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STRUCTURES OF THE HAKEN MANIFOLDS WITH HEEGAARD SPLITTINGS OF GENUS TWO

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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1. Introduction

In this paper we will give a complete list of the closed, orientable 3-manifolds with Heegaard splittings of genus two and admitting non-trivial torus decompositions. We use the following notations.

D(n) (A(n), Mö(n) resp.): *the collection of the Seifert fibered manifolds the orbit manifold of each of which is a disk (annulus, Möbius band resp.) with* n *exceptional fibers.*

M_K (M_L resp.) : *the collection of the exteriors of the two bridge knots (links resp.).*

L_K : *the collection of the exteriors of the one bridge knots in lens spaces each of which admits a complete hyperbolic structure or admits a Seifert fibration whose regular fiber is not a meridian loop.*

For the definitions of the one bridge knots in lens spaces see section 5. Then our main result is

Theorem. *Let M be a closed, connected Haken manifold with a Heegaard splitting of genus two. If M has a nontrivial torus decomposition then either*

(i) *M is obtained from $M_1 \in D(2)$ and $M_2 \in L_K$ by identifying their boundaries where the regular fiber of M_1 is identified with the meridian loop of M_2,*

(ii) *M is obtained from $M_1 \in Mö(n)$ (n=0, 1 or 2) and $M_2 \in M_K$ by identifying their boundaries where the regular fiber of M_1 is identified with the meridian loop of M_2,*

(iii) *M is obtained from $M_1 \in D(n)$ (n=2 or 3) and $M_2 \in M_K$ by identifying their boundaries where the regular fiber of M_1 is identified with the meridian loop of M_2,*

(iv) *M is obtained from M_1, $M_2 \in D(2)$ and $M_3 \in M_L$ by identifying their boundaries where the regular fiber of M_i (i=1, 2) is identified with the meridian loop of M_2 or*

(v) *M is obtained from $M_1 \in A(n)$ (n=0, 1 or 2) and $M_2 \in M_L$ by ident-*

ifying their boundaries where the regular fiber of M_1 *is identified with the meridian loop of* M_2.

Conversely if a 3*-manifold has a decomposition as in* (i)~(v) *then it has a Heegaard splitting of genus two.*

For the structures of the elements of L_K, M_K or M_L see Lemma 4.2, 4.4, 5.2.

In [9] Thurston listed eight 3-dimensional geometries with compact stabilizers and conjectured that every closed 3-manifold admits a geometric decomposition. Thurston's recent result [10] asserts that every closed, orientable 3-manifold with a Heegaard splitting of genus two has a geometric decomposition. Then our Theorem together with this result implies

Corollary. *If* M *is a closed, orientable* 3*-manifold with a Heegaard splitting of genus two then either*

(i) M *admits one of the eight geometric structures stated in* [9], *or*
(ii) M *is one of* (i)~(v) *in the above theorem.*

We note that for each of the eight geometric structures there is a 3-manifold which has a Heegaard splitting of genus two and admits the geometric structure. See section 7.

2. Preliminaries

Throughout this paper we will work in the piecewise linear category.

For the definitions of *irreducible* 3*-manifolds, incompressible surfaces* we refer to [1]. For the definitions of *Haken manifolds* we refer to [4].

Let M be a closed, connected 3-manifold. $(V_1, V_2; F)$ is called a *Heegaard splitting* of M if each V_i is a 3-dimensional handlebody, $M=V_1\cup V_2$ and $V_1\cap V_2=\partial V_1=\partial V_2=F$. Then F is called a *Heegaard surface* of M. The first Betti number of V_i is called the *genus* of the Heegaard splitting.

For the definitions of *Seifert fibered manifolds, orbit manifold, an isotopy of type* A, *hierarchy* for a surface, *an essential arc* in a surface and other definitions of standard terms in three dimensional topology we refer to [4]. The 3-manifold M is *simple* if every incompressible torus in M is boundary parallel.

By [4] every closed Haken manifold contains a unique, maximal, perfectly embedded Seifert fibered manifold Σ which is called a characteristic Seifert pair for M. The components of the closure of $M-\Sigma$ are simple. The boundary of Σ consists of tori in M. If some components of them are parallel in M then we eliminate one of them from the system of tori. By proceeding this step we get a system of tori in M which are mutaually non-parallel. We get simple manifolds and Seifert fibered manifolds by cutting M along these tori. In this paper, we call this decomposition a *torus decomposition* of M.

3. Essential annuli in genus two handlebody

Let F be a 2-sided surface properly embedded in a 3-manifold M. F is *essential* if it is incompressible and not parallel to a surface in ∂M. Let M' be a 3-manifold obtained by cutting M along F. Then there are copies of F on $\partial M'$ and we denote the component of the copies also by F.

In this section we will classify the system of essential annuli in the genus two handlebody.

Lemma 3.1 *If A is an incompressible annulus properly embedded in the solid torus V, the genus one handlebody, then A is boundary parallel.*

Proof. First, we claim that A cuts V into two solid tori. ∂A cuts ∂V into two annuli A_1, A_2. Then $A \cup A_i$ $(i=1, 2)$ is a torus in V. Since $\pi_1(V) \cong \boldsymbol{Z}$, $A \cup A_i$ is compressible in V. By the loop theorem [1] and the irreducibility of V we see that $A \cup A_i$ bounds a solid torus V_i. Let p_i $(i=1, 2)$ be a positive integer such that $\mathrm{Im}\,(i_*; \pi_1(A) \rightarrow \pi_1(V_i)) = \langle a_i^{p_i} \rangle$, where a_i is a generator of $\pi_1(V_i)$. Then $\pi_1(V) \cong \langle a_1, a_2 : a_1^{p_1} = a_2^{p_2} \rangle$. Then $p_1 = 1$ or $p_2 = 1$ for $\pi_1(V) \cong \boldsymbol{Z}$. If $p_i = 1$ then A is parallel to A_i.

This completes the proof of Lemma 3.1.

Let D be a disk properly embedded in a handlebody V. D is *a meridian disk* of V if D does not separate V. Let $\{D_1, \cdots, D_n\}$ be a system of mutually disjoint properly embedded disks in V. $\{D_1, \cdots, D_n\}$ is *a complete system of meridian disks* of V if $\bigcup_{i=1}^{n} D_i$ cuts V into a 3-cell.

Lemma 3.2 *If A is an essential annulus in a genus two handlebody V then either*

(i) *A cuts V into a solid torus V_1 and a genus two handlebody V_2 and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_2 such that $D_1 \cap A = \phi$ and $D_2 \cap A$ is an essential arc of A, or*

(ii) *A cuts V into a genus two handlebody V' and there is a complete system of meridian disks $\{D_1, D_2\}$ of V' such that $D_1 \cap A$ is an essential arc of A.*

See Fig. 1.

