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A REMARK ON ODD-PRIMARY COMPONENTS OF SPECIAL UNITARY GROUPS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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Let G be a compact connected Lie group of dimension $d>0$, and let us assume that an orientation of G is chosen. Let $\mathfrak{L}$ denote the left invariant framing of the tangent bundle of G. For the pair $(G, \mathfrak{L})$ we obtain by the Pontrjagin-Thom construction an element $[G, \mathfrak{L}]$ in π_d^S. Ossa[6] proved that $72[G, \mathfrak{L}]=0$. Of course this implies that the p-primary component of $[G, \mathfrak{L}]$ is zero for any prime $p>3$. As for information on the 3-primary part of general nature we have the following results of Becker-Schultz: For $G=SO(2n)$, $Spin(2n)$ or $U(n)$ the 3-primary component of $[G, \mathfrak{L}]$ is zero [2]. For the exceptional Lie groups Knapp[4] proved that the 3-primary component of $[F_4, \mathfrak{L}]$ is zero. In this note we give the following additional information:

*) *For* $n\equiv 0$ *or* 3 mod 4 *the* 3-*primary component of* $[SU(n), \mathfrak{L}]$ *vanishes.*

Let τ be an involutive automorphism of G and let K denote the closed subgroup of G consisting of all elements fixed by τ. Then using the equivariant stable homotopy theory for involutions, we have

Proposition 1. *If K is of odd codimension in G, then*

$$[G, \mathfrak{L}]_{(odd)}=0$$

where $a_{(odd)}$ denotes the odd-primary part of a.

The assertion *) is an immediate corollary of this proposition. According to the classification theorem of irreducible Riemannian symmetric spaces, examples of Lie groups to which this proposition applies are $SU(4n)$, $SU(4n+3)$, $Spin(2n)$ and $SO(2n)$. For $SU(n)$ (resp. $Spin(n+1)$ and $SO(n+1)$) we adopt the involutive automorphism corresponding to the symmetric space of AI-type (resp. of BDII-type), whose fixed point set is of codimension $(n-1)(n+2)/2$ (resp. n).

In the final section we make a remark on the real Adams e-invariant e'_R and we show

$$e'_R[SO(16n+6), \mathfrak{L}] = e'_R[Spin(16n+6), \mathfrak{L}] = e'_R[SU(4n), \mathfrak{L}] = 0 \ [1, 4].$$

1. Proposition 1 is generalized as follows. Let $R^{p,q}$ denote the euclidean space R^{p+q} with the linear involution which reverses the first p coordinates and fixes the last q. Let M be a closed smooth manifold with a smooth involution τ. If there exist integers r, $s \geq 0$ and an isomorphism of Z_2-vector bundles

$$\Phi_\tau \colon TM \oplus (M \times R^{r,s}) \to M \times R^{p+r, q+s}$$

for r, s where TM is the tangent bundle of M, then we say that M is (p, q)-*framed*. (This terminology stands for Segal's $R^{p,q}$-*framed* in [7].) Let $\pi^S_{p,q}$ be the (p, q)-th equivariant stable homotopy group of sphere with involution of Landweber [5].

When M is (p, q)-framed, the pair (M, Φ_τ) defines an element $[M, \Phi_\tau]^{Z_2}$ in $\pi^S_{p,q}$ via the equivariant Pontrjagin-Thom construction. Here forget the involution, then Φ_τ becomes the usual stable framing of M which we denote by Φ. We also denote as usual by $[M, \Phi]$ the element of π^S_{p+q} defined by the pair (M, Φ) similarly.

Proposition 2. *Suppose that M is (p, q)-framed and p is odd. Then*

$$[M, \Phi]_{(odd)} = 0 .$$

Before we prove Proposition 2 we observe that G is (d-dim K, dim K)-framed. Recall that identifying R^d with T_eG in the orientation preserving way, the left invariant framing $\mathfrak{L}\colon TG \to G \times R^d$ of G is given by $\mathfrak{L}(v_g) = (g, (L_{g^{-1}})_* v_g)$ $v_g \in T_gG$, where T_gG denotes the tangent space at $g \in G$ and $L_{g^{-1}}\colon G \to G$ the left multiplication by g^{-1}. If we consider here the action on TG induced by τ, then it is easy to check that $\mathfrak{L}$ itself is an isomorphism of Z_2-vector bundles $TG \to G \times R^{d-h,h}$ where h denotes the dimension of K. Thus we see that Proposition 1 is obtained as a corollary of Proposition 2.

