots$ denote general elements of V^{+}, T^{+}, N^{+} and Q^{+}, respectively.

Lemma 4.1.

(4.1) $\alpha(A_{\xi}\bar{X}, A_{\alpha(Y,Y)}\bar{X})=\alpha(A_{\xi}A_{\alpha(\bar{X},\bar{X})}Y, Y)$.

(4.2) $$2\{\alpha(A_{\xi}A_{\bar{\eta}}R(X, \bar{Y})Z, Z)+\alpha(A_{\xi}A_{\bar{\eta}}Z, R(X, \bar{Y})Z)\}$$
$$=c\langle X, \bar{Y}\rangle\alpha(A_{\xi}A_{\bar{\eta}}Z, Z)+\alpha(A_{\xi}A_{\bar{\eta}}X, A_{\alpha(Z,Z)}\bar{Y})$$
$$+\alpha(A_{\xi}A_{\bar{\eta}}A_{\alpha(Z,Z)}\bar{Y}, X).$$

(4.3) $[A_{\xi}A_{\bar{\eta}}, R(X, \bar{Y})]=R(A_{\xi}A_{\bar{\eta}}X, \bar{Y})-R(X, A_{\bar{\eta}}A_{\xi}\bar{Y})$.

(4.4) $A_{\xi}A_{\bar{\eta}}A_{\alpha(X,X)}+A_{\alpha(X,X)}A_{\bar{\eta}}A_{\xi}=2A_{\alpha(A_{\xi}A_{\bar{\eta}}X,X)}$.

(4.5) $A_{\alpha(X,A_{\xi}\bar{Y})}\bar{Z}+A_{\xi}A_{\alpha(\bar{Y},\bar{Z})}X=R(X, \bar{Y})A_{\xi}\bar{Z}+\frac{c}{2}\langle X, \bar{Z}\rangle A_{\xi}\bar{Y}$.

(4.6) $$\alpha(R(X, \bar{Y})Z, W)+\frac{c}{2}\langle\bar{Y}, W\rangle\alpha(X, Z)$$
$$=\alpha(X, A_{\alpha(Z,W)}\bar{Y})+\alpha(Z, A_{\alpha(X,W)}\bar{Y}).$$

Proof. By (1.8) we have

$$\begin{aligned}
&\alpha(A_\xi A_{\alpha(\bar X,\bar X)}Y,\ Y)-\alpha(A_\xi\bar X,\ A_{\alpha(Y,Y)}\bar X)\\
&\quad= c\langle\bar X,\ Y\rangle\alpha(A_\xi\bar X,\ Y)-\alpha(A_\xi R(\bar X,\ Y)\bar X,\ Y)\\
&\qquad -c\langle Y,\ \bar X\rangle\alpha(A_\xi\bar X,\ Y)+\alpha(A_\xi\bar X,\ R(Y,\ \bar X)Y)\\
&\quad= \alpha(A_\xi R(Y,\ \bar X)\bar X,\ Y)+\alpha(A_\xi\bar X,\ R(Y,\ \bar X)Y)\\
&\quad= R^\perp(Y,\ R(Y,\ \bar X)\bar X)\xi-\frac{c}{2}\langle Y,\ R(Y,\ \bar X)\bar X\rangle\xi\\
&\qquad +R^\perp(R(Y,\ \bar X)Y,\ \bar X)\xi-\frac{c}{2}\langle R(Y,\ \bar X)Y,\ \bar X\rangle\xi \qquad \text{by (1.9)}\\
&\quad= R^\perp(Y,\ R(Y,\ \bar X)\bar X)\xi+R^\perp(R(Y,\ \bar X)Y,\ \bar X)\xi\\
&\quad= R^\perp(Y,\ \bar X)R^\perp(Y,\ \bar X)\xi-R^\perp(Y,\ \bar X)R^\perp(Y,\ \bar X)\xi \quad \text{since } R^\perp \text{ is parallel}\\
&\quad= 0\,.
\end{aligned}$$

This proves (4.1).

Because of the surjectivity of $\alpha\colon T^+\otimes T^+\to N^+$, any $\zeta\in N^+$ is written as

$$\zeta=\sum\alpha(U_i,\ V_i)\,,\quad U_i,\ V_i\in T^+\,.$$

We define

$$(4.7)\quad (\xi\square\eta)\zeta=\sum\{\alpha(A_\xi A_{\bar\eta}U_i,\ V_i)+\alpha(A_\xi A_{\bar\eta}V_i,U_i)\}\in N^+.$$

By the polarized form of (4.1) the right hand side is equal to

$$\alpha(A_\xi\bar X,\ A_{\Sigma\alpha(U_i,V_i)}\bar X)\qquad \text{for}\quad \eta=\alpha(X,\ X)\,,$$

and hence $(\xi\square\eta)\zeta$ is well defined. Incidentally we know

$$(4.8)\quad (\xi\square\eta)\zeta=\alpha(A_\xi\bar X,\ A_\zeta\bar X)\qquad \text{for}\quad \eta=\alpha(X,\ X)\,.$$

Now we proceed to the proof of (4.2). If we put $\zeta=\alpha(Z,Z)$, the right hand side of (4.2) is

$$\begin{aligned}
&(\xi\square\eta)\,(\alpha(X,\ A_\zeta\bar Y)+\frac{c}{2}\langle X,\ \bar Y\rangle\zeta)\\
&\quad= (\xi\square\eta)R^\perp(X,\ \bar Y)\alpha(Z,\ Z) \qquad \text{by (1.9)}\\
&\quad= 2(\xi\square\eta)\alpha(R(X,\ \bar Y)Z,\ Z) \qquad \text{since } \alpha \text{ is parallel}\\
&\quad= 2\{\alpha(A_\xi A_{\bar\eta}R(X,\ \bar Y)Z,\ Z)+\alpha(A_\xi A_{\bar\eta}Z,\ R(X,\ \bar Y)Z)\}\,.
\end{aligned}$$

This proves (4.2).

By (1.9) we have

$$\langle A_\xi A_{\bar\eta}Z,\ \bar W\rangle=\langle R^\perp(Z,\ \bar W)\xi,\ \bar\eta\rangle-\frac{c}{2}\langle Z,\ \bar W\rangle\langle\xi,\ \bar\eta\rangle\,.$$

Therefore

$$
\begin{aligned}
&\langle [A_\xi A_{\bar\eta}, R(X, \bar Y)]Z, \bar W\rangle \\
&\quad = \langle A_\xi A_{\bar\eta}\, R(X, \bar Y)Z, \bar W\rangle + \langle A_\xi A_{\bar\eta} Z, R(X, \bar Y)\bar W\rangle \\
&\quad = \langle R^\perp(R(X, \bar Y)Z, \bar W)\xi, \bar\eta\rangle - \frac{c}{2}\langle R(X, \bar Y)Z, \bar W\rangle\langle \xi, \bar\eta\rangle \\
&\qquad + \langle R^\perp(Z, R(X, \bar Y)\bar W)\xi, \bar\eta\rangle - \frac{c}{2}\langle Z, R(X, \bar Y)\bar W\rangle\langle \xi, \bar\eta\rangle \\
&\quad = \langle [R^\perp(X, \bar Y), R^\perp(Z, \bar W)]\xi, \bar\eta\rangle, \qquad \text{since } R^\perp \text{ is parallel.}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\langle R(A_\xi A_{\bar\eta} X, \bar Y)Z, \bar W\rangle - \langle R(X, A_{\bar\eta} A_\xi \bar Y)Z, \bar W\rangle \\
&\quad = \langle R(Z, \bar W)A_\xi A_{\bar\eta} X, \bar Y\rangle - \langle R(Z, \bar W)X, A_{\bar\eta} A_\xi \bar Y\rangle \\
&\quad = -\langle [A_\xi A_{\bar\eta}, R(Z, \bar W)]X, \bar Y\rangle = \langle [R^\perp(X, \bar Y), R^\perp(Z, \bar W)]\xi, \bar\eta\rangle
\end{aligned}
$$

by the equality just proved. This proves (4.3).

$$
\begin{aligned}
&2\langle A_{\alpha(A_\xi A_{\bar\eta} X, X)}\bar Y, \bar Z\rangle \\
&\quad = c\langle A_\xi A_{\bar\eta} X, \bar Y\rangle\langle X, \bar Z\rangle + c\langle X, \bar Y\rangle\langle A_\xi A_{\bar\eta} X, \bar Z\rangle \\
&\qquad - 2\langle R(A_\xi A_{\bar\eta} X, \bar Y)X, \bar Z\rangle \qquad \text{by (1.8)} \\
&\quad = c\langle X, \bar Z\rangle\langle A_\xi A_{\bar\eta} X, \bar Y\rangle + c\langle X, \bar Y\rangle\langle A_\xi A_{\bar\eta} X, \bar Z\rangle \\
&\qquad - \langle R(A_\xi A_{\bar\eta} X, \bar Y)X, \bar Z\rangle - \langle R(X, \bar Y)A_\xi A_{\bar\eta} X, \bar Z\rangle, \qquad \text{by (1.7).}
\end{aligned}
$$

On the other hand, by (1.8)

$$
\begin{aligned}
&\langle A_\xi A_{\bar\eta} A_{\alpha(X,X)}\bar Y, \bar Z\rangle \\
&\quad = c\langle X, \bar Y\rangle\langle A_\xi A_{\bar\eta} X, \bar Z\rangle - \langle A_\xi A_{\bar\eta} R(X, \bar Y)X, \bar Z\rangle, \\
&\langle A_{\alpha(X,X)} A_{\bar\eta} A_\xi \bar Y, \bar Z\rangle = \langle A_\xi A_{\bar\eta} A_{\alpha(X,X)}\bar Z, \bar Y\rangle \\
&\quad = c\langle X, \bar Z\rangle\langle A_\xi A_{\bar\eta} X, \bar Y\rangle - \langle R(X, \bar Z)X, A_{\bar\eta} A_\xi \bar Y\rangle \\
&\quad = c\langle X, \bar Z\rangle\langle A_\xi A_{\bar\eta} X, \bar Y\rangle - \langle R(X, A_{\bar\eta} A_\xi \bar Y)X, \bar Z\rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2\langle A_{\alpha(A_\xi A_{\bar\eta}, XX)}\bar Y, \bar Z\rangle - \langle A_\xi A_{\bar\eta} A_{\alpha(X,X)}\bar Y, \bar Z\rangle - \langle A_{\alpha(X,X)} A_{\bar\eta} A_\xi \bar Y, \bar Z\rangle \\
&\quad = \langle [A_\xi A_{\bar\eta}, R(X, \bar Y)]X, \bar Z\rangle - \langle R(A_\xi A_{\bar\eta} X, \bar Y)X, \bar Z\rangle + \langle R(X, A_{\bar\eta} A_\xi \bar Y)X, \bar Z\rangle \\
&\quad = 0, \qquad \text{by (4.3).}
\end{aligned}
$$