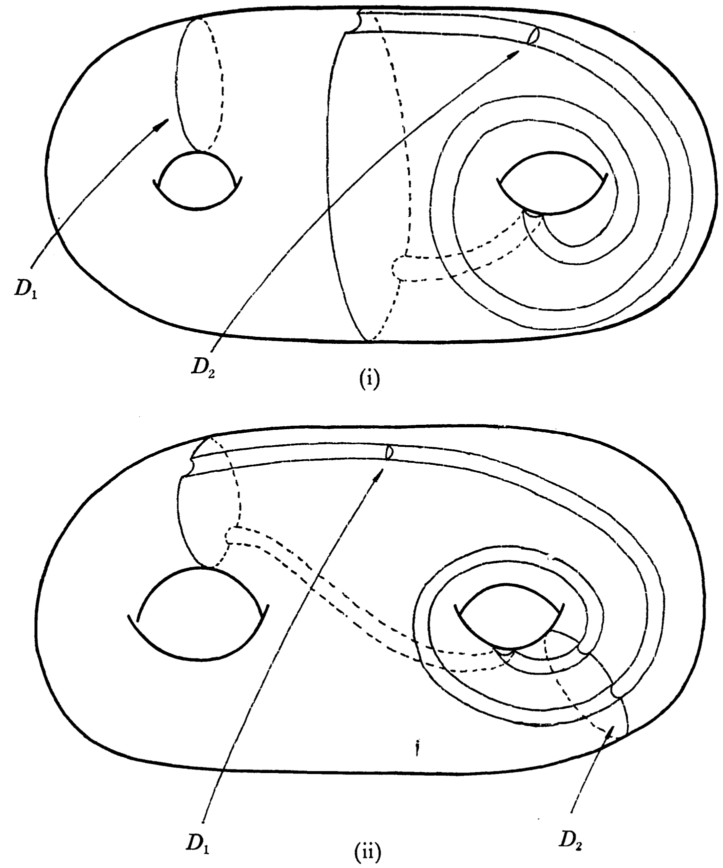

Fig. 1

Proof. Since A is incompressible in V, by using the complete system of meridian disks of V we can find a disk Δ in V such that $\Delta \cap A = a$ is an essential arc of A, $\Delta \cap \partial V = b$ is an arc such that $\partial a = \partial b$, $a \cup b = \partial \Delta$. Then we can perform a surgery on A along Δ to get a disk D properly embedded in V. Since A is essential, D is essential, say D is a meridian disk of V or D cuts V into two solid tori.

If D cuts V into two solid tori V', V'' then there are copies Δ', Δ'' of Δ on $\partial V'$. Then there is a meridian disk D_1 of V' such that $D_1 \cap (\Delta' \cup \Delta'') = \phi$. Since Δ' and Δ'' are identified in V cut along A, A cuts V into a solid torus V_1 and a genus two handlebody V_2, where Δ is a meridian disk of V_2 such that $\Delta \cap A$ is an essential arc of A. Then we set $D_2 = \Delta$.

If D is a meridian disk of V then D cuts V into a solid torus V_1. There are copies Δ', Δ'' of Δ on ∂V_1. Since Δ' and Δ'' are identified in V cut along A, A cuts V into a genus two handlebody V'. Then we set $D_1 = \Delta$ and we have the conclusion (ii).

This completes the proof of Lemma 3.2.

Let M be a 3-manifold and S be a 2-manifold contained in ∂M. Let F be a surface properly embedded in M. Then F is *essential in* (M, S) if F is incompressible, $\partial F \subset S$ and F is not parallel to a surface in S.

Lemma 3.3 *Let V be a genus two handlebody and A_1, A_2 be a system of mutually disjoint annuli in ∂V such that there is a complete system of meridian disks $\{D_1, D_2\}$ of V which satisfies $D_i \cap A_j = \phi$ $(i \neq j)$ and $D_i \cap A_i$ is an essential arc of A_i $(i=1, 2)$. If A is an essential annulus in $(V, cl\,(\partial V-(A_1 \cup A_2)))$ then A is parallel to A_1 or A_2.*

Proof. Since A is incompressible in V, by using $\{D_1, D_2\}$ we can find a disk Δ in V such that $\Delta \cap A = a$ is an essential arc of A, $\Delta \cap cl\,(\partial V-(A_1 \cup A_2)) = b$ is an arc such that $\partial a = \partial b$, $a \cup b = \partial\Delta$. Then we can perform a surgery on A along Δ to get an essential disk D such that $D \cap (A_1 \cup A_2) = \phi$. Since $D \cap (A_1 \cup A_2) = \phi$, D cuts V into two solid tori V_1, V_2. We may suppose that $A_i \subset \partial V_i$. By assumption there is a meridian disk D_i' of V_i such that $D_i' \cap A_i$ is an essential arc of A_i. Then by the proof of Lemma 3.2 A cuts V into a genus two handlebody V_1' and a solid torus V_2'. We may suppose that $A_2 \subset \partial V_2'$. Then $\mathrm{Im}\,(i_* : \pi_1(A_2) \rightarrow \pi_1(V_2)) = \pi_1(V_2)$ and $A_2 \cap A = \phi$. Hence A is parallel to A_2.

This completes the proof of Lemma 3.3.

For the two essential annuli in the genus two handlebody we have

Lemma 3.4 *Let $\{A_1, A_2\}$ be a system of mutually disjoint, non-parallel, essential annuli in the genus two handlebody V. Then either*

(i) *$A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2. Then $A_1 \cup A_2 \subset \partial V_1$, $A_1 \cup A_2 \subset \partial V_2$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_2 such that $D_i \cap A_j = \phi$ $(i \neq j)$ and $D_i \cap A_i$ $(i=1, 2)$ is an essential arc of A_i,*

(ii) *$A_1 \cup A_2$ cuts V into two solid tori V_1, V_2 and a genus two handlebody V_3. Then $A_1 \subset \partial V_1$, $A_2 \subset \partial V_2$, $A_1 \cup A_2 \subset \partial V_3$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_3 such that $D_i \cap A_j = \phi$ $(i \neq j)$ and $D_i \cap A_i$ $(i=1, 2)$ is an essential arc of A_i or*

(iii) *$A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2. Then $A_i \subset \partial V_1$ $(i=1$ or 2, say $1)$, $A_2 \cap V_1 = \phi$, $A_1 \subset \partial V_2$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_2 such that $D_1 \cap A_2$ is an essential arc of A_2 and $D_2 \cap A_i$ $(i=1, 2)$ is an essential arc of A_i.*

See Fig. 2.

