

## A NOTE ON THE NUMBER OF IRREDUCIBLE CHARACTERS IN A $p$ -BLOCK OF A FINITE GROUP

Dedicated to Professor Hiroshi Nagao on his 60th birthday

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### Introduction

Let  $G$  be a finite group and  $n$  a divisor of  $|G|$ , the order of  $G$ . We set  $L_n(G) = \{x \in G \mid x^n = 1\}$ . G. Frobenius gave the following conjecture.

(F): If  $|L_n(G)| = n$ , then  $L_n(G)$  is a normal subgroup of  $G$ .

Concerning this problem the following results are known.

- (1) In order to prove (F) we may assume that  $n$  and  $|G|/n$  are relatively prime. (See [7], [12])
- (2) Let  $G$  be a minimal counterexample to (F). Then  $G$  is a simple group. (See [1], [12], [17])

Since the classification of finite simple groups has been completed, it may be possible to verify (F) by checking simple groups, by using (2). Indeed such verifications have been carried out for certain classes of simple groups by several authors ([1], [8], [9], [14], [16]).

However, it may also be desirable to investigate (F) from a more general standpoint. The purpose of this paper is to provide an example of such investigations.

For a group  $G$  and a prime  $p$ , we denote by  $G_{p'}$ , the set of  $p'$ -elements, or  $p$ -regular elements, of  $G$  and by  $|G|_p$  the highest power of  $p$  dividing  $|G|$ . We set  $|G|_{p'} = |G| / |G|_p$ . We are interested in the following special case of (F).

(F <sub>$p$</sub> ): If  $|G_{p'}| = |G|_{p'}$ , then  $G$  is  $p$ -nilpotent.

In this case Brauer and Nesbitt [3] found the following fact. Let  $c_{11}$  be the Cartan invariant corresponding to the principal representation (mod  $p$ ) of  $G$ . Then if  $c_{11}$  is not larger than  $|G|_p$ , then (F <sub>$p$</sub> ) is true for  $G$ . So it may be interesting to know whether  $c_{11}$  is not larger than  $|G|_p$  for a finite group. But Landrock [11] showed that  $c_{11}$  can be larger than  $|G|_p$  for a certain group (see also [6], p. 168). In this paper we consider another generalization of (F <sub>$p$</sub> ). Let

$p$  be a prime. Let  $B_0(G)$  denote the principal  $p$ -block of a group  $G$  and let  $l(B_0(G))$  be the number of irreducible modular characters in  $B_0(G)$ . We set  $m(G) = |L_n(G)|/n$ , where  $n$  denotes  $|G|_p'$ . Then we conjecture the following.

$$(ML): \quad m(G) \geq l(B_0(G)).$$

In fact  $(F_p)$  is a simple consequence of (ML) (see § 1).

Furthermore (ML) is related to a conjecture of R. Brauer (see Remark below). Let  $B$  denote a  $p$ -block of a group with a defect group  $D$ . We denote by  $k(B)$  the number of ordinary irreducible characters in  $B$ . Then Brauer [2] conjectured

$$(K): \quad k(B) \leq |D|.$$

We shall see in § 1 that if (ML) is true, then (K) is true for the principal  $p$ -block  $B$  of a group. Therefore by Nagao's reduction [13], it would follow from (ML) that (K) is true for  $p$ -solvable groups.

In this paper we shall give a partial answer to the conjecture (ML):

**Theorem 3.2.** *Let  $G$  be a  $p$ -solvable group. If  $p$  is sufficiently large compared with the sectional  $p$ -rank of  $G$ , then (ML) is true for  $G$ .*

As a consequence we have the following.

**Theorem 3.3.** *Let  $G$  be a  $p$ -solvable group and let  $B$  be a  $p$ -block of  $G$  with a defect group  $D$ . Let  $n$  be the sectional rank of  $D$ . Then, if  $p$  is sufficiently large compared with  $n$ , we have  $k(B) \leq |D|$ .*

This theorem sharpens results of Gow [10].

Besides the ones mentioned above we use the following notation. We denote by  $cl(G)$  the number of conjugacy classes in  $G$  and by  $p$ -reg  $cl(G)$  the number of conjugacy classes in  $G$  consisting of  $p$ -regular elements. For a prime  $p$ , we set  $m(G) = |G_p'|/|G|_p'$ . The number  $m(G)$  is fundamental in this paper.

REMARK. M. Fujii investigated the problem (ML) in connection with the conjecture (K) (see [18], § 6, Pb. 2). The author owes this information to the referee.

The author would like to express his hearty thanks to the referee for many valuable suggestions and, in particular, for correcting an error in the original proof of Theorem 3.1.

## 1. Motivations for the conjecture

**Proposition 1.1.** *If (ML) is true, then  $(F_p)$  is true.*

Proof. Let  $G, p$  be the same as in  $(F_p)$ . Therefore  $m(G)=1$ . By  $(ML)$ ,  $l(B_0(G))=1$ , which implies that  $G$  is  $p$ -nilpotent ([6], Lemma 4.12 (iv)).