This proves (4.4).

$$
A_{\alpha(X, A_\xi \bar Y)}\bar Z = A_{R^\perp(X, \bar Y)\xi}\bar Z - \frac{c}{2}\langle X, \bar Y\rangle A_\xi \bar Z \qquad \text{by (1.9),}
$$

$$
A_\xi A_{\alpha(\bar Y, \bar Z)} X = \frac{c}{2}\langle \bar Y, X\rangle A_\xi \bar Z + \frac{c}{2}\langle \bar Z, X\rangle A_\xi \bar Y - A_\xi R(\bar Y, X)\bar Z
$$

by (1.8). Therefore

$$A_{\alpha(X,A_\xi\bar{Y})}\bar{Z}+A_\xi A_{\alpha(\bar{Y},\bar{Z})}X-R(X,\bar{Y})A_\xi\bar{Z}-\frac{c}{2}\langle X,\bar{Z}\rangle A_\xi\bar{Y}$$
$$=A_{R^\perp(\bar{X},\bar{Y})\xi}\bar{Z}+A_\xi R(X,\bar{Y})\bar{Z}-R(X,\bar{Y})A_\xi\bar{Z}=0\,,$$

since α is parallel. This proves (4.5).

$$\alpha(X,A_{\alpha(Z,W)}\bar{Y})=R^\perp(X,\bar{Y})\alpha(Z,W)-\frac{c}{2}\langle X,\bar{Y}\rangle\alpha(Z,W) \qquad \text{by (1.9)}$$
$$=\alpha(R(X,\bar{Y})Z,W)+\alpha(Z,R(X,\bar{Y})W)-\frac{c}{2}\langle X,\bar{Y}\rangle\alpha(Z,W),$$

since α is parallel. On the other hand,

$$\alpha(Z,A_{\alpha(X,W)}\bar{Y})=\frac{c}{2}\langle X,\bar{Y}\rangle\alpha(Z,W)+\frac{c}{2}\langle W,\bar{Y}\rangle\alpha(X,Z)-\alpha(Z,R(X,\bar{Y})W)$$

by (1.8). This proves (4.6). q.e.d.

We define a triple product $\{x,y,z\}=(x\Box y)z$ on $V^+=Q^++T^++N^+$ by the table in §3, except for $(\xi\Box\eta)\zeta$, which is defined by (4.7). The triple system $(V^+,\Box)$ is said to be *associated to* (M,o). We want to show that $(V^+,\Box)$ becomes a positive definite hermitian Jordan triple system. The condition (i) in §3 is obvious from the table. The condition (ii) is also clear from the table, except for $(\xi\Box\eta)\zeta=(\zeta\Box\eta)\xi$. This is verified as follows: We may assume $\eta=\alpha(X,X)$. Then by (4.8) we have $(\xi\Box\eta)\zeta=\alpha(A_\xi\bar{X},A_\zeta\bar{X})$, which is symmetric in ξ,ζ. This implies the required equality. The conditions (iii) and (iv) will be proved in the next section. The following lemma will be used there.

Lemma 4.2. *For an endomorphism ϕ of V^+, let ϕ^* denote the adjoint endomorphism of V^+ with respect to $\langle\!\langle x,y\rangle\!\rangle=\langle x,\bar{y}\rangle$. Then*

$$(x\Box y)^*=y\Box x \qquad \textit{for}\quad x,y\in V^+\,.$$

Proof. This is clear for $X\Box\xi, X\Box A$ and $\xi\Box A$ by the arguments in §3. So we prove this for $X\Box Y$, $\xi\Box\eta$ and $A\Box B$.

(I) $X\Box Y$

$$\langle\!\langle(X\Box Y)Z,W\rangle\!\rangle=\langle R(X,\bar{Y})Z,\bar{W}\rangle=-\langle Z,R(X,\bar{Y})\bar{W}\rangle$$
$$=\langle Z,\overline{R(Y,\bar{X})W}\rangle=\langle\!\langle Z,(Y\Box X)W\rangle\!\rangle\,.$$
$$\langle\!\langle(X\Box Y)\zeta,\xi\rangle\!\rangle=\langle\alpha(X,A_\zeta\bar{Y}),\bar{\xi}\rangle=\langle A_{\bar{\xi}}X,A_\zeta\bar{Y}\rangle$$
$$=\langle\zeta,\alpha(A_{\bar{\xi}}X,\bar{Y})\rangle=\langle\zeta,\overline{\alpha(A_\xi\bar{X},Y)}\rangle=\langle\!\langle\zeta,(Y\Box X)\xi\rangle\!\rangle\,.$$
$$\langle\!\langle(X\Box Y)C,A\rangle\!\rangle=\frac{c}{2}\langle X,\bar{Y}\rangle\langle C,\bar{A}\rangle=\langle C,\overline{\frac{c}{2}\langle\bar{X},Y\rangle A}\rangle$$
$$=\langle\!\langle C,(Y\Box X)A\rangle\!\rangle\,.$$

Thus we get $(X\Box Y)^*=Y\Box X$.

(II) $\xi\Box\eta$

$$\langle\!\langle(\xi\Box\eta)Z, X\rangle\!\rangle = \langle A_\xi A_{\bar\eta} Z, \bar X\rangle = \langle Z, A_{\bar\eta}A_\xi\bar X\rangle$$
$$= \langle Z, \overline{A_\eta A_{\bar\xi}X}\rangle = \langle\!\langle Z, (\eta\Box\xi)X\rangle\!\rangle .$$

For $\xi=\alpha(X, X)$ we have by (ii)

$$\langle\!\langle(\xi\Box\eta)\zeta, \lambda\rangle\!\rangle = \langle\!\langle(\zeta\Box\eta)\xi, \lambda\rangle\!\rangle = 2\langle\alpha(A_\zeta A_{\bar\eta}X, X), \bar\lambda\rangle$$
$$= 2\langle A_{\bar\lambda}A_\zeta A_{\bar\eta}X, X\rangle = 2\langle A_{\bar\eta}X, A_\zeta A_{\bar\lambda}X\rangle$$
$$= 2\langle\zeta, \alpha(A_{\bar\lambda}X, A_{\bar\eta}X)\rangle = 2\langle\zeta, \overline{\alpha(A_\lambda\bar X, A_\eta\bar X)}\rangle$$
$$= \langle\!\langle\zeta, (\eta\Box\xi)\lambda\rangle\!\rangle , \qquad \text{by (4.8).}$$
$$\langle\!\langle(\xi\Box\eta)C, A\rangle\!\rangle = \langle\!\langle C, (\eta\Box\xi)A\rangle\!\rangle = 0 .$$

Thus we get $(\xi\Box\eta)^*=\eta\Box\xi$.

(III) $A\Box B$

$$\langle\!\langle(A\Box B)Z, X\rangle\!\rangle = \frac{c}{2}\langle A, \bar B\rangle\langle Z, \bar X\rangle = \langle Z, \overline{\frac{c}{2}\langle B, \bar A\rangle X}\rangle$$
$$= \langle\!\langle Z, (B\Box A)X\rangle\!\rangle .$$
$$\langle\!\langle(A\Box B)\zeta, \xi\rangle\!\rangle = \langle\!\langle\zeta, (B\Box A)\xi\rangle\!\rangle = 0 .$$
$$\langle\!\langle(A\Box B)C, D\rangle\!\rangle = c\langle A, \bar B\rangle\langle C, \bar D\rangle = \langle C, \overline{c\langle\bar B, A\rangle D}\rangle$$
$$= \langle\!\langle C, (B\Box A)D\rangle\!\rangle .$$

Thus we get $(A\Box B)^*=B\Box A$. q.e.d.

5. Jordan triple systems associated to projective manifolds with parallel second fundamental form

Let $M\subset P_N(c)$ be a full complex submanifold with parallel second fundamental form and take a point $o\in M$. We retain the notation in §4. We prove first that the condition (iii) holds for the triple system $(V^+, \Box)$ associated to (M, o). We put

$$T(l, m; x, y) = [l\Box m, x\Box y]-((l\Box m)x)\Box y+x\Box((m\Box l)y)$$

for $x, y, l, m\in V^+$. The condition (iii) is equivalent to the vanishing of T.

Lemma 5.1. *$T(l, m; x, y)=0$ implies $T(m, l; y, x)=0$.*

Proof. Applying the anti-automorphism $\phi\mapsto\phi^*$ of $\mathfrak{gl}(V^+)$ to $T(l, m; x, y)=0$, by Lemma 4.2 we get

$$[y\Box x, m\Box l]-y\Box((l\Box m)x)+((m\Box l)y)\Box x = 0 .$$

Since the left hand side equals $-T(m, l; y, x)$, we get $T(m, l; y, x)=0$. q.e.d.

In the following computations, $z \in V^+$ is always assumed to be

$$z = C+Z+\zeta\,.$$

(I) $T(L, M; x, y)=0$.

$$\begin{aligned}
&(L\square M)(X\square Y)z \\
&\quad = R(L, \bar{M})R(X, \bar{Y})Z+\alpha(L, A_{\alpha(X, A_\zeta \bar{Y})}\bar{M})+\frac{c^2}{4}\langle L, \bar{M}\rangle\langle X, \bar{Y}\rangle C\,. \\
&(X\square Y)(L\square M)z \\
&\quad = R(X, \bar{Y})R(L, \bar{M})Z+\alpha(X, A_{\alpha(L, A_\zeta \bar{M})}\bar{Y})+\frac{c^2}{4}\langle X, \bar{Y}\rangle\langle L, \bar{M}\rangle C\,. \\
&\{((L\square M)X)\square Y\}z \\
&\quad = R(R(L, \bar{M})X, \bar{Y})Z+\alpha(R(L, \bar{M})X, A_\zeta \bar{Y})+\frac{c}{2}\langle R(L, \bar{M})X, \bar{Y}\rangle C\,. \\
&\{X\square((M\square L)Y)\}z \\
&\quad = -R(X, R(L, \bar{M})\bar{Y})Z-\alpha(X, A_\zeta R(L, \bar{M})\bar{Y})-\frac{c}{2}\langle X, R(L, \bar{Y})\rangle C\,.
\end{aligned}$$