2. Proof of Proposition 2. According to [5] there are different equivariant stable homotopy groups of spheres with involutions $\lambda^S_{p,q}$ and an exact sequence involving them

$$(1) \qquad \to \lambda^S_{p,q} \to \pi^S_{p,q} \xrightarrow{\phi} \pi^S_q \to \lambda^S_{p,q-1} \to$$

where ϕ is the fixed-point homomorphism. Furthermore we have by [5], Proposition 6.1 and Corollary 6.3 an isomorphism

$$\lambda^S_{p,q} \cong \pi^S_{2^a+q}(P^\infty / P^{2^a-p-1})$$

for a suitably large a (depending on p and q) where P^k is the k-dimensional real projective space. Using the Atiyah-Hirzebruch spectral sequence we therefore obtain

$$\lambda^S_{p,q(odd)} = 0 \qquad \text{for } p \text{ odd.} \tag{2}$$

(For a finite abelian group A, $A_{(odd)}$ denotes the odd components of A.) From (1) and (2) we have an isomorphism

$$\phi\colon \pi^S_{p,q(odd)} \cong \pi^S_{q(odd)} \qquad \text{for } p \text{ odd [3]}. \tag{3}$$

We now prove Proposition 2. Let I be an involution of $R^{p+r,q+s}$ defined by $I(x_1, x_2, \cdots, x_{p+q+r+s}) = (-x_1, x_2, \cdots, x_{p+q+r+s})$. Assuming that M has the equivariant stable framing Φ_τ as above, we see by definition that $\phi[M, \Phi_\tau]^{Z_2} = \phi[M, (1\times I)\Phi_\tau]^{Z_2}$. So it follows from (3) that

$$[M, \Phi_\tau]^{Z_2}{}_{(odd)} = [M, (1\times I)\Phi_\tau]^{Z_2}{}_{(odd)}\,. \tag{4}$$

By definition we also have $\psi[M, (1\times I)\Phi_\tau]^{Z_2} = -[M, \Phi]$ where ψ is the forgetful homomorphism. By this and (4) we therefore have $[M, \Phi]_{(odd)} = 0$, which completes the proof of Proposition 2.

3. We prove here the following

Proposition 3. *Suppose that* $4n-1>p\geq 0$ *and* $p\equiv 1, 5 \bmod 8$, *n even*; $p\equiv 3, 7 \bmod 8$. *Then the composite*

$$e'_R\psi\colon \pi^S_{p,4n-p-1} \to Q/Z$$

is zero.

Proof. By $\Sigma^{p,q}$ we denote the one-point compactification of $R^{p,q}$ with ∞ as base point. Let $f\colon \Sigma^{8r+p,8r+4n-p-1} \to \Sigma^{8r,8r}$ be a base point preserving Z_2-map for large r and $\bar f$ be the map obtained from f by forgetting the action. Applying $\widetilde{KO}$ to the cofibre sequence

$$S^{16r+4n-1} \xrightarrow{\bar f} S^{16r} \xrightarrow{\bar i} C_{\bar f} \to S^{16r+4n} \to S^{16r+1}$$

where $C_{\bar f}$ is the mapping cone of $\bar f$ and $\bar i$ is the injection map, we have an exact sequence

$$0 \leftarrow \underset{\cong Z}{\widetilde{KO}(S^{16r})} \xleftarrow{\bar i^*} \widetilde{KO}(C_{\bar f}) \leftarrow \underset{\cong Z}{\widetilde{KO}(S^{16r+4n})} \leftarrow 0\,. \tag{1}$$

Using the periodicity theorem and Lemma 4.1 in [5] we get

$$\widetilde{KO}_{Z_2}(\Sigma^{8r+p,8r+4n-p-\varepsilon}) \cong \begin{cases} 0 \text{ or } Z_2 & \text{if } \varepsilon = 0 \\ 0 \text{ or } Z & \text{if } \varepsilon = 1\,. \end{cases} \tag{2}$$

Furthermore it follows from the argument in §2 that $\lambda^S_{p,4n-p-1}\otimes Q = 0$ and hence $\pi^S_{p,4n-p-1}$ is a finite group. So applying $\widetilde{KO}_{Z_2}$ to the cofibre sequence induced by f, we see that

$$i^*: \widetilde{KO}_{Z_2}(C_f) \rightarrow \widetilde{KO}_{Z_2}(\Sigma^{8r,8r}) \text{ is onto}$$

where i is the injection map $\Sigma^{8r,8r} \subset C_f$. Therefore we can choose a generator ξ of $\widetilde{KO}_{Z_2}(C_f)$ such that $\bar{i}^*\psi(\xi)$ generates $\widetilde{KO}(S^{16r})$ where ψ denotes the forgetful homomorphism in K-theory. Let ψ^k be the k-th Adams operation in KO- or KO_{Z_2}-theory. Since $\psi\psi^k = \psi^k\psi$ we have by [5], Lemma 3.3 and (2)

$$\psi^k\psi(\xi) = k^{8r}\psi(\xi),$$

which implies that (1) splits as an exact sequence of groups admitting operations ψ^k. Hence by definition

$$e'_R[\bar{f}] = 0$$

where $[\bar{f}]$ is the stable homotopy class of $\bar{f}$. This completes the proof.

Again considering the classification of involutive automorphisms of Lie groups, we have the examples above to which this proposition applies.

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