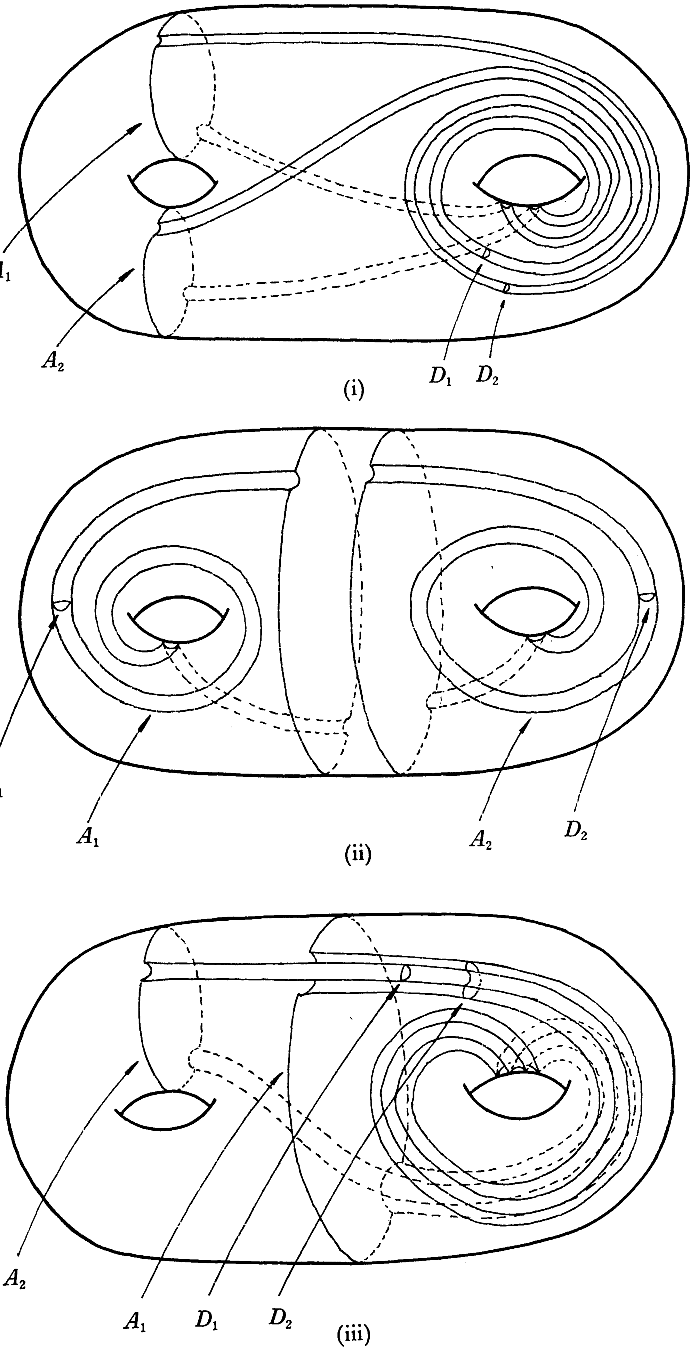

Fig. 2

Proof. There is a disk Δ in V such that $\Delta\cap A_i=\phi$ ($i=1$ or 2, say 2), $\Delta\cap A_1=a$ is an essential arc of A_1, $\Delta\cap\partial V=b$ is an arc in $\partial\Delta$ such that $a\cap b=\partial a=\partial b$, $a\cup b=\partial\Delta$. We can perform a surgery on A_1 along Δ to get an essential disk D' properly embedded in V. Then there is a disk Δ' in V such that $\Delta'\cap D'=\phi$, $\Delta'\cap A_2=a'$ is an essential arc of A_2, $\Delta'\cap\partial V=b'$ is an arc in $\partial\Delta'$ such that $a'\cap b'=\partial a'=\partial b'$, $a'\cup b'=\partial\Delta'$. By performing a surgery on A_2 along Δ' we have an essential disk D'' in V, which is disjoint from D'.

We claim that $\{D', D''\}$ is not a complete system of meridian disks of V. Assume that $\{D', D''\}$ is a complete system of meridian disks of V. Then we can move A_2 by a small isotopy into V cut along $D'\cup D''$. This contradicts the fact that A_2 is incompressible in V.

Then we have the following three cases.

Case 1. D' and D'' are parallel and D' (hence, D'') does not separate V. In this case, we have the conclusion (i).

Case 2. D' and D'' are parallel and D' (hence, D'') cuts V into two solid tori. In this case, we have the conclusion (ii).

Case 3. D' and D'' are not parallel. We claim that D' does not separate V. Assume that D' separate V into two solid tori V' and V''. Then we may suppose that $A_2\subset V'$. By Lemma 3.1 A_2 is parallel to an annulus A_2' in $\partial V'$. Then $A_2'\cap D'=\phi$ for D' and D'' are not parallel. But this contradicts the fact that A_2 is essential.

Then since $\{D', D''\}$ is not a complete system of meridian disks, D'' separates V into two solid tori and we have the conclusion (iii).

This completes the proof of Lemma 3.4.

Lemma 3.5 *Let $\{A_1, A_2, A_3\}$ be a system of pairwise disjoint, non-parallel essential annuli in the genus two handlebody V. Then $A_1\cup A_2\cup A_3$ cuts V into two solid tori V_1, V_2 and a genus two handlebody V_3 which satisfies*

1. $A_i\subset\partial V_1$ ($i=1$, 2 *or* 3, *say* 3), $A_1, A_2\subset\partial V_3$, $A_1, A_2, A_3\subset\partial V_2$.
2. *there is a complete system of meridian disks $\{D_1, D_2\}$ of V_3 such that $D_i\cap A_j=\phi$ ($i\neq j$) and $D_i\cap A_i$ ($i=1$, 2) is an essential arc of A_i and*
3. *there is a meridian disk D_3 of V_2 such that $D_3\cap A_i$ ($i=1$, 2, 3) is an essential arc of A_i.*

See Fig. 3.

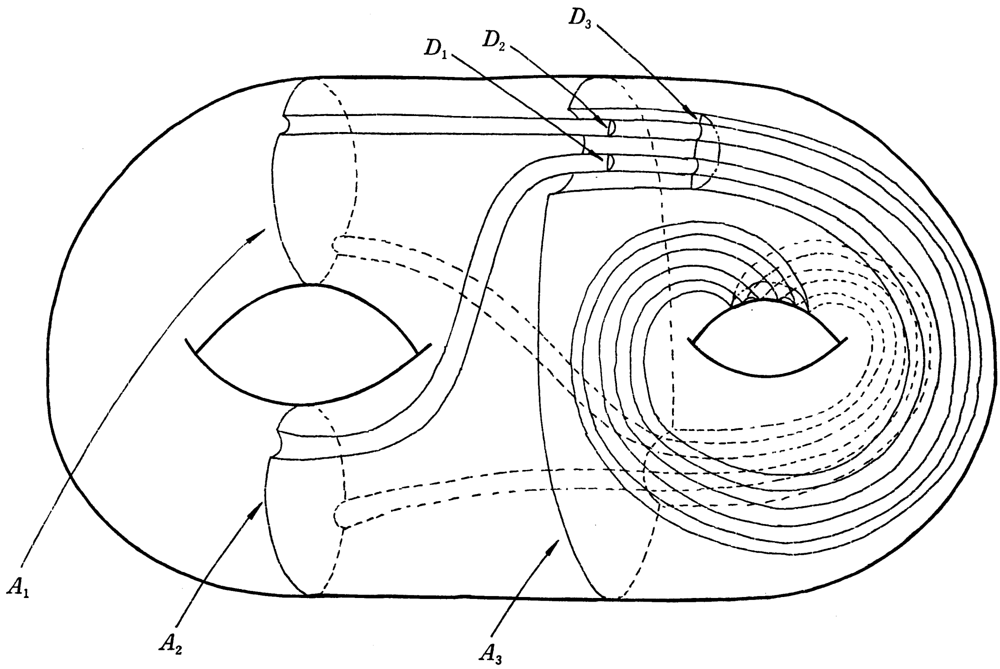

Fig. 3

Proof. $\{A_1, A_2\}$ satisfies one of the conclusions of Lemma 3.4. First, we claim that $\{A_1, A_2\}$ does not satisfy (ii). Assume that $\{A_1, A_2\}$ satisfies (ii). Then $A_1 \cup A_2$ separates V into two solid tori V_1, V_2 and a genus two handlebody V_3. If $A_3 \subset V_1$ or V_2 then by Lemma 3.1 A_3 is parallel to A_1 or A_2, which is a contradiction. If $A_3 \subset V_3$ then by Lemma 3.3 A_3 is parallel to A_1 or A_2, which is a contradiction and the claim is established.