Thus we have by (1.9)

$$\begin{aligned}
&T(L, M; X, Y)z \\
&\quad = (R(L, \bar{M})\cdot R)(X, \bar{Y})Z+R^\perp(L, \bar{M})\alpha(X, A_\zeta \bar{Y})-\frac{c}{2}\langle L, \bar{M}\rangle\alpha(X, A_\zeta \bar{Y}) \\
&\qquad -\alpha(X, A_{R^\perp(L,\bar{M})\zeta}\bar{Y})+\frac{c}{2}\langle L, \bar{M}\rangle\alpha(X, A_\zeta \bar{Y})-\alpha(R(L, \bar{M})X, A_\zeta \bar{Y}) \\
&\qquad -\alpha(X, A_\zeta R(L, \bar{M})\bar{Y}) \\
&\quad = (R(L, \bar{M})\cdot R)(X, \bar{Y})Z+R^\perp(L, \bar{M})\alpha(X, A_\zeta \bar{Y})-\alpha(X, A_{R^\perp(L,\bar{M})\zeta}\bar{Y}) \\
&\qquad -\alpha(R(L, \bar{M})X, A_\zeta \bar{Y})-\alpha(X, A_\zeta R(L, \bar{M})\bar{Y}) = 0\,,
\end{aligned}$$

since R and α are parallel. For $\zeta=\alpha(U, U)$ we have

$$\begin{aligned}
(L\square M)(\xi\square\eta)z &= R(L, \bar{M})A_\xi A_{\bar{\eta}}Z+2\alpha(L, A_{\alpha(A_\xi A_{\bar{\eta}}U, U)}\bar{M})\,, \\
(\xi\square\eta)(L\square M)z &= A_\xi A_{\bar{\eta}}R(L, \bar{M})Z+\alpha(A_\xi A_{\bar{\eta}}L, A_\zeta \bar{M}) \\
&\quad +\alpha(A_\xi A_{\bar{\eta}}A_\zeta \bar{M}, L)\,, \\
\{((L\square M)\xi)\square\eta\}z &= A_{\alpha(L, A_\zeta \bar{M})}A_{\bar{\eta}}Z+2\alpha(A_{\alpha(L, A_\xi \bar{M})}A_{\bar{\eta}}U, U)\,, \\
\{\xi\square((M\square L)\eta)\}z &= A_\xi A_{\alpha(A_{\bar{\eta}}L, \bar{M})}Z+2\alpha(A_\xi A_{\alpha(A_{\bar{\eta}}L, \bar{M})}U, U)\,.
\end{aligned}$$

Thus by (1.9) we have

$$\begin{aligned}
T(L, M; \xi, \eta)z &= R(L, \bar{M})A_\xi A_{\bar{\eta}}Z-A_\xi A_{\bar{\eta}}R(L, \bar{M})Z \\
&\quad -A_{R^\perp(L,\bar{M})\xi}A_{\bar{\eta}}Z+\frac{c}{2}\langle L, \bar{M}\rangle A_\xi A_{\bar{\eta}}Z-A_\xi A_{R^\perp(L,\bar{M})\bar{\eta}}Z \\
&\quad -\frac{c}{2}\langle L, \bar{M}\rangle A_\xi A_{\bar{\eta}}Z+2R^\perp(L, \bar{M})\alpha(A_\xi A_{\bar{\eta}}U, U)-c\langle L, \bar{M}\rangle\alpha(A_\xi A_{\bar{\eta}}U, U)
\end{aligned}$$

$$
\begin{aligned}
&-\alpha(A_\xi A_{\bar\eta} L, A_\zeta \bar M)-\alpha(A_\xi A_{\bar\eta} A_\zeta \bar M, L)-2\alpha(A_{R^\perp(L,\bar M)\xi} A_{\bar\eta} U, U)\\
&+c\langle L, \bar M\rangle \alpha(A_\xi A_{\bar\eta} U, U)-2\alpha(A_\xi A_{R^\perp(L,\bar M)\bar\eta} U, U)-c\langle L, \bar M\rangle\alpha(A_\xi A_{\bar\eta} U, U)\\
&= 2\{\alpha(A_\xi A_{\bar\eta} R(L, \bar M)U, U)+\alpha(A_\xi A_{\bar\eta} U, R(L, \bar M)U)\}\\
&-\{c\langle L, \bar M\rangle \alpha(A_\xi A_{\bar\eta} U, U)+\alpha(A_\xi A_{\bar\eta} L, A_\zeta \bar M)+\alpha(A_\xi A_{\bar\eta} A_\zeta \bar M, L)\}\\
&=0, \qquad \text{by (4.2).}
\end{aligned}
$$

Making use of (1.9) we have

$$
\begin{aligned}
&T(L, M; X, \eta)z\\
&= R(L, \bar M)A_\zeta A_{\bar\eta} X-A_{R^\perp(L,\bar M)\zeta} A_{\bar\eta} X-A_\zeta A_{\bar\eta} R(L, \bar M)X-A_\zeta A_{R^\perp(L,\bar M)\bar\eta} X\\
&+\frac{c}{2}\langle R(L, \bar M)A_{\bar\eta} X-A_{\bar\eta} R(L, \bar M)X-A_{R^\perp(L,\bar M)\bar\eta} X, Z\rangle E^+ = 0,
\end{aligned}
$$

since α is parallel. We have

$$
\begin{aligned}
&T(L, M; X, B)z\\
&= \frac{c}{2}\langle \bar B, E^+\rangle\{R^\perp(L, \bar M)\alpha(X, Z)-\alpha(X, R(L, \bar M)Z)-\alpha(R(L, \bar M)X, Z\}=0,
\end{aligned}
$$

since α is parallel. A straightforward calculation shows $T(L, M; A, B)= T(L, M; \xi, B)=0$.

Now these vanishings together with Lemma 5.1 imply $T(L, M; x, y)=0$.

(II) $T(\lambda, \mu; x, y)=0$.

For $\zeta=\alpha(U, U)$ we have

$$
\begin{aligned}
&(\lambda\Box\mu)(X\Box Y)z = A_\lambda A_{\bar\mu} R(X, \bar Y)Z+\alpha(A_\lambda A_{\bar\mu} X, A_\zeta \bar Y)+\alpha(A_\lambda A_{\bar\mu} A_\zeta \bar Y, X),\\
&(X\Box Y)(\lambda\Box\mu)z = R(X, \bar Y)A_\lambda A_{\bar\mu} Z+2\alpha(X, A_{\alpha(A_\lambda A_{\bar\mu} U, U)}\bar Y),\\
&\{((\lambda\Box\mu)X)\Box Y\}z = R(A_\lambda A_{\bar\mu} X, \bar Y)Z+\alpha(A_\lambda A_{\bar\mu} X, A_\zeta \bar Y)+\frac{c}{2}\langle A_\lambda A_{\bar\mu} X, \bar Y\rangle C,\\
&\{X\Box((\mu\Box\lambda)Y)\}z=R(X, A_{\bar\mu} A_\lambda \bar Y)Z+\alpha(X, A_\zeta A_{\bar\mu} A_\lambda \bar Y)+\frac{c}{2}\langle X, A_{\bar\mu} A_\lambda \bar Y\rangle C.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&T(\lambda, \mu; X, Y)z\\
&= [A_\lambda A_{\bar\mu}, R(X, \bar Y)]Z-R(A_\lambda A_{\bar\mu} X, \bar Y)Z+R(X, A_{\bar\mu} A_\lambda \bar Y)Z\\
&+\alpha(X, A_\lambda A_{\bar\mu} A_\zeta \bar Y)+\alpha(X, A_\zeta A_{\bar\mu} A_\lambda \bar Y)-2\alpha(X, A_{\alpha(A_\lambda A_{\bar\mu} U, U)}\bar Y)\\
&=0, \qquad \text{by (4.3), (4.4).}
\end{aligned}
$$

For $\xi=\alpha(V, V)$, $\eta=\alpha(W, W)$, $\zeta=\alpha(U, U)$ we have

$$
\begin{aligned}
&(\lambda\Box\mu)(\xi\Box\eta)z = A_\lambda A_{\bar\mu} A_\xi A_{\bar\eta} Z+2\alpha(A_\lambda A_{\bar\mu} A_\xi A_{\bar\eta} U, U)+2\alpha(A_\lambda A_{\bar\mu} U, A_\xi A_{\bar\eta} U),\\
&(\xi\Box\eta)(\lambda\Box\mu)z = A_\xi A_{\bar\eta} A_\lambda A_{\bar\mu} Z+2\alpha(A_\xi A_{\bar\eta} A_\lambda A_{\bar\mu} U, U)+2\alpha(A_\xi A_{\bar\eta} U, A_\lambda A_{\bar\mu} U),
\end{aligned}
$$

$$\{((\lambda\Box\mu)\xi)\Box\eta\}z=2A_{\alpha(A_\lambda A_{\bar\mu}V,V)}A_{\bar\eta}Z+4\alpha(A_{\alpha(A_\lambda A_{\bar\mu}V,V)}A_{\bar\eta}U,\ U)\,,$$
$$\{\xi\Box((\mu\Box\lambda)\eta)\}z=2A_\xi A_{\alpha(A_{\bar\mu}A_\lambda\bar W,\bar W)}Z+4\alpha(A_\xi A_{\alpha(A_{\bar\mu}A_\lambda\bar W,\bar W)}U,\ U)\,.$$

Thus by (4.4)

$$\begin{aligned}T(\lambda,\ \mu;\ \xi,\ \eta)z=\ &A_\lambda A_{\bar\mu}A_\xi A_{\bar\eta}Z-A_\xi A_{\bar\eta}A_\lambda A_{\bar\mu}Z-A_\lambda A_{\bar\mu}A_\xi A_{\bar\eta}Z\\&-A_\xi A_{\bar\mu}A_\lambda A_{\bar\eta}Z+A_\xi A_{\bar\mu}A_\lambda A_{\bar\eta}Z+A_\xi A_{\bar\eta}A_\lambda A_{\bar\mu}Z\\&+2\alpha(A_\lambda A_{\bar\mu}A_\xi A_{\bar\eta}U,\ U)-2\alpha(A_\xi A_{\bar\eta}A_\lambda A_{\bar\mu}U,\ U)-2\alpha(A_\lambda A_{\bar\mu}A_\xi A_{\bar\eta}U,\ U)\\&-2\alpha(A_\xi A_{\bar\eta}A_\lambda A_{\bar\mu}U,\ U)+2\alpha(A_\xi A_{\bar\mu}A_\lambda A_{\bar\eta}U,\ U)+2\alpha(A_\xi A_{\bar\eta}A_\lambda A_{\bar\mu}U,\ U)\\=\ &0\,.\end{aligned}$$

For $\eta=\alpha(V,\ V)$, $\zeta=\alpha(U,\ U)$ we have

$$(\lambda\Box\mu)(X\Box\eta)z=A_\lambda A_{\bar\mu}A_\xi A_{\bar\eta}X\,,$$
$$(X\Box\eta)(\lambda\Box\mu)z=\frac{c}{2}\langle A_{\bar\eta}X,\ A_\lambda A_{\bar\mu}Z\rangle E^{+}+2A_{\alpha(A_\lambda A_{\bar\mu}U,U)}A_{\bar\eta}X\,,$$
$$\{((\lambda\Box\mu)X)\Box\eta\}z=\frac{c}{2}\langle A_{\bar\eta}A_\lambda A_{\bar\mu}X,\ Z\rangle E^{+}+A_\zeta A_{\bar\eta}A_\lambda A_{\bar\mu}X\,,$$
$$\{X\Box((\mu\Box\lambda)\eta)\}z=c\langle A_{\alpha(A_{\bar\mu}A_\lambda\bar V,\bar V)}X,\ Z\rangle E^{+}+2A_\zeta A_{\alpha(A_{\bar\mu}A_\lambda\bar V,\bar V)}X\,.$$

Therefore by (4.4)

$$\begin{aligned}T(\lambda,\ \mu;\ X,\ \eta)z=&-\frac{c}{2}\langle A_{\bar\mu}A_\lambda A_{\bar\eta}X,\ Z\rangle E^{+}-\frac{c}{2}\langle A_{\bar\eta}A_\lambda A_{\bar\mu}X,\ Z\rangle E^{+}\\&+\frac{c}{2}\langle A_{\bar\mu}A_\lambda A_{\bar\eta}X,\ Z\rangle E^{+}+\frac{c}{2}\langle A_{\bar\eta}A_\lambda A_{\bar\mu}X,\ Z\rangle E^{+}\\&+A_\lambda A_{\bar\mu}A_\zeta A_{\bar\eta}X-A_\lambda A_{\bar\mu}A_\zeta A_{\bar\eta}X-A_\zeta A_{\bar\mu}A_\lambda A_{\bar\eta}X\\&-A_\zeta A_{\bar\eta}A_\lambda A_{\bar\mu}X+A_\zeta A_{\bar\mu}A_\lambda A_{\bar\eta}X+A_\zeta A_{\bar\eta}A_\lambda A_{\bar\mu}X=0\,.\end{aligned}$$

Straightforward calculations show $T(\lambda,\mu;A,B)=T(\lambda,\mu;X,B)=T(\lambda,\mu;\xi,B)=0$.
Thus by Lemma 5.1 we get $T(\lambda,\ \mu;\ x,y)=0$.

(III) $T(R,S;x,y)=0$.
This follows by straightforward calculations.

(IV) $T(L,\ \mu;\ x,y)=0$.
By a straightforward calculation we have

$$\begin{aligned}&T(L,\ \mu;\ X,\ Y)z\\&=\frac{c}{2}\langle R(\bar Y,\ X)A_{\bar\mu}L-\frac{c}{2}\langle\bar Y,\ X\rangle A_{\bar\mu}L-\frac{c}{2}\langle A_{\bar\mu}L,\ X\rangle\bar Y+A_{\alpha(A_{\bar\mu}L,\bar Y)}X,\ Z\rangle E^{+}\\&\quad+A_{\alpha(X,A_\zeta\bar Y)}A_{\bar\mu}L-R(X,\ \bar Y)A_\zeta A_{\bar\mu}L-\frac{c}{2}\langle A_{\bar\mu}L,\ X\rangle A_\zeta\bar Y+A_\zeta A_{\alpha(A_{\bar\mu}L,\bar Y)}X\\&=0\,,\qquad\text{by (1.8), (4.5).}\end{aligned}$$

For $\zeta=\alpha(U, U)$ we have

$$T(L, \mu; \xi, \eta)z$$
$$= 2A_{\alpha(A_\xi A_{\bar\eta}U,U)}A_{\bar\mu}L - A_\xi A_{\bar\eta}A_\zeta A_{\bar\mu}L - A_\zeta A_{\bar\eta}A_\xi A_{\bar\mu}L$$
$$= 0, \qquad \text{by (4.4).}$$

For $\zeta=\alpha(U, U)$ we have

$$T(L, \mu; \xi, Y)z$$
$$= A_{\alpha(A_\xi \bar Y,Z)}A_{\bar\mu}L - \frac{c}{2}\langle A_{\bar\mu}L, Z\rangle A_\xi \bar Y - R(Z, \bar Y)A_\xi A_{\bar\mu}L + A_\xi A_{\alpha(A_{\bar\mu}L,\bar Y)}Z$$
$$-\alpha(A_\xi \bar Y, A_\zeta A_{\bar\mu}L) - \alpha(A_\xi A_{\bar\mu}L, A_\zeta \bar Y) + 2\alpha(A_\xi A_{\alpha(A_{\bar\mu}L,\bar Y)}U, U)$$
$$= 0, \qquad \text{by (1.7), (4.5), (4.1).}$$

$$T(L, \mu; X, B)z$$
$$= \frac{c}{2}\langle \bar B, E^+\rangle\{A_{\alpha(X,Z)}A_{\bar\mu}L - \frac{c}{2}\langle A_{\bar\mu}L, Z\rangle X - \frac{c}{2}\langle A_{\bar\mu}L, X\rangle Z + R(X, A_{\bar\mu}L)Z\}$$
$$= 0, \qquad \text{by (1.8).}$$

Straightforward calculations show $T(L, \mu; A, B)=T(L, \mu; X, \eta)=T(L, \mu; A, Y)$ $=T(L, \mu; \xi, B)=T(L, \mu; A, \eta)=0$. Thus we get $T(L, \mu; x, y)=0$.

(V) $T(L, S; x, y)=0$.

$$T(L, S; X, Y)z$$
$$= \frac{c}{2}\langle \bar S, C\rangle\left\{\frac{c}{2}\langle X, \bar Y\rangle L - R(X, \bar Y)L - A_{\alpha(L,X)}\bar Y + \frac{c}{2}\langle L, \bar Y\rangle\right\}$$
$$+\frac{c}{2}\langle \bar S, E^+\rangle\{\alpha(L, R(X, \bar Y)Z) - \alpha(X, A_{\alpha(L,Z)}\bar Y) - \alpha(A_{\alpha(L,X)}\bar Y, Z)$$
$$+\frac{c}{2}\langle L, \bar Y\rangle \alpha(X, Z)\} = 0, \qquad \text{by (1.8), (4.6).}$$

For $\zeta=\alpha(U, U)$ we have

$$T(L, S; X, \eta)z$$
$$= \frac{c}{2}\langle \bar S, E^+\rangle\left\{\frac{c}{2}\langle A_{\bar\eta}X, Z\rangle L - A_{\alpha(L,Z)}A_{\bar\eta}X - A_{\alpha(L,X)}A_{\bar\eta}Z + R(X, A_{\bar\eta}L)Z\right\}$$
$$+\frac{c}{2}\langle \bar S, E^+\rangle\{\alpha(L, A_\zeta A_{\bar\eta}Z) - 2\alpha(A_{\alpha(L,X)}A_{\bar\eta}U, U) + \alpha(X, A_\zeta A_{\bar\eta}L)\}$$
$$= 0, \qquad \text{by (4.6), (4.1).}$$

$$T(L, S; A, Y)z$$
$$= \frac{c}{2}\langle A, \bar S\rangle\left\{\frac{c}{2}\langle \bar Y, Z\rangle L - A_{\alpha(L,Z)}\bar Y - R(L, \bar Y)Z + \frac{c}{2}\langle L, \bar Y\rangle Z\right\}$$
$$= 0, \qquad \text{by (1.8).}$$

Straightforward calculations show $T(L, S; \xi, \eta)=T(L, S; A, B)=T(L, S; \xi, Y)=T(L, S; X, B)=T(L, S; \xi, B)=T(L, S; A, \eta)=0$. Thus we get $T(L, S; x, y)=0$.

(VI) $T(\lambda, S; x, y)=0$.
This follows by $\lambda\Box S=S\Box\lambda=0$.

Now (I)$\sim$(VI) together with Lemma 5.1 implies that $T(l, m; x, y)=0$ for all $l, m, x, y\in V^+$.

We prove next the condition (iv): $\tau(x, y)=\mathrm{Tr}(x\Box y)$ is a hermitian inner product on V^+. By Lemma 4.2 τ is a hermitian form on V^+. Moreover, it is seen from the definition of $x\Box y$ that $V^+=Q^++T^++N^+$ is an orthogonal sum with respect to τ. Thus it suffices to show that τ is positive definite on each of Q^+, T^+ and N^+.

Choose orthonormal basis $\{Z_i\}$ and $\{\zeta_\alpha\}$ of T^+ and N^+, respectively, with respect to the hermitian inner product $\langle\langle x, y\rangle\rangle=\langle x, \bar{y}\rangle$. For $A\in Q^+$ we have

$$\begin{aligned}\tau(A, A) &= \sum \langle\langle (A\Box A)Z_i, Z_i\rangle\rangle+c\langle A, \bar{A}\rangle \\ &= \sum\frac{c}{2}\langle A, \bar{A}\rangle\langle\langle Z_i, Z_i\rangle\rangle+c\langle A, \bar{A}\rangle \\ &= \frac{n+2}{2}c\|A\|^2,\end{aligned}$$

where $n=\dim_{\mathbf{C}} M$. Thus τ is positive definite on Q^+. For $X\in T^+$ we have

$$\begin{aligned}\tau(X, X) &= \sum\langle\langle (X\Box X)Z_i, Z_i\rangle\rangle+\sum\langle\langle (X\Box X)\zeta_\alpha, \zeta_\alpha\rangle\rangle+\frac{c}{2}\|X\|^2 \\ &= \sum\langle\langle R(X, \bar{X})Z_i, Z_i\rangle\rangle+\sum\langle\langle \alpha(X, A_{\zeta_\alpha}\bar{X}), \zeta_\alpha\rangle\rangle+\frac{c}{2}\|X\|^2,\end{aligned}$$

where

$$\begin{aligned}\sum\langle\langle R(X, \bar{X})Z_i, Z_i\rangle\rangle &= \sum\langle R(X, \bar{X})Z_i, \bar{Z}_i\rangle \\ &= \sum\left\langle \frac{c}{2}\langle X, \bar{X}\rangle Z_i+\frac{c}{2}\langle Z_i, \bar{X}\rangle X-A_{\alpha(X,Z_i)}\bar{X}, \bar{Z}_i\right\rangle \qquad \text{by (1.8)} \\ &= \frac{c}{2}n\langle X, \bar{X}\rangle+\frac{c}{2}\langle X, \bar{X}\rangle-\sum\langle A_{\alpha(X,Z_i)}\bar{X}, \bar{Z}_i\rangle \\ &= \frac{n+1}{2}c\|X\|^2-\sum\langle \alpha(X, Z_i), \alpha(\bar{X}, \bar{Z}_i)\rangle \\ &= \frac{n+1}{2}c\|X\|^2-\sum_{i,\alpha}|\langle \alpha(X, Z_i), \bar{\zeta}_\alpha\rangle|^2 \\ &= \frac{n+1}{2}c\|X\|^2-\sum_{i,\alpha}|\langle A_{\bar{\zeta}_\alpha}X, Z_i\rangle|^2,\end{aligned}$$

and

$$\sum\langle\!\langle\alpha(X, A_{\zeta_\alpha}\bar{X}), \zeta_\alpha\rangle\!\rangle = \sum\langle\alpha(X, A_{\zeta_\alpha}\bar{X}), \bar{\zeta}_\alpha\rangle$$
$$= \sum\langle A_{\bar{\zeta}_\alpha}X, A_{\zeta_\alpha}\bar{X}\rangle = \sum_{i,\alpha}|\langle A_{\bar{\zeta}_\alpha}X, Z_i\rangle|^2 .$$