If $\{A_1, A_2\}$ satisfies the conclusion (i) of Lemma 3.4 then $A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2 where $A_1, A_2 \subset \partial V_1$, $A_1, A_2 \subset \partial V_2$. By Lemma 3.3 we see that A_3 is not contained in V_2. Then $A_3 \subset V_1$ and by Lemma 3.1 A_3 is parallel to an annulus A' in ∂V_1. Since A_3 is essential and is not parallel to A_i $(i=1, 2)$, $\partial A_1 \cup \partial A_2$ is contained in A'. Then we easily verify that $\{A_1, A_2, A_3\}$ satisfies the conclusions of Lemma 3.5.

If $\{A_1, A_2\}$ satisfies the conclusion (iii) then $A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2, where $A_1, A_2 \subset \partial V_2$ and $A_i \cap \partial V_1 = \phi$ $(i=1$ or 2, say $1)$. By Lemma 3.1 we see that A_3 is contained in V_2. Since $A_3 \cap (A_1 \cup A_2) = \phi$, by Lemma 3.3 we see that A_3 is parallel to an annulus A' in ∂V_2. Since A_3 is essential and is not parallel to A_i $(i=1, 2)$, $\partial A_1 \cup \partial A_2$ is contained in A'. Then by changing the suffix we easily verify that $\{A_1, A_2, A_3\}$ satisfies the conclusions of Lemma 3.5.

This completes the proof of Lemma 3.5.

4. Two bridge knot, link complements

A *knot* is a simple closed curve in the 3-sphere S^3. A *link* is a union of mutually disjoint simple closed curves in S^3 with more than one component. For the definitions of the *two bridge knots* and *links* we refer to [8]. A *exterior* $Q(K)$ ($Q(L)$ resp.) of a knot K (link L resp.) is the closure of the complement of a regular neighborhood of K (L resp.). The *meridian* of K (L resp.) is a simple loop in $\partial Q(K)$ ($\partial Q(L)$ resp.) which bounds a meridian disk of the regular neighborhood of K (L resp.). A knot (link resp.) is *simple* if the exterior is a simple 3-manifold.

Lemma 4.1 *Let V_i ($i=1, 2$) be the genus two handlebody and A_1^i, A_2^i ($\subset \partial V_i$) be a system of pairwise disjoint, incompressible annuli such that there is a complete system of meridian disks $\{D_1^i, D_2^i\}$ of V_i which satisfies* (i) $D_k^i \cap A_l^i = \phi$ ($k \neq l$) *and* (ii) $D_k^i \cap A_k^i$ *is an essential arc of A_k^i ($k=1, 2$). If M is obtained from V_1 and V_2 by identifying their boundaries by a homeomorphism $h: cl(\partial V_1 - (A_1^1 \cup A_2^1)) \rightarrow cl(\partial V_2 - (A_1^2 \cup A_2^2))$ then M is homeomorphic to certain two bridge knot complement or a two bridge link complement, where the component of ∂A_j^i corresponds to a meridian loop.*

Proof. This can be proved by using the similar arguments of the section 4 of [5].

Lemma 4.2 *If K is a non-trivial two bridge knot then $Q(K)$ admits a complete hyperbolic structure or is a Seifert fibered manifold with orbit manifold a disk with two exceptional fibers.*

Proof. Since K is a simple knot [8], by [9] and the torus theorem [4] we see that $Q(K)$ admits a complete hyperbolic structure or is a special Seifert fibered manifold. If $Q(K)$ is a special Seifert fibered manifold then the orbit manifold is a disk or a Möbius band for $\partial Q(K)$ has one component (see 155p. of [4]). If the orbit manifold of $Q(K)$ is a Möbius band then it has no exceptional fibers. Hence $Q(K)$ is the twisted I-bundle over the Klein bottle but this is impossible for $Q(K)$ does not contain the Klein bottle.

This completes the proof of Lemma 4.2.

Let $\{a_1, \cdots, a_n\}$ be a system of mutually disjoint, essential arcs in a punctured torus T. We say that a_i is of *type* 1 if a_i joins distinct components of ∂T, a_i is of *type* 2 if a_i joins one component of ∂T and a_i separates T, a_i is of *type* 3 if a_i joins one component of ∂T and a_i does not separate T. We say that a_i is a *d-arc* if a_i is of type 1 and there is a component S of ∂T such that a_i is the only arc that joins S.

The next Lemma is perhaps known but no reference could be found.

Lemma 4.3 *Every two bridge link is a simple link.*

Proof. Let L be a two bridge link. Since L is a union of two trivial tangles with two arcs, $Q(L)$ has a decomposition as in Lemma 4.1 (see section 4 of [5]). Then we use the notations in Lemma 4.1. Let T be an incompressible torus in $Q(L)$. We may suppose that the components of $T\cap V_1$ are all disks and that the number of the components of $T\cap V_1$ is minimum among all tori which are isotopic to T and the components of the intersection of each of which with V_1 are all disks. Since T is incompressible, $T\cap V_1\neq\phi$.

Let $T_2=T\cap V_2$. Then by using D_1^2, D_2^2 we have a hierarchy $(T_2^{(0)}, a_0),\cdots,(T_2^{(m)}, a_m)$ of T_2 and a sequence of isotopies of type A which realizes the hierarchy as in [4]. Let $T^{(1)}$ be the image of T after an isotopy of type A at a_0 and $T^{(k+1)}$ $(k\geq 1)$ be the image of $T^{(k)}$ after an isotopy of type A at a_k.

Then we will show that $T\cap V_1$ consists of a disk.

Assume that $T\cap V_1$ consists of $n(\geq 2)$ disks $D_1, \cdots, D_n$. We claim that $D_1, \cdots, D_n$ are mutually parallel in V_1 and each D_i cuts V_1 into two solid tori. If some D_i does not separate V_1 then $D_i\cap(A_1^1\cup A_2^1)\neq\phi$ for $\mathrm{Im}\,(i_*\colon \pi_1(A_1^1\cup A_2^1)\to\pi_1(V_1))=\pi_1(V_1)$, which is a contradiction. By the minimality of T it follows that each D_i cuts V_1 into two solid tori. Hence $D_1 \cdots, D_n$ are mutually parallel and the claim is established.

Then we show

(*) $a_0, \cdots, a_{n-1}$ are of type 3 and a_i and a_j joins pairwise distinct components of ∂T_2 if $i\neq j$.

By Lemma 3.3 each essential annulus in $(V_1, cl\,(\partial V_1-(A_1^1\cup A_2^1))$ is parallel to A_1^1 or A_2^1. By Lemma 3.3 of [5] we see that a_0, a_1 are of type 3 and we may suppose that a_0, a_1 joins D_1, D_2 respectively. Note that in [5] we considered the non-separating incompressible torus, but in Lemma 3.1, 3.2, 3.3 of [5] which are proved by using the argument of the inverse operation of isotopy of type A in [2] the non-separating property is not essential.

Assume that (*) does not hold then there is such i (≥ 3) that a_i is not of type 3 or a_i is of type 3 and a_i joins D_k that some a_l $(l<i)$ joins. Then we may suppose that a_j $(j<i)$ is of type 3 and joins D_j. Then $T^{(i-1)}\cap V_1=A_1\cup\cdots\cup A_{i-1}\cup D_i\cup\cdots\cup D_n$, where each A_i is an essential annulus in $(V_1, cl\,(\partial V_1-(A_1^1\cup A_1^2)))$.