Therefore we get

$$\tau(X, X) = \frac{n+2}{2}c\|X\|^2 ,$$

and hence τ is positive definite on T^+. For $\xi, \eta \in N^+$ we have

$$\tau(\xi, \eta) = \sum\langle\!\langle(\xi\square\eta)Z_i, Z_i\rangle\!\rangle + \sum\langle\!\langle(\xi\square\eta)\zeta_\alpha, \zeta_\alpha\rangle\!\rangle ,$$

where

$$\Sigma\langle\!\langle(\xi\square\eta)Z_i, Z_i\rangle\!\rangle = \Sigma\langle A_\xi A_{\bar{\eta}}Z_i, \bar{Z}_i\rangle = \Sigma\langle A_{\bar{\eta}}Z_i, A_\xi\bar{Z}_i\rangle$$

is a positive semi-definite hermitian form on N^+, and $\sum\langle\!\langle(\xi\square\xi)Z_i, Z_i\rangle\!\rangle=0 \Rightarrow A_{\bar{\xi}}Z_i=0$ for each $i \Rightarrow \langle\alpha(X, Y), \bar{\xi}\rangle=0$ for each $X, Y\in T^+ \Rightarrow \xi=0$ by $\alpha(T^+\otimes T^+)=N^+$. Thus τ is positive definite on N^+, if

$$\sigma(\xi, \eta) = \sum\langle\!\langle(\xi\square\eta)\zeta_\alpha, \zeta_\alpha\rangle\!\rangle \qquad \text{for} \quad \xi, \eta\in N^+$$

is a positive semi-definite hermitian form on N^+, which will be proved in the following. It is a hermitian form on N^+ by Lemma 4.2. Let $\hat{T}=T^+\otimes T^+$ equipped with the natural hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ which extends $\langle\!\langle\ ,\ \rangle\!\rangle$ on T^+, thus α is a surjective linear map $\hat{T}\to N^+$. Let $\alpha^*: N^+\to\hat{T}$ denote the adjoint of α. We define linear endomorphisms of β of T^+ and $\hat{\beta}$ of $\hat{T}$ by

$$\beta = \sum A_{\zeta_\alpha}A_{\bar{\zeta}_\alpha} , \quad \hat{\beta} = \beta\otimes 1 + 1\otimes\beta .$$

Since

$$\langle\!\langle\beta X, Y\rangle\!\rangle = \sum\langle A_{\zeta_\alpha}A_{\bar{\zeta}_\alpha}X, \bar{Y}\rangle = \sum\langle A_{\bar{\zeta}_\alpha}X, A_{\zeta_\alpha}\bar{Y}\rangle$$

for $X, Y\in T^+$, β is a semi-positive hermitian endomorphism of T^+, and hence $\hat{\beta}$ is also a semi-positive hermitian endomorphism of $\hat{T}$. We define

$$\hat{\sigma}(u, v) = \langle\!\langle\hat{\beta}\alpha^*\alpha u, v\rangle\!\rangle \qquad \text{for} \quad u, v\in\hat{T} .$$

Lemma 5.2. $\hat{\sigma}(u, v)=\sigma(\alpha(u), \alpha(v))$ *for* $u, v\in\hat{T}$.

Proof. We may assume $u=U\otimes U'$, $v=V\otimes V'$, where $U, U', V, V'\in T^+$. We put $\xi=\alpha(u)=\alpha(U, U')$ and $\eta=\alpha(v)=\alpha(V, V')$. Then

$$\begin{aligned}
\hat{\sigma}(u, v) &= \langle\!\langle\hat{\beta}\alpha^*\alpha(U\otimes U'), V\otimes V'\rangle\!\rangle , \\
&= \langle\!\langle\alpha(U\otimes U'), \alpha(\beta V\otimes V' + V\otimes\beta V')\rangle\!\rangle \\
&= \langle\!\langle\xi, \alpha(\beta V, V') + \alpha(V, \beta V')\rangle\!\rangle \\
&= \sum\langle\xi, \alpha(A_{\bar{\zeta}_\alpha}A_{\zeta_\alpha}\bar{V}, \bar{V}') + \alpha(\bar{V}, A_{\bar{\zeta}_\alpha}A_{\zeta_\alpha}\bar{V}')\rangle \\
&= \sum\{\langle A_{\bar{\zeta}_\alpha}A_{\zeta_\alpha}\bar{V}, A_\xi\bar{V}'\rangle + \langle A_{\bar{\zeta}_\alpha}A_{\zeta_\alpha}\bar{V}', A_\xi\bar{V}\rangle\} \\
&= \sum\langle\alpha(A_{\zeta_\alpha}\bar{V}, A_\xi\bar{V}') + \alpha(A_{\zeta_\alpha}\bar{V}', A_\xi\bar{V}), \bar{\zeta}_\alpha\rangle \\
&= \sum\langle(\xi\square\eta)\zeta_\alpha, \bar{\zeta}_\alpha\rangle \qquad \text{by (4.8)}
\end{aligned}$$

$$= \sigma(\xi, \eta) = \sigma(\alpha(u), \alpha(v)) . \qquad \text{q.e.d.}$$

Now by the above lemma $\hat{\sigma}$ is a hermitian form on $\hat{T}$, and hence $\hat{\beta}\alpha^*\alpha$ is a hermitian endomorphism on $\hat{T}$, i.e., $(\hat{\beta}\alpha^*\alpha)^* = \hat{\beta}\alpha^*\alpha$. Thus we have $(\alpha^*\alpha)\hat{\beta} = \hat{\beta}(\alpha^*\alpha)$. Since both $\hat{\beta}$ and $\alpha^*\alpha$ are semi-positive, $\hat{\beta}\alpha^*\alpha$ is also semi-positive. Therefore $\hat{\sigma}$ is positive semi-definite on $\hat{T}$. Since α is surjective, the above lemma implies that σ is positive semi-definite on N^+.

Thus we have proved the following

Theorem 5.3. *Let M be a full complex submanifold of $P_N(c)$ with parallel second fundamental form, and take a point $o \in M$. Then the triple system $(V^+, \square)$ associated to (M, o) is a positive definite hermitian Jordan triple system.*

Let $(V, \square)$ and $(V', \square')$ be positive definite hermitian Jordan triple systems. They are said to be *isomorphic*, if there exists a linear isomorphism $\phi: V \to V'$ such that $\phi\{x, y, z\} = \{\phi x, \phi y, \phi z\}'$ for each $x, y, z \in V$. Then it is easily verified that the isomorphism class of $(V^+, \square)$ in the above theorem does not depend on the choice of $E \in V$ with $[E] = o$ and $\langle E, E \rangle = 4/c$.

6. Classification of projective manifolds with parallel second fundamental form

Theorem 6.1 (Koecher [9], cf. also Satake [13]).

1) *For each positive definite hermitian Jordan triple system $(V, \square)$ there exists an effective hermitian symmetric Lie algebra $(\mathfrak{g}, \sigma, J)$ of compact type such that we have an identification $V = \mathfrak{p}^+$ with*

$$(6.1) \qquad X \square Y = -\rho_*^+([X, \bar{Y}]) \qquad \textit{for} \quad X, Y \in \mathfrak{p}^+ ,$$

where $\rho_*^+: \mathfrak{k}^C \to \mathfrak{gl}(\mathfrak{p}^+)$ is the faithful representation given by $\rho_*^+ = ad_{\mathfrak{p}^+}$.

2) *The correspondence $(V, \square) \rightsquigarrow (\mathfrak{g}, \sigma, J)$ induces a bijection from the set of all isomorphism classes of positive definite hermitian Jordan triple systems to the set of all isomorphism classes of effective hermitian symmetric Lie algebras of compact type.*

Let $M \subset P_N(c)$ be a complete full connected complex submanifold with parallel second fundamental form, and $o \in M$. As in §4 we choose $E \in V$ with $[E] = o$ and $\langle E, E \rangle = 4/c$, and put $E^+ = \varpi^+(E) \in V^+$. Let $(V^+, \square)$ be the positive definite hermitian Jordan triple system associated to (M, o). We use the notation in §4 for $(V^+, \square)$. Let $(\mathfrak{g}, \sigma, J)$ be the effective hermitian symmetric Lie algebra of compact type with $V^+ = \mathfrak{p}^+$ and (6.1). Note that then we have an identification $V_{\mathbf{R}} = \mathfrak{p}$ together with the complex structures J. We use the notation in §2 for $(\mathfrak{g}, \sigma, J)$.

Lemma 6.2. *$E^+ \in \mathfrak{p}^+$ satisfies the conditions* (i) *and* (ii) *in Lemma* 2.1.

Therefore $(\mathfrak{g}, \sigma, J)$ *is irreducible.*

Proof. By (6.1) we have $ad_{\mathfrak{p}^+}(H_0)=E^+\square E^+$. Since $\langle E^+, \bar{E}^+\rangle=\frac{2}{c}$, by the table of $x\square y$ we have

$$(E^+\square E^+)(C+Z+\zeta)=2C+Z \qquad \text{for} \quad C\in Q^+, Z\in T^+, \zeta\in N^+ .$$

Thus we have

$$\mathfrak{p}_2^+=Q^+, \quad \mathfrak{p}_1^+=T^+, \quad \mathfrak{p}_0^+=N^+,$$

and hence

$$\text{(6.2)} \qquad \mathfrak{p}^+=\mathfrak{p}_2^++\mathfrak{p}_1^++\mathfrak{p}_0^+, \quad \text{with} \quad \mathfrak{p}_2^+=\boldsymbol{C}E^+ .$$

For each $A\in\mathfrak{p}_2^+=Q^+$ and $\xi\in\mathfrak{p}_0^+=N^+$ we have $A\square\xi=0$, and hence by (6.1) $\rho_*^+([A, \bar{\xi}])=0$. Thus we have $[\mathfrak{p}_2^+, \bar{\mathfrak{p}}_0^+]=\{0\}$, which implies also $[\mathfrak{p}_0^+, \bar{\mathfrak{p}}_2^+]=\{0\}$. Therefore by (6.2) $\mathfrak{k}^C=[\mathfrak{p}^+, \bar{\mathfrak{p}}^+]$ is decomposed as

$$\text{(6.3)} \qquad \mathfrak{k}^C=\mathfrak{k}_0^C+\mathfrak{k}_1^C+\mathfrak{k}_{-1}^C .$$

Now (6.2) and (6.3) imply the decomposition

(i) $\mathfrak{g}^C=\mathfrak{g}_0^C+\mathfrak{g}_1^C+\mathfrak{g}_{-1}^C+\mathfrak{g}_2^C+\mathfrak{g}_{-2}^C$, with $\mathfrak{g}_2^C=\boldsymbol{C}E^+$.