Assume that a_i is of type 1. If a_i joins some A_k and D_l $(l\geq i)$ or D_k and D_l $(k, l\geq i)$ as an arc on $T^{(i-1)}\cap V_2$ then $T^{(i)}\cap V_1$ consists of $i-1$ annuli and $n-i$ disks. Then by performing a sequence of isotopies of type A on $T^{(i)}$ we have such a torus T' that $T'\cap V_1$ consists of $n-1$ disks, which contradicts the minimality of T. If a_i joins some A_k and A_l then A_k is parallel to A_l in V_1 for D_i separates V_1 into two solid tori. Then $T^{(i)}\cap V_1$ consists of $i-2$

annuli, $n-i$ disks and one disk with two holes B. Some component l of ∂B bounds a disk on ∂V. Since T is incompressible and $Q(L)$ is irreducible, we see that l bounds a disk on $T^{(i)}$ and there is an ambient isotopy h_t $(0\leq t\leq 1)$ of $Q(L)$ such that $h_1(T^{(i)})\cap V_1$ consists of $i-1$ annuli and $n-i$ disks. Then we have a contradiction as above.

Assume that a_i is of type 2. Then there is an arc a in ∂T_2 such that $a\cap a_i=\partial a=\partial a_i$, $a\cup a_i$ bounds a planar surface P in T_2. We easily see that some a_j $(\subset P)$ is a d-arc. Hence by Lemma 3.1 of [5] T is ambient isotopic to such a torus T' that $T'\cap V_1$ consists of $n-1$ disks, which is a contradiction.

Assume that a_i is of type 3 and a_i joins D_j $(j<i)$. Then there are two arcs b_1, b_2 in ∂T_2 such that $a_j\cup b_1\cup a_i\cup b_2$ is a simple loop in T_2 and $a_j\cup b_1\cup a_i\cup b_2$ bounds a planar surface P in T_2. Then see that some a_k $(\subset P)$ is a d-arc and we have a contradiction as above.

Hence (*) is established.

Then $T^{(n)}\cap V_1$ $(T^{(n)}\cap V_2$ resp.) consists of n essential annuli $A_1, \cdots, A_n$ $(A'_1, \cdots, A'_n$ resp.). By Lemma 3.3 each A_i is parallel to either A^1_1 or A^1_2. We may suppose that A_n is outermost in $(V_1, cl(\partial V_1-(A^1_1\cup A^1_2)))$ and is parallel to A^1_1. Then some A'_j is parallel to A^2_k $(k=1$ or $2)$ and $\partial A_n=\partial A'_j$. This contradicts the fact that $n\geq 2$.

Hence $T\cap V_1$ consists of a disk. Then $T^{(1)}\cap V_i$ consists of an annulus A^i which is parallel to A^i_j $(j=1$ or $2)$. Hence $T^{(1)}$ is parallel to a component of $\partial Q(L)$.

This completes the proof of Lemma 4.3.

Lemma 4.4 *If L is a two bridge link then $Q(L)$ admits a complete hyperbolic structure or is a Seifert fibered manifold with orbit manifold an annulus with at most one exceptional fiber.*

Proof. By Lemma 4.3 together with [4] and [9] $Q(L)$ is a hyperbolic manifold or a special Seifert fibered manifold. If $Q(L)$ is a special Seifert fibered manifold then the orbit manifold of $Q(L)$ is an annulus and it has at most one exceptional fiber for $\partial Q(L)$ has two components.

5. One bridge knots in lens spaces

Let us give the definition of the *one bridge knot in a lens space*. For the definition of lens spaces we refer to 20p. of [1]. In this paper we think that S^3, $S^2\times S^1$ are lens spaces. Let V be a solid torus and let a be an arc properly embedded in V. We say that a is trivially embedded in V if there is a disk D in V such that $D\cap\partial V=b$ an arc, $cl(\partial D-b)=a$. It is easily seen that if a' is another trivially embedded arc in V then there is an ambient isotopy h_t $(0\leq t\leq 1)$ of V such that $h_1(a)=a'$. Let K be a knot in a lens space L_n. We say that K is a one bridge knot in L_n if there is a Heegaard splitting $(V_1, V_2;$

F) of L_n of genus one such that $V_i \cap K$ ($i=1, 2$) is an arc trivially embedded in V_i. We denote the exterior of K also by $Q(K)$. Then we can naturally define a meridian loop on $Q(K)$.

Lemma 5.1 *Let V_i ($i=1, 2$) be a genus two handlebody and A_i ($\subset \partial V_i$) be an incompressible annulus such that there is a complete system of meridian disks $\{D_1^i, D_2^i\}$ of V_i which satisfies* (i) *$D_1^i \cap A_i = \phi$ and* (ii) *$D_2^i \cap A_i$ is an essential arc of A_i. If M is obtained from V_1 and V_2 by identifying their boundaries by a homeomorphism h: $cl(\partial V_1 - A_1) \to cl(\partial V_2 - A_2)$ then M is homeomorphic to certain one bridge knot complement in lens space, where the component of ∂A_i corresponds to a meridian loop.*

Proof. This is proved by using the similar arguments of the proof of Lemma 4.1.

Lemma 5.2 *Let K be a one bridge knot in a lens space L_n. If $Q(K)$ is a Seifert fibered manifold with incompressible boundary, whose regular fiber is not a meridian loop then either*

(i) *$Q(K) \in D(2)$ where the regular fiber in $\partial Q(K)$ intersects their meridian loop transversely in a single point,*

(ii) *$Q(K) \in Mö(1)$ where the regular fiber in $\partial Q(K)$ intersects the meridian loop transversely in a single point or*

(iii) *$Q(K)$ is homeomorphic to the twisted I-bundle over the Klein bottle.*

Proof. We fix the fiber structure of $Q(K)$. Since an incompressible torus in $Q(K)$ is separating, the orbit manifold of $Q(K)$ is a disk or a Möbius band.

Suppose that the orbit manifold of $Q(K)$ is a disk. First we claim that L_n does not admit such a Seifert fibration that the orbit manifold is a 2-sphere with n (≥ 3) exceptional fibers. $n \geq 4$ implies that L_n contains an incompressible torus, which is a contradiction. By Theorem 12.2 of [1] $n=3$ implies that there is an epimorphism from $\pi_1(L_n)$ to the group

$$G = \langle a, b;\ a^p = b^q = (ab)^r = 1 \rangle \quad (p, q, r > 1)\,.$$

This is impossible for G is not a cyclic group [7] and the claim is established.

Assume that $Q(K)$ contains m (≥ 3) exceptional fibers. Then since the regular fiber of $Q(K)$ is not a meridian loop, L_n admits such a Seifert fibration that the orbit manifold is a 2-sphere with m or $m+1$ exceptional fibers, which contradicts the above claim. Hence $Q(K)$ contains two exceptional fibers. Then if the regular fiber is not isotopic to a loop which intersects the meridian loop transversely in a single point then L_n admits such a Seifert fibration that the orbit manifold is a 2-sphere with three exceptional fibers, which contradicts the above claim.

Then we have the conclusion (i).