For the proof of (ii): $\mathfrak{p}_0^C\subset[\mathfrak{g}_1^C, \mathfrak{g}_{-1}^C]$, it is enough to show $[\mathfrak{k}_{-1}^C, \mathfrak{p}_1^+]=\mathfrak{p}_0^+$, since $\mathfrak{g}_1^C=\mathfrak{k}_1^C+\mathfrak{p}_1^+$, $\mathfrak{g}_{-1}^C=\mathfrak{k}_{-1}^C+\bar{\mathfrak{p}}_1^+$ and $\mathfrak{p}_0^C=\mathfrak{p}_0^++\bar{\mathfrak{p}}_0^+$. Let $X, Y\in\mathfrak{p}_1^+=T^+$. Then $h(Y)=[\bar{E}^+, Y]\in\mathfrak{k}_{-1}^C$ and

$$\text{(6.4)} \qquad \begin{aligned}[h(Y), X] &= -[[Y, \bar{E}^+], X]=(Y\square E^+)X \qquad \text{by (6.1)}\\ &= \frac{c}{2}\langle\bar{E}^+, E^+\rangle\alpha(Y, X)=\alpha(X, Y) .\end{aligned}$$

Now $[\mathfrak{k}_{-1}^C, \mathfrak{p}_1^+]=\mathfrak{p}_0^+$ follows from $\alpha(T^+\otimes T^+)=N^+$. q.e.d.

Lemma 6.3. *The inner product* $\langle\ ,\ \rangle$ *on* $V_{\boldsymbol{R}}$ *is invariant under the action of* $\mathfrak{k}$.

Proof. It suffices to show that $\rho_*^+(\mathfrak{k})$ leaves invariant the hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathfrak{p}^+$. Let $F=[X+\bar{X}, Y+\bar{Y}]\in\mathfrak{k}$ with $X, Y\in\mathfrak{p}^+$. Then

$$\begin{aligned}\rho_*^+(F) &= \rho_*^+([X, \bar{Y}])+\rho_*^+([\bar{X}, Y])\\ &= -X\square Y+Y\square X \qquad \text{by (6.1)}\\ &= -X\square Y+(X\square Y)^* , \qquad \text{by Lemma 4.2.}\end{aligned}$$

Thus $\rho_*^+(F)$ leaves $\langle\!\langle\ ,\ \rangle\!\rangle$ invariant. Since $[\mathfrak{p}, \mathfrak{p}]=\mathfrak{k}$, this holds for each $F\in\mathfrak{k}$. q.e.fi.

REMARK. The trace form τ for $(V^+, \square)$ is actually given by

$$\tau(x, y) = \frac{n+2}{2} c\langle\langle x, y\rangle\rangle \qquad \text{for} \quad x, y \in V^+ .$$

In fact, $\mathfrak{k}$ leaves $\langle\langle\ ,\ \rangle\rangle$ invariant (Lemma 6.3). Also it leaves τ invariant, since it preserves the triple product. Thus the assertion follows from the $\mathfrak{k}$-irreducibility of V^+ (Lemma 6.2) and $\tau(A, A) = \frac{n+2}{2} c\|A\|^2$ $(A \in Q^+)$.

Lemma 6.4. *Let $M' = K\cdot[E] \subset (P(\mathfrak{p}, J), g)$ be the complex submanifold associated to $(\mathfrak{g}, \sigma, J)$, where g is defined by the $\mathfrak{k}$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{p} = V_R$ with $\langle E, E\rangle = 4/c$. By means of the identification $(\mathfrak{p}, J) = V$, we identify as $(P(\mathfrak{p}, J), g) = P_N(c)$, $[E] = o$ and $M' = K\cdot o \subset P_N(c)$. Then $M = M'$.*

Proof. Submanifolds M and M' are complete connected submanifolds of $P_N(c)$ with parallel second fundamental form, passing through the same point o and having the same tangent space at o. Moreover, they have the same second fundamental form at o, which follows by Lemma 2.5 and (6.4). Then the uniqueness of Frenet curves implies (Naitoh [11]) that $M = M'$. q.e.d.

Let D be an irreducible symmetric bounded domain with $\dim_C D = N+1$ $(N \geqq 1)$. Let G^* be the identity component of the group of holomorphic automorphisms of D. Fix a point $p \in D$ and put

$$K = \{a \in G^*;\ a\cdot p = p\} .$$

Let $\rho: K \rightarrow GL(V)$ be the isotropy representation on the holomorphic tangent space V of D at p. Let $P_N(c)$ denote the complex projective space associated to V, endowed with a Kähler metric with constant holomorphic sectional curvature $c > 0$, invariant under the natural action of K through ρ. Taking a highest weight vector $E \in V$ $(E \neq 0)$ of the irreducible K-module V, we define

$$M = K\cdot[E] \subset P_N(c) .$$

Theorem 6.5. *The correspondence $D \rightsquigarrow M$ induces a bijection Φ from the set $\mathcal{D}$ of all holomorphic equivalence classes of irreducible symmetric bounded domains D with $\dim_C D \geqq 2$ to the set $\mathcal{M}$ of all equivalence classes of complete full connected complex submanifolds M of Fubini-Study spaces (with curvature $c > 0$) with parallel second fundamental form. Here $M \subset P_N(c)$ and $M' \subset P_{N'}(c)$ are said to be* equivalent *if $N = N'$ and there exists a holomorphic isometry ϕ of $P_N(c)$ such that $\phi(M) = M'$.*

Proof. Let $(\mathfrak{g}^*, \sigma^*, J^*)$ be the effective hermitian symmetric Lie algebra associated to (D, p). Then the dual effective hermitian symmetric Lie algebra $(\mathfrak{g}, \sigma, J)$ of compact type is irreducible, and $M = K\cdot[E] \subset P_N(c)$ defined by (D, p) is equivalent to the complex submanifold of $P_N(c)$ associated to $(\mathfrak{g}, \sigma, J)$.

Thus by Theorem 2.7 M is a compact full connected complex submanifold of $P_N(c)$ with parallel second fundamental form.

Let D and D' be holomorphically equivalent, thus $N=N'$, and let $M=K\cdot[E]\subset P_N(c)$ and $M'=K'\cdot[E']\subset P_N(c)$ be constructed in the above way from D and D', respectively. Then there exist a Lie isomorphism $\psi: K\to K'$ and a linear isomorphism $\phi: V\to V'$ with $\phi(E)=E'$ such that

$$\rho'(\psi(k))\phi(v)=\phi(\rho(k)v) \qquad \text{for} \quad k\in K,\, v\in V\,.$$

Recalling that a K-invariant hermitian inner product on V is unique up to positive constant multiple, we know that ϕ induces an equivalence between $M\subset P_N(c)$ and $M'\subset P_N(c)$. Thus our correspondence $D \rightsquigarrow M$ induces a map $\Phi: \mathcal{D}\to\mathcal{M}$.

Conversely, let $M\subset P_N(c)$ be a complete full connected complex submanifold with parallel second fundamental form. Take a point $o\in M$ and let $(V^+, \Box)$ be the positive definite hermitian Jordan triple system associated to (M, o). Let $(\mathfrak{g}, \sigma, J)$ be the effective hermitian symmetric Lie algebra of compact type corresponding to $(V^+, \Box)$. It is irreducible by Lemma 6.2. Let finally D be the irreducible symmetric bounded domain corresponding to the hermitian symmetric Lie algebra $(\mathfrak{g}^*, \sigma^*, J^*)$ which is dual to $(\mathfrak{g}, \sigma, J)$.

Let $M\subset P_N(c)$ and $M'\subset P_{N'}(c)$ be equivalent, thus $N=N'$, and let D and D' be constructed in the above way from M and M', respectively. By Takeuchi [20] (M', g') is a symmetric Kähler manifold and any holomorphic isometry of (M', g') is extended to a holomorphic isometry of $P_N(c)$. Therefore we may assume that there exists a holomorphic isometry ϕ of $P_N(c)$ such that $\phi(M)=M'$ and $\phi(o)=o'$. Thus $(V^+, \Box)$ and $(V^{+\prime}, \Box')$ are isomorphic, and hence $(\mathfrak{g}, \sigma, J)$ and $(\mathfrak{g}', \sigma', J')$ are isomorphic by Theorem 6.1. Therefore D and D' are holomorphically equivalent. Thus our correspondence $M \rightsquigarrow D$ induces a map $\Psi: \mathcal{M}\to\mathcal{D}$.

Now we have $\Phi\circ\Psi=I_{\mathcal{M}}$ by Lemma 6.4, and $\Psi\circ\Phi=I_{\mathcal{D}}$ by the construction. Thus Φ is a bijection. q.e.d.

Here we list up all our submanifolds $M\subset P_N(c)$.

$D=(\mathrm{I})_{1,n+1}$: unit ball in $\boldsymbol{C}^{n+1}$, $N=n$.
$M=P_n(\boldsymbol{C})\subset P_n(c)$: identity map.

$D=(\mathrm{I})_{p+1,q+1}\ (1\leqq p\leqq q)$, $N=(p+1)(q+1)-1=pq+p+q$.
$M=P_p(\boldsymbol{C})\times P_q(\boldsymbol{C})\subset P_N(c)$: Segre imbedding.

$D=(\mathrm{II})_{n+2}\ (n\geqq 3)$, $N=\dfrac{1}{2}(n+2)(n+1)-1$, $\dim_C M=2n$.
$M=G_{2,n}(\boldsymbol{C})\subset P_N(c)$: Plücker imbedding.

$D = (\mathrm{III})_{n+1}\,(n\geqq 1),\quad N = \frac{1}{2}(n+1)(n+2)-1\,.$

$M = P_n(\boldsymbol{C})\subset P_N(c)$: 2nd Veronese imbedding.

$D = (\mathrm{IV})_{n+2}\,(n\geqq 3)\,,\quad N = n+1\,.$

$M = Q_n(\boldsymbol{C})\subset P_{n+1}(c)$: standard imbedding.

$D = (\mathrm{V})\,,\quad N = 15\,,\quad \dim_C M = 10\,.$

$M = SO(10)/U(5)\subset P_{15}(c)$: canonical imbedding.

$D = (\mathrm{VI})\,,\quad N = 26\,,\quad \dim_C M = 16\,.$

$M = E_6/T\cdot \mathrm{Spin}(10)\subset P_{26}(c)$: canonical imbedding.

Here, $G_{2,n}(\boldsymbol{C})$ is the complex Grassmann manifold of all 2-planes in $\boldsymbol{C}^{n+2}$; $Q_n(\boldsymbol{C})$ is the complex quadric of dimension n; See Sakane-Takeuchi [15] for the canonical imbedding.