Suppose that the orbit manifold of $Q(K)$ is a Möbius band. Since L_n does not contain an incompressible torus $Q(K)$ contains at most one exceptional fibers. If $Q(K)$ contains one exceptional fiber then the regular fiber intersects the meridian loop transversely in a single point and we have the conclusion (ii). If $Q(K)$ contains no exceptional fibers then we have the conclusion (iii).

This completes the proof of Lemma 5.2.

6. Proof of Theorem

Lemma 6.1 *Let M be a simple manifold whose boundary components are all tori. If M contains an essential annulus then M is a Seifert fibered manifold.*

Proof. This is a consequence of the characteristic Seifert pair theorem [4].

We shall divide the proof of Theorem into several cases.

Case 1. M contains a non-separating incompressible torus. In this case by Theorem 2 of [5] we have the conclusion (v) of the Theorem.

Hereafter, we will suppose that each incompressible torus in M is separating.

Case 2. M is decomposed into two components M_1, M_2 by the torus decomposition. Let T be the torus which cuts M into M_1, M_2 and $(V_1, V_2; F)$ be a genus two Heegaard splitting of M. We may suppose that the components of $T\cap V_1$ are all disks and that the number of the components of $T\cap V_1$ is minimum among all tori which are isotopic to T and the components of the intersection of each of which with V_1 are all disks. Let $T_2=T\cap V_2$. Then as in [4] we have a hierarchy $(T_2^{(0)}, a_0)$, $\cdots$, $(T_2^{(m)}, a_m)$ of T_2 and a sequence of isotopies of type A which realizes the hierarchy. Let $T^{(1)}$ be the image of T after an isotopy of type A at a_0 and $T^{(k+1)}$ $(k\geq 1)$ be the image of $T^{(k)}$ after an isotopy of type A at a_k.

Then we claim that $T\cap V_1$ consists of at most two components. Assume that $T\cap V_1$ consists of n (≥ 3) disks $D_1, \cdots, D_n$. Then by [5] a_0, a_1 are of type 3 and, hence, $T^{(1)}\cap V_1=A_1\cup D_2\cup\cdots\cup D_n$, $T^{(2)}\cap V_1=A_1\cup A_2\cup D_3\cup\cdots\cup D_n$, where A_i $(i=1, 2)$ is an essential annulus in V_1. If D_1, D_2 are separating in V_1 and A_1, A_2 are parallel in V_1 then there are two annuli A', A'' in ∂V_1 such that $A'\cap(A_1\cup A_2)=A'\cap A_1=\partial A'=\partial A_1$, $A''\cap(A_1\cup A_2)=\partial A''$, $A'\cap A''$ is a component of ∂A_1. We may suppose that $A'\subset M_1$ and $A''\subset M_2$. See Fig. 4. By the minimality of T, A' (A'' resp.) is an essential annulus in M_1 (M_2 resp.). Hence by Lemma 6.1 and Theorem VI. 34 of [4] M_1 and M_2 admit such Seifert fibrations that the component of ∂A_1 is a regular fiber. Hence M admits a Serfert fibration, which is a contradiction. If D_1, D_2 are separating in V_1 and A_1 is not parallel to A_2 in V_1 then $D_1, \cdots, D_n$ are parallel in V_1. See Fig.

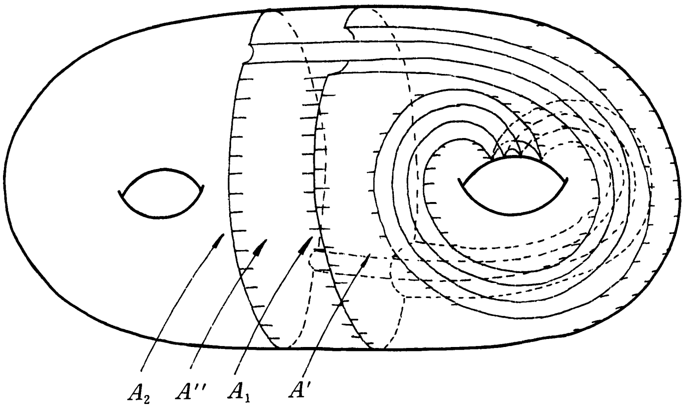

Fig. 4

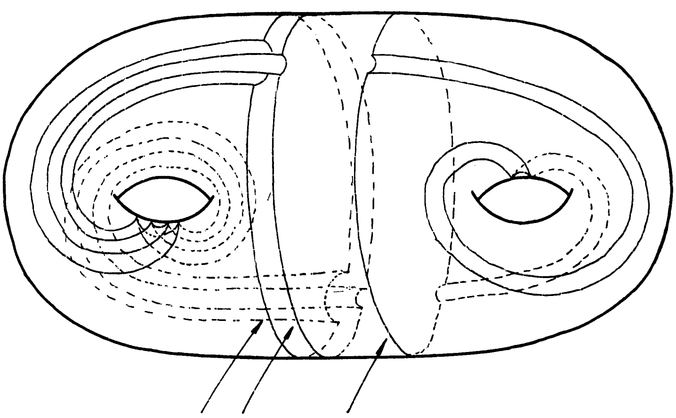

Fig. 5

5. Since each a_i is not a d-arc [5], a_2 is of type 3 and we may suppose that $T^{(3)} \cap V_1 = A_1 \cup A_2 \cup A_3 \cup \cdots \cup D_n$ where A_3 is an essential annulus in V_1. Then A_3 is parallel to A_1 or A_2 (Lemma 3.3) and we have a contradiction as above. If D_1 is separating and D_2 is non-separating in V_1 then there exists annuli A', A'' as above and we have a contradiction. Since A_1 is incompressible, the case of D_1 being non-separating and D_2 being separating cannot occur. If D_1, D_2 are non-separating in V_1 then $D_1, \cdots, D_n$ are mutually parallel in V_1. Since each a_i is not a d-arc, a_2 is of type 3 and we may suppose that $T^{(3)} \cap V_1 = A_1 \cup A_2 \cup A_3 \cup \cdots \cup D_n$, where A_3 is an essential annulus in V_1. Then there exists annuli A', A'' in ∂V_1 such that $A' \cap (A_1 \cup A_2 \cup A_3) = \partial A'$, $A'' \cap (A_1 \cup A_2 \cup A_3) = \partial A''$ and $A' \cap A''$ is a component of $\partial A'$. Then we have a contradiction as above and we establish the claim.

Now, we have two subcases.

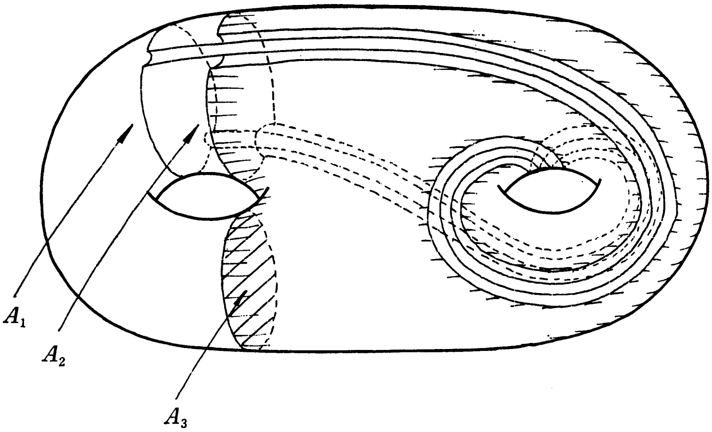

Fig. 6

Case 2.1. $T \cap V_1$ consists of a disk D_1. Since T separates M, D_1 cuts V_1 into two solid tori. Let $A_1 = V_1 \cap T^{(1)}$, $A_2 = V_2 \cap T^{(1)}$. Then by Lemma 3.2 A_i ($i=1, 2$) cuts V_i into a solid torus V_i^1 and a genus two handlebody V_i^2. By attaching V_1^1 and V_2^1 along $cl(\partial V_i^1 - A_i)$ we have M_1 ($\in D(2)$) and by attaching V_1^2 and V_2^2 along $cl(\partial V_i^2 - A_i)$ we have M_2 ($\in L_K$) (Lemma 5.1). Then we have the conclusion (i) of the Theorem.