7. Non-singular hyperplane sections of projective manifolds with parallel second fundamental form

Theorem 7.1. *Let $M\subset P_N(\boldsymbol{C})$ be a full compact connected complex submanifold with parallel second fundamental form (with respect to a Fubini-Study metric). Then a non-singular hyperplane section of M is unique up to holomorphic automorphisms of M.*

Proof. By Theorem 6.5 we may assume that $M\subset P_N(\boldsymbol{C})$ is the projective manifold $M=K^C\cdot[E^+]\subset P(\mathfrak{p}^+)$ associated to an irreducible hermitian symmetric Lie algebra $(\mathfrak{g}, \sigma, J)$ of compact type. Furthermore we may assume that $E^+\in\mathfrak{p}^+$ satisfies the condition 3) in Lemma 2.2. We use the notation in §2. We choose root vectors $E_\alpha\in\mathfrak{g}_\alpha^C$ such that

$$[E_\alpha, E_{-\alpha}] = -\frac{2}{(\alpha,\alpha)}\alpha\,,\quad \bar{E}_\alpha = E_{-\alpha}\qquad (\alpha\in\Sigma)\,.$$

Let $\Delta=\{\gamma_1, \cdots, \gamma_r\}\subset\Sigma^+$ be as in the proof of Lemma 2.2, and put

$$E_i^+ = E_{\gamma_i}\,,\quad E_i^- = \bar{E}_i^+\,,\quad E_i = E_i^+ + E_i^-\,,$$
$$H_i = -[E_i^+, E_i^-] = \frac{2}{(\gamma_i,\gamma_i)}\gamma_i\qquad (1\leqq i\leqq r)\,.$$

Thus we may assume $E^+=E_1^+$. Note that $(\gamma_i, H_j)=2\delta_{ij}\,(1\leqq i,j\leqq r)$. Then $\mathfrak{a}=\{E_1, \cdots, E_r\}_{\boldsymbol{R}}$ is a maximal abelian subalgebra in $\mathfrak{p}$, and the Weyl group W of $(\mathfrak{g}, \sigma, J)$ relative to $\mathfrak{a}$ consists of linear maps $E_i\mapsto\pm E_{s(i)}\,(1\leqq i\leqq r)$, $s\in\mathfrak{S}_r$ (Takeuchi [19]). Thus

$$C = \{\textstyle\sum h_i E_i;\ h_1\geqq\cdots\geqq h_r\geqq 0\}$$

is a closed Weyl chamber in $\mathfrak{a}$, and hence we have $\mathfrak{p}=K\cdot C$. Since the projection $\varpi^-:\mathfrak{p}\to\mathfrak{p}^-$ is a K-equivariant $\boldsymbol{R}$-isomorphism, we get

$$\mathfrak{p}^- = K\cdot C^- , \tag{7.1}$$

where

$$C^- = \varpi^-(C) = \{\sum h_i E_i^- ;\ h_1 \geqq \cdots \geqq h_r \geqq 0\} .$$

Moreover, for $H=\sum a_i H_i \in \mathfrak{k}^C$ we have

$$(\exp H)\cdot \sum h_i E_i^- = \sum h_i \exp(-2a_i) E_i^- . \tag{7.2}$$

Therefore by (7.1) and (7.2) we get

$$\mathfrak{p}^- - \{0\} = K^C\cdot\{X_1^-, \cdots, X_r^-\} , \tag{7.3}$$

where $X_i^- = E_1^- + \cdots + E_i^-$ $(1\leqq i\leqq r)$.

Now let $\mathcal{H}(\mathfrak{p}^+)$ denote the set of all hyperplanes of $P(\mathfrak{p}^+)$ and $\mathcal{S}_M$ the set of all hyperplane sections of M, i.e.,

$$\mathcal{S}_M = \{M\cap H;\ H\in\mathcal{H}(\mathfrak{p}^+)\} .$$

Let moreover $\mathcal{S}_M^0$ denote the set of all non-singular hyperplane sections in $\mathcal{S}_M$. The group K^C acts on $P(\mathfrak{p}^-)$, $\mathcal{H}(\mathfrak{p}^+)$, $\mathcal{S}_M$ and $\mathcal{S}_M^0$ in a natural way, and the natural maps

$$P(\mathfrak{p}^-) \xrightarrow{\cong} \mathcal{H}(\mathfrak{p}^+) \longrightarrow \mathcal{S}_M$$

are K^C-equivariant. Here the first map is defined by $[X]\mapsto H_{[X]}$, where

$$H_{[X]} = \{[Y]\in P(\mathfrak{p}^+);\ (Y, X) = 0\} ,$$

and the second one is defined by $H\mapsto M\cap H$. Therefore by (7.3) the orbit space $K^C\backslash\mathcal{S}_M$ is given by

$$K^C\backslash\mathcal{S}_M = \{K^C\cdot S_i;\ 1\leqq i\leqq r\} , \tag{7.4}$$

where $S_i = M\cap H_{[X_i^-]}$ $(1\leqq i\leqq r)$.

We shall show that for each i with $1\leqq i\leqq r-1$ S_i has a singular point. Denoting by $\pi:\mathfrak{p}^+-\{0\}\to P(\mathfrak{p}^+)$ the canonical projection, we put $\hat{M}=\pi^{-1}(M)$. Then we have $\hat{M}=K^C\cdot E^+ = K^C\cdot E_1^+$. Since we may assume $r\geqq 2$, there exists $s\in W$ with $sE_1=E_r$, $sE_r=E_1$ and $sE_j=E_j$ $(2\leqq j\leqq r-1)$. Take $k\in K$ which normalizes $\mathfrak{a}$ and induces s on $\mathfrak{a}$. We have then

$$k\cdot E_1^\pm = E_r^\pm ,\quad k\cdot E_r^\pm = E_1^\pm ,\quad k\cdot E_j^\pm = E_j^\pm \quad (2\leqq j\leqq r-1) . \tag{7.5}$$

Since $(E_r^+, X_i^-)=0$ $(1\leqq i\leqq r-1)$ and $E_r^+ = k\cdot E_1^+ \in \hat{M}$, $[E_r^+]\in P(\mathfrak{p}^+)$ is a point of S_i. We show that $[E_r^+]$ is a singular point of S_i. For this purpose we define a linear form λ_i on $\mathfrak{p}^+$ by

$$\lambda_i(X) = (X_i^-, X) \qquad \text{for} \quad X \in \mathfrak{p}^+ ,$$

and prove that $(d\lambda_i) | \hat{M}$ vanishes at $E_r^+ \in \hat{M}$. Since

$$T_{E_r^+}\hat{M} = k\cdot T_{E^+}\hat{M} = k\cdot[\mathfrak{k}^C, E^+] = k\cdot(\mathfrak{p}_1^+ + \mathfrak{p}_2^+) ,$$

any $Y \in T_{E_r^+}\hat{M}$ is written as $Y = k\cdot X$ by an element $X \in \mathfrak{p}_1^+ + \mathfrak{p}_2^+$. Then

$$(d\lambda_i)_{E_r^+}(Y) = (X_i^-, Y) = (k^{-1}\cdot X_i^-, X) = 0 ,$$

since by (7.5) $k^{-1}\cdot X_i^- = E_2^- + \cdots + E_i^- + E_r^- \in \mathfrak{p}_0^-$.

Therefore by (7.4) the orbit space $K^C \backslash \mathcal{S}_M^0$ consists of the single orbit $K^C\cdot S_r$. On the other hand, as mentioned in §2, the action of K^C and that of $\mathrm{Aut}^0(M)$ are the same on M. Thus $\mathrm{Aut}^0(M) \backslash \mathcal{S}_M^0$ consists of the single orbit $\mathrm{Aut}^0(M)\cdot S_r$. This proves the theorem. q.e.d.

Theorem 7.2. *Let $M \subset P_N(\boldsymbol{C})$ be the same as in Theorem 7.1 with $\dim_{\boldsymbol{C}} M \geqq 2$ and $S \subset M$ a non-singular hyperplane section of M. Then the Lie algebra $\mathfrak{a}(S)$ of holomorphic vector fields on S is reductive if and only if the irreducible symmetric bounded domain D corresponding to M is a unit ball or of tube type.*

Lemma 7.3. *Let $\mathfrak{a}(M)$ denote the Lie algebra of holomorphic vector fields on M, and put*

$$\mathfrak{a}(M, S) = \{X \in \mathfrak{a}(M);\ X | S \text{ is tangent to } S\} .$$

If D is not a unit ball, then $\mathfrak{a}(M, S)$ is naturally isomorphic to $\mathfrak{a}(S)$.

Proof. If $D = (\mathrm{III})_{n+1}$ $(n \geqq 2)$, we have $S = Q_{n-1}(\boldsymbol{C}) \subset M = P_n(\boldsymbol{C})$, and hence the assertion follows from Takeuchi [20]. The assertion was proved by Sakane [13], Hano [2] for $D = (\mathrm{I})_{p+1,p+1}$ $(1 \leqq p \leqq q)$, by Kimura [5] for $D = (\mathrm{II})_{n+2}$ $(n \geqq 3)$, $(\mathrm{IV})_{n+2}$ $(n \geqq 3)$, (V) or (VI). q.e.d.

Lemma 7.4. *A hyperplane H of $P_N(\boldsymbol{C})$ such that $M \cap H = S$ is unique.*

Proof. Recall the exact sequence (Hirzebruch [4])

$$0 \to \boldsymbol{C} \to \Gamma(M, \{S\}) \to \Gamma(S, \{S\} | S) \to 0 ,$$

where $\{S\}$ denotes the homomorphic line bundle on M associated to the non-singular divisor S of M, and $\Gamma(\cdot)$ the space of holomorphic sections. Here $\Gamma(M, \{S\})$ is naturally isomorphic to the dual space of $\boldsymbol{C}^{N+1}$ (Takeuchi [20]). Thus the space of linear forms on $\boldsymbol{C}^{N+1}$ vanishing on S is 1-dimensional. This proves the assertion. q.e.d.

Proof of Theorem 7.2. If $D=(\mathrm{I})_{1,n+1}$ $(n\geqq 2)$: the unit ball in $\boldsymbol{C}^{n+1}$, then $S=P_{n-1}(\boldsymbol{C})\subset M=P_n(\boldsymbol{C})$, and hence $\mathfrak{a}(S)=\mathfrak{sl}(n, \boldsymbol{C})$, which is reductive. So we may assume that D is not a unit ball.