Case 2.2. $T \cap V_1$ consists of two disks D_1, D_2. In this case $T^{(2)} \cap V_1$ ($T^{(2)} \cap V_2$ resp.) consists of two essential annuli A_1, A_2 (A_1', A_2' resp.).

We claim that if A_1 is parallel to A_2 then A_1, A_2 satisfies the conclusion (i) of Lemma 3.4. First, we show that A_1 is non-separating in V_1. If A_1 is separating in V_1 then there are annuli A', A'' in ∂V_1 such that $A' \cap (A_1 \cup A_2) = A' \cap A_1 = \partial A' = \partial A_1$, $A'' \cap (A_1 \cup A_2) = \partial A''$, $A' \cap A''$ is a component of $\partial A'$. Hence we have a contradiction as in Case 2. Then by Lemma 3.2 A_1 cuts V_1 into a genus two handlebody V'. Let A_1', A_1'' be the copies of A_1 on V'. By the proof of Lemma 3.4 we can show that there is a complete system of meridian disks $\{D_1', D_2'\}$ of V' such that $(D_1' \cup D_2') \cap A_1' = D_1' \cap A_1'$ is an essential arc of A_1', $(D_1' \cup D_2') \cap A_1'' = D_2' \cap A_1''$ and each component of $D_2' \cap A_1''$ is an essential arc of A_1''. If needed by exchanging A_1 and A_2 we may suppose that A_2 is parallel to A_1'' in V'. We will show that D_2' can be taken so that $D_2' \cap A_1''$ is an essential arc of A_1''. If this is done then the claim is established. Since A_1 is parallel to A_2 there is an annulus A''' in ∂V_1 such that $A''' \cap A_i$ ($i=1, 2$) is a component of $\partial A'''$. We may suppose that $A''' \subset M_1$. Then A''' is an essential annulus in M_1 and by Lemma 6.1 and [4] M_1 admits such a Seifert fibration that A''' is a union of regular fibers. If the meridian disk D_2' as above cannot taken then there is an essential annulus A_3 in V' such that $A_3 \cap T^{(2)} = A_3 \cap A_2 = \partial A_3 = \partial A_2$, $A_3 \cap D_1' = \phi$, A_3 is not parallel to A_2 and $A_2 \cup A_3$ bounds a solid torus T' in V'. See Fig. 6. Then $M_1' = M_1 \cup T'$

admits a Seifert fibration and M_1' is not homotopic into M_1. This contradicts the maximal property of the characteristic Seifert pair.

By the above claim and Lemma 3.4 we see that $\{A_1, A_2\}$ ($\{A_1', A_2'\}$ resp.) satisfies one of the conclusions of Lemma 3.4.

Note that $\{A_1, A_2\}$ ($\{A_1', A_2'\}$ resp.) does not satisfy the conclusion (iii) of Lemma 3.4 for $T^{(2)}$ is separating in M.

We claim that either $\{A_1, A_2\}$ or $\{A_1', A_2'\}$ does not satisfy the conclusion (ii) of Lemma 3.4. Assume that both $\{A_1, A_2\}$ and $\{A_1', A_2'\}$ satisfy (ii) of Lemma 3.4. Then $A_1 \cup A_2$ ($A_1' \cup A_2'$ resp.) cuts V_1 (V_2 resp.) into two solid tori and a genus two handlebody, but this contradicts the fact that $T^{(2)}$ is connected.

Then we have two subcases.

Case 2.2.1. Both $\{A_1, A_2\}$ and $\{A_1', A_2'\}$ satisfy the conclusion (i) of Lemma 3.4. In this case $A_1 \cup A_2$ ($A_1' \cup A_2'$ resp.) cuts V_1 (V_2 resp.) into a solid torus $V_1^{(1)}$ ($V_1^{(2)}$ resp.) and a genus two handlebody $V_2^{(1)}$ ($V_2^{(2)}$ resp.) where $A_1 \cup A_2 \subset \partial V_1^{(1)}$, $A_1 \cup A_2 \subset \partial V_2^{(1)}$ ($A_1' \cup A_2' \subset \partial V_1^{(2)}$, $A_1' \cup A_2' \subset \partial V_2^{(2)}$ resp.). By attaching $V_1^{(1)}$ and $V_1^{(2)}$ along $cl(\partial V_1^{(i)} - (A_1 \cup A_2))$ we get M_1 ($\in Mo(n)$, $n=0$, 1 or 2) and by attaching $V_2^{(1)}$ and $V_2^{(2)}$ along $cl(\partial V_2^{(i)} - (A_1 \cup A_2))$ we get M_2 ($\in M_K$) (Lemma 4.1). Then we have the conclusion (ii) of the Theorem.

Case 2.2.2. $\{A_1, A_2\}$ satisfies the conclusion (i) and $\{A_1', A_2'\}$ satisfies the conclusion (ii) of Lemma 3.4. In this case $A_1 \cup A_2$ cuts V_1 into a solid torus $V_1^{(1)}$ and a genus two handlebody $V_2^{(1)}$, $A_1' \cup A_2'$ cuts V_2 into two solid tori $V_1^{(2)}$, $V_2^{(2)}$ and a genus two handlebody $V_3^{(2)}$. By attaching $V_1^{(1)}$ and $V_1^{(2)} \cup V_2^{(2)}$ along $cl(\partial V_1^{(1)} - (A_1 \cup A_2))$ and $cl(\partial V_1^{(2)} - A_1) \cup cl(\partial V_2^{(2)} - A_2)$ we get M_1 ($\in D(n)$, $n=2$ or 3) and by attaching $V_2^{(1)}$ and $V_3^{(2)}$ along $cl(\partial V_2^{(1)} - (A_1 \cup A_2))$ and $cl(\partial V_3^{(2)} - (A_1 \cup A_2))$ we get M_2 ($\in M_K$) (Lemma 4.1).

Then we have the conclusion (iii) of the Theorem.

Case 3. M is decomposed into three components M_1, M_2 and M_3 by the torus decomposition. Let T_1, T_2 be the pair of tori which cuts M into M_1, M_2 and M_3 and let $T = T_1 \cup T_2$. Then we may suppose that $T_1 \subset \partial M_1$, $T_2 \subset \partial M_3$ and $T \subset \partial M_2$. Let (V_1, V_2; F) be a genus two Heegaard splitting of M. Then we may suppose that the components of $T \cap V_1$ are all disks and that the number of the components of $T \cap V_1$ is minimum among all the pair of tori which are isotopic to T and the components of the intersection of each of which with V_1 are all disks. Let $T' = T \cap V_2$. Then we have a hierarchy $(T'^{(0)}, a_0), \cdots, (T'^{(m)}, a_m)$ of T' and a sequence of isotopies of type A which realizes the hierarchy.