We may assume that $M\subset P_N(\boldsymbol{C})$ is the same as in the proof of Theorem 7.1 and $S=S_r$. In general, for a subalgebra $\mathfrak{b}$ of $\mathfrak{g}^C$ and $X\in\mathfrak{g}^C$, the normalizer of $\boldsymbol{C}X$ in $\mathfrak{b}$ will be denoted by $\mathfrak{n}_{\mathfrak{b}}(X)$, i.e.,

$$\mathfrak{n}_{\mathfrak{b}}(X)=\{Y\in\mathfrak{b};\ [Y, X]\in \boldsymbol{C}X\}\,.$$

Let $\mathfrak{k}'^C$ denote the derived algebra $[\mathfrak{k}^C, \mathfrak{k}^C]$, which is isomorphic to $\mathfrak{a}(M)$ as mentioned in §2. Now by Lemmas 7.3 and 7.4 $\mathfrak{a}(S)$ is isomorphic to $\mathfrak{n}_{\mathfrak{k}'^C}(X_r^-)$. We define

$$X_r^+=\bar{X}_r^-=E_1^++\cdots+E_r^+,\quad F=H_1+\cdots+H_r\,.$$

We have then $[X_r^+, X_r^-]=-F$, $[F, X_r^\pm]=\pm 2X_r^\pm$, and hence $\mathfrak{s}^C=\{X_r^+, X_r^-, F\}_C$ is a 3-dimensional simple subalgebra of $\mathfrak{g}^C$, which is the complexification of the subalgebra $\mathfrak{s}=\mathfrak{g}\cap\mathfrak{s}^C$ of $\mathfrak{g}$. By (2.3) and (2.4) the possible eigenvalues of $ad(F)$ are 0, ± 1 on $\mathfrak{k}^C$, ± 2, ± 1 on $\mathfrak{p}^C$. The corresponding eigenspaces are denoted by $\mathfrak{k}^C_{(0)}$, $\mathfrak{k}^C_{(\pm 1)}$, $\mathfrak{p}^C_{(\pm 2)}$, $\mathfrak{p}^C_{(\pm 1)}$, respectively. Then we have the decomposition

$$\mathfrak{g}^C=\mathfrak{k}^C_{(0)}+\mathfrak{k}^C_{(1)}+\mathfrak{k}^C_{(-1)}+\mathfrak{p}^C_{(1)}+\mathfrak{p}^C_{(-1)}+\mathfrak{p}^C_{(2)}+\mathfrak{p}^C_{(-2)}\,.$$

It is known (Korányi-Wolf [10]) that D is of tube type if and only if

$$\mathfrak{k}^C_{(-1)}=\{0\}\,, \tag{7.6}$$

which is also equivalent to that F belongs to the center $\mathfrak{c}$ of $\mathfrak{k}^C$. We define

$$\mathfrak{z}^C=\{X\in\mathfrak{k}^C;\ [X, \mathfrak{s}^C]=\{0\}\}\,.$$

Then $\mathfrak{z}^C$ is reductive, since it is the complexification of the compact algebra $\mathfrak{z}=\{X\in\mathfrak{k};\ [X, \mathfrak{s}]=\{0\}\}$. We put

$$(\mathfrak{z}^C)^\perp=\{X\in\mathfrak{k}^C_{(0)};\ (X, \mathfrak{z}^C)=\{0\}\}\,,$$
$$\mathfrak{k}'^C_{(0)}=\mathfrak{k}'^C\cap\mathfrak{k}^C_{(0)}\,.$$

We have then

$$\mathfrak{k}^C_{(0)}=\mathfrak{z}^C+(\mathfrak{z}^C)^\perp=\mathfrak{c}\oplus\mathfrak{k}'^C_{(0)}\,.$$

Note here that $F\in(\mathfrak{z}^C)^\perp$, since we have

$$(\mathfrak{z}^C, F)=(\mathfrak{z}^C, [X_r^+, X_r^-])=([\mathfrak{z}^C, X_r^+], X_r^-)=\{0\}\,.$$

Now applying the representation theory of 3-dimensional complex simple Lie algebras to the $\mathfrak{s}^C$-module $\mathfrak{g}^C$, we know that $ad(X_r^-)$ annihilates $\mathfrak{z}^C$ and $\mathfrak{k}^C_{(-1)}$, and it induces linear isomorphisms $\mathfrak{k}^C_{(1)}\to\mathfrak{p}^C_{(-1)}$ and $(\mathfrak{z}^C)^\perp\to\mathfrak{p}^C_{(-2)}$. Recall that in particular we have $ad(X_r^-)F=2X_r^-$. Therefore we have

$$\mathfrak{n}_{\mathfrak{k}^C}(X_r^-)=\mathfrak{n}_{\mathfrak{k}^C_{(0)}}(X_r^-)+\mathfrak{k}^C_{(-1)},$$

where

$$\mathfrak{n}_{\mathfrak{k}^C_{(0)}}(X_r^-)=\mathfrak{z}^C\oplus \boldsymbol{C}F\,,\quad \mathfrak{k}^C_{(-1)}\subset\mathfrak{k}'^C\,.$$

Furthermore $\mathfrak{a}(S)=\mathfrak{n}_{\mathfrak{k}'^C}(X_r^-)$ is given by

$$\mathfrak{a}(S)=\mathfrak{n}_{\mathfrak{k}'^C_{(0)}}(X_r^-)+\mathfrak{k}^C_{(-1)}\,. \tag{7.7}$$

Here $\mathfrak{n}_{\mathfrak{k}'^C_{(0)}}(X_r^-)\cong\mathfrak{n}_{\mathfrak{k}^C_{(0)}}(X_r^-)/\mathfrak{c}$ is reductive since $\mathfrak{n}_{\mathfrak{k}^C_{(0)}}(X_r^-)=\mathfrak{z}^C\oplus\boldsymbol{C}F$ is reductive, and $\mathfrak{k}^C_{(-1)}$ is an abelian ideal of $\mathfrak{a}(S)$. In particular, $\mathfrak{k}^C_{(-1)}$ is contained in the radical of $\mathfrak{a}(S)$.

Assume first that D is not of tube type. Decompose F as $F=F''+F'$ ($F''\in\mathfrak{c}$, $F'\in\mathfrak{k}'^C$). Then $F'\in\mathfrak{n}_{\mathfrak{k}'^C_{(0)}}(X_r^-)$. We may find $X\in\mathfrak{k}^C_{(-1)}$ with $X\neq 0$ by (7.6.). Then $[F', X]=[F, X]=-X\neq 0$. Thus the radical of $\mathfrak{a}(S)$ is not contained in the center of $\mathfrak{a}(S)$, and hence $\mathfrak{a}(S)$ is not reductive.

Assume next that D is of tube type. Then $\mathfrak{k}^C_{(-1)}=\{0\}$. Thus by (7.7) we get $\mathfrak{a}(S)=\mathfrak{n}_{\mathfrak{k}'^C_{(0)}}(X_r^-)$, which is reductive. Actually, by $F\in\mathfrak{c}$ we have $\mathfrak{a}(S)=\mathfrak{z}^C$.

q.e.d.

Corollary. *Let $S\subset M\subset P_N(\boldsymbol{C})$ be the same as in Theorem* 7.2.

1) *In the following cases, S admits no Kähler metric with constant scalar curvature.*

$M=P_p(\boldsymbol{C})\times P_q(\boldsymbol{C})\subset P_{pq+p+q}(\boldsymbol{C})$ $(1\leqq p<q)$: *Segre imbedding,*
$M=G_{2,n}(\boldsymbol{C})\subset P_{(n+2)(n+1)/2-1}(\boldsymbol{C})$ ($n\geqq 3$, odd): *Plücker imbedding,*
$M=SO(10)/U(5)\subset P_{15}(\boldsymbol{C})$: *canonical imbedding.*

2) *Otherwise, S is a kählerian C-space, and therefore it has an Einstein Kähler metric.*

Proof. 1) These are the all cases where D is not a unit ball nor of tube type. Suppose that S admits a Kähler metric with constant scalar curvature. Then by a theorem of Matsushima-Lichnerowicz (cf. Kobayashi [7]) $\mathfrak{a}(S)$ is reductive. This is a contradiction to Theorem 7.2.

2) This was proved by Hano [2] for $D=(\mathrm{I})_{p+1,p+1}$ $(p\geqq 1)$, by Sakane [14] for $D=(\mathrm{II})_{2m+2}$ $(m\geqq 2)$, by Kimura [6] for $D=(\mathrm{VI})$. In the remaining cases we have $S=P_{n-1}(\boldsymbol{C})$ or $Q_{n-1}(\boldsymbol{C})$. Thus the assertion is obvious. q.e.d.

REMARK. The assertion 1) was proved by Hano [2], Sakane [14] by explicit computation of $\mathfrak{a}(S)$ for each S.

References

[1] A. Borel and F. Hirzebruch: *Characteristic classes and homogeneous spaces* I, Amer. J. Math. **80** (1958), 459–538.

[2] J. Hano: *Examples of projective manifolds not admitting Kähler metric with constant scalar curvature*, Osaka J. Math. **20** (1983), 787–791.

[3] S. Helgason: Differential geometry and symmetric spaces, Academic Press, New York, 1962.

[4] F. Hirzebruch: Topological methods in algebraic geometry, Springer-Verlag, New York, 1966.

[5] Y. Kimura: *On the hypersurfaces of Hermitian symmetric spaces of compact type*, Osaka J. Math. **16** (1979), 97–116.

[6] ————: *A hypersurface of the irreducible Hermitian symmetric space of type EIII*, Osaka J. Math. **16** (1979), 431–438.

[7] S. Kobayashi: Transformation groups in differential geometry, Springer-Verlag, New York, 1972.

[8] ———— and K. Nomizu: Foundations of differential geometry II, Interscience, New York, 1969.

[9] M. Koecher: An elementary approach to bounded symmetric domains, Lect. Notes Rice Univ. Houston, 1969.

[10] A. Korányi and J.A. Wolf: *Realization of hermitian symmetric spaces as generalized half-planes*, Ann. of Math. **81** (1965), 265–288.

[11] H. Naitoh: *Totally real parallel submanifolds in $P^n(C)$*, Tokyo J. Math. **4** (1981), 279–306.

[12] H. Nakagawa and R. Takagi: *On locally symmetric Kaehler submanifolds in a complex projective space*, J. Math. Soc. Japan **28** (1976), 638–669.

[13] Y. Sakane: *On compact Einstein Kähler manifolds with abundant holomorphic transformations*, Manifolds and Lie Groups, Papers in honor of Yozô Matsushima, Birkhäuser, Boston, 1981, 337–358.

[14] ————: *On a non-singular hyperplane section of a Hermitian symmetric space*, to appear in Osaka J. Math.

[15] ———— and M. Takeuchi: *Einstein hypersurfaces of kählerian C-spaces*, Nagoya Math. J. **78** (1980), 153–175.

[16] I. Satake: Algebraic structures of symmetric domains, Iwanami Shoten and Princeton Univ. Press, Tokyo-Princeton, 1981.

[17] J.P. Serre: Algèbres de Lie semi-simple complexes, Benjamin, New York, 1966.

[18] R. Takagi and M. Takeuchi: *Degree of symmetric Kählerian submanifolds of a complex projective space*, Osaka J. Math. **14** (1977), 501–518.

[19] M. Takeuchi: *Polynomial representations associated with symmetric bounded domains*, Osaka J. Math. **10** (1973), 441–475.

[20] ————: *Homogeneous Kähler submanifolds in complex projective spaces*, Japan. J. Math. **4** (1978), 171–219.

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