We will show that we may suppose that a_0 and a_1 are of type 3 and a_1 joins distinct component of $\partial T'$ that a_0 joins. By the argument of section 3 of [5] both a_0 and a_1 are of type 3. Suppose that a_1 joins the same component of

$\partial T'$ that a_0 joins. We may suppose that a_0, $a_1 \subset T_1$. Then $T_1 \cap V_1$ consists of a disk D_1 for if $T_1 \cap V_1$ has more than one component then $a_0 \cup a_1$ cuts $T_1 \cap V_2$ into a planar surface and hence some a_i ($\subset T_1 \cap V_2$) is a d-arc, which contradicts the minimality of T (see Lemma 3.1 of [5]). Let T_1' be the image of T_1 after an isotopy of type A at a_0. Then $T_1' \cap V_1 = A_1$ ($T_1' \cap V_2 = A_1'$ resp.) is an essential annulus in V_1 (V_2 resp.). Since T_1 is separating in M, A_1' cuts V_2 into a solid torus $V_1^{(2)}$ and a genus two handlebody $V_2^{(2)}$ where there is a complete system of meridian disks $\{D_1, D_2\}$ of $V_2^{(2)}$ such that $D_1 \cap A_1' = \phi$, $D_2 \cap A_1'$ is an essential arc of A_1'. Since $T_2 \cap V_2$ is incompressible in V_2, $(T_2 \cap V_2) \subset V_2^{(2)}$. Then by using $\{D_1, D_2\}$ we can define an isotopy of type A at some essential arc b in $T_2 \cap V_2$. Then by the minimality of T, b is of type 3. Hence by taking b as a_1 we may suppose that a_0, a_1 are of type 3 and a_1 joins distinct component of $\partial T'$ that a_0 joins.

Then by the argument of Case 2 we see that $T \cap V_1$ consists of two disks. Let T^1 be the image of T after an isotopy of type A at a_0, T^2 be the image of T^1 after an isotopy of type A at a_1. Then $T^2 \cap V_1$ ($T^2 \cap V_2$ resp.) consists of two essential annuli A_1, A_2 (A_1', A_2' resp.) where $\partial A_1 = \partial A_1'$ and $\partial A_2 = \partial A_2'$. By the argument of Case 2.2 $\{A_1, A_2\}$ ($\{A_1', A_2'\}$ resp.) satisfies one of the conclusions of Lemma 3.4.

Since T_1 and T_2 are separating in M, each A_i (A_i' resp.) is esparating in V_1 (V_2 resp.). Hence both $\{A_1, A_2\}$ and $\{A_1', A_2'\}$ satisfy the conclusion (ii) of Lemma 3.4. $A_1 \cup A_2$ ($A_1' \cup A_2'$ resp.) cuts V_1 (V_2 resp.) into two solid tori $V_1^{(1)}$, $V_2^{(1)}$ ($V_1^{(2)}$, $V_2^{(2)}$ resp.) and a genus two handlebody $V_3^{(1)}$ ($V_3^{(2)}$ resp.) where $A_i \subset \partial V_i^{(1)}$ ($A_i' \subset \partial V_i^{(2)}$ resp.) ($i=1$, 2). By attaching $V_1^{(1)}$ and $V_1^{(2)}$ ($V_2^{(1)}$ and $V_2^{(2)}$ resp.) along $cl(\partial V_1^{(1)} - A_1)$ and $cl(\partial V_1^{(2)} - A_1')$ ($cl(\partial V_2^{(1)} - A_2)$ and $cl(\partial V_2^{(2)} - A_2')$ resp.) we get M_1 ($\in D(2)$) ($M_2 \in D(2)$ resp.). By attaching $V_3^{(1)}$ and $V_3^{(2)}$ along $cl(\partial V_3^{(1)} - (A_1 \cup A_2))$ and $cl(\partial V_3^{(2)} - (A_1' \cup A_2'))$ we get M_3 ($\in M_L$) (Lemma 4.1).

Then we have the conclusion (iv) of the Theorem.

Note that M does not have such a torus decomposition that M decomposed into more than three components. Assume that M has such a decomposition. Let $T_1, \cdots, T_n$ ($n \geq 3$) be a system of tori which gives the decomposition. We may suppose that each component of $(T_1 \cup \cdots \cup T_n) \cap V_1$ is a disk. Note that $(T_1 \cup \cdots \cup T_n) \cap V_1$ has more than two components and we can derive a contradiction by using the arguments of Case 2.

If M admits a decomposition as in (i)$\sim$(v) of Theorem then by tracing the above arguments conversely we can show that M has a Heegaard splitting of genus two.

This completes the proof of Theorem.

7. Geometric structures of the 3-manifolds with Heegaard splittings of genus two

In this section we show that for each of eight geometric structures in [9] there exists a 3-manifold M which has a Heegaard splitting of genus two and admits the geometric structure.

Lemma 7.1 *If M is a Seifert fibered manifold with orbit manifold a 2-sphere with three exceptional fibers then M has a Heegaard splitting of genus two.*

Proof. Let f be an exceptional fiber in M and Q be the closure of the complement of a regular neighborhood of f. Then Q contains such an essential annulus A that A cuts Q into two solid tori. Let a be an essential arc in A and V_1 be a regular neighborhood of $N \cup a$ in M. Then V_1 is a genus two handlebody. We easily see that $cl(M-V_1)$ is also a genus two handlebody.

This completes the proof of Lemma 7.1.

Let M be a Seifert fibered manifold as in Lemma 7.1. Then by Theorem 12.1 of [**1**] $\pi_1(M)$ has the presentation

$$\langle a, b, c, t;\ [a, t] = [b, t] = [c, t] = 1,\ a^p = t^{p'},\ b^q = t^{q'},\ c^r = t^{r'},\ abc = 1 \rangle$$

where $p>1$, $q>1$, $r>1$. Then for the geometric structure of M the following theorem is given by Kojima [6].

Theorem. *If M is a Seifert fibered manifold as above then M admits a geometric structure according to the table*:

$1/p+1/q+1/r$ \\ $p'/p+q'/q+r'/r$	>1	$=1$	<1
$=0$	ϕ	type 2	type 5
$\neq 0$	type 1	type 7	type 6

where the type of geometries appears in [9].

Then by Lemma 7.1 we see that for each of the geometries that appeared in the above Theorem there is a 3-manifold with a Heegaard splitting of genus two, which has the geometric structure.

The examples of the 3-manifolds with Heegaard splittings of genus two in the hyperbolic geometry (type 3) are obtained by Dehn surgery on the figure eight knot [11].

The closed 3-manifolds in the type 4 geometry are only either $S^2 \times S^1$ or $P^3 \# P^3$, each of which has a Heegaard splitting of genus two.

Any torus bundle over S^1 with a hyperbolic monodromy has type 8 geometric structure [9]. Then the torus bundle whose monodromy is $\begin{pmatrix} 0 & 1 \\ -1 & m \end{pmatrix}$

($m \geq 3$) has type 8 geometric structure and by [3] it has a Heegaard splitting of genus two.

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