**Proposition 1.2.** *If  $(ML)$  is true, then  $(K)$  is true in case  $B$  is the principal  $p$ -block of  $G$ .*

Proof. Let  $\{x_i\}$  be a complete set of representatives of the conjugacy classes in  $G$  consisting of  $p$ -elements. Then

$$(1.2.1) \quad k(B_0(G)) = \sum_i l(B_0(C_G(x_i))).$$

On the other hand, the number of elements in the  $p$ -section containing  $x_i$  is  $|G|/|C_G(x_i)| \cdot |C_G(x_i)_p|$ , for each  $i$ . Therefore by counting the number of elements in the  $p$ -sections of  $G$ , it follows that

$$(1.2.2) \quad |G|_p = \sum_i \frac{|G|_p}{|C_G(x_i)|_p} m(C_G(x_i)).$$

Comparing these equations, we obtain the result.

For  $p$ -solvable groups, Nagao [13] reduced  $(K)$  to the case where  $B$  in  $(K)$  is the principal  $p$ -block of  $G$ . Therefore (1.2) yields the following

**Proposition 1.3.** *If  $(ML)$  is true (for  $p$ -solvable groups), then  $(K)$  is true for any  $p$ -block of  $p$ -solvable groups.*

Let  $G$  be a group with an abelian  $p$ -Sylow subgroup  $S$  and let  $\{x_i\}$  be as in the proof of (1.2.). We may assume that  $\{x_i\} \subset S$ . By the fusion lemma of Burnside,  $\{x_i\}$  plays the same role for  $N_G(S)$  as for  $G$ . Since  $S$  is abelian, starting from (1.2.2), an induction argument implies the following.

**Proposition 1.4.** *Let  $G$  be a group with an abelian  $p$ -Sylow subgroup  $S$ . Then  $m(G)=m(N_G(S))$ .*

REMARK. Proposition (1.4) implies that  $(F_p)$  is true if  $G$  has an abelian  $p$ -Sylow subgroup  $S$ . In fact we obtain, by (1.4),  $m(N_G(S))=m(G)=1$ . Since  $N_G(S)$  has a  $p$ -complement, this implies that  $N_G(S)$  is  $p$ -nilpotent. Hence  $G$  is  $p$ -nilpotent by the transfer theorem of Burnside. This generalizes [5], Theorem 7.

## 2. Reduction

The following lemma was shown in [12]. For the sake of completeness, we quote it here with a proof.

**Lemma 2.1.** *Let  $N$  be a normal subgroup of a group  $G$ . Then,  $m(G) \cong m(G/N)$ .*

Proof. We define a class function  $\theta$  on  $G$  as follows:

$$\theta(x) = \begin{cases} |G|_p & \text{if } x \text{ is } p\text{-regular,} \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\theta$  is a generalized character of  $G$  ([6], (IV. 1.3)). Therefore if we define a class function  $\bar{\theta}$  on  $G/N$  by setting

$$\bar{\theta}(xN) = \frac{1}{|N|} \sum_{y \in xN} \theta(y),$$

for each  $xN \in G/N$ , then  $\bar{\theta}$  is a generalized character of  $G/N$  ([15], p. 106). Thus  $\bar{\theta}(xN) = |N|^{-1} |G|_p |xN \cap G_{p'}|$  is an algebraic integer, hence a rational integer. Therefore  $|xN \cap G_{p'}|$  is divisible by  $|N|_{p'}$ . On the other hand, it is easy to verify that  $xN$  is a  $p$ -regular element in  $G/N$  if and only if the coset  $xN$  contains a  $p$ -regular element in  $G$ . Thus we obtain  $|G_{p'}| \geq |(G/N)_{p'}| |N|_{p'}$ , as was to be shown.

Let  $G$  be a  $p$ -solvable group. It is known that if  $O_{p'}(G)$  equals 1, then  $G$  has only one  $p$ -block, so that  $l(B_0(G))$  equals  $p\text{-reg } cl(G)$  ([6], (X. 1.5)). This fact will be used without explicit reference. We need a definition ([10]). Let  $p$  be a prime. A group  $G$  is said to be  $p$ -primitive if  $G = LV$ , where  $V$  is a normal elementary abelian  $p$ -subgroup of  $G$ , and where  $L$  is a  $p'$ -group which acts (by conjugation) faithfully on  $V$ .

**Proposition 2.2.** *Let  $S$  be a class of  $p$ -solvable groups that is subgroup and factor group closed. In order to show that (ML) is true for any group  $G$  in  $S$ , we may assume that  $G$  is  $p$ -primitive.*

Proof. We set  $\bar{G} = G/O_{p'}(G)$ . By (2.1),  $m(G) \geq m(\bar{G})$ . It is known that  $l(B_0(\bar{G})) = l(B_0(G))$ . Thus we may assume that  $O_{p'}(G) = 1$ . Let  $L$  be a  $p$ -complement of  $G$  and set  $H = LO_p(G)$ . Since  $|H|$  is divisible by  $|L| = |G|_{p'}$ , we see that  $m(G) \geq m(H)$ . By the Hall-Higman lemma  $O_{p'}(H) = 1$ , so that  $l(B_0(H)) = p\text{-reg } cl(H) \geq p\text{-reg } cl(G) = l(B_0(G))$ . (Here the inequality follows since any  $p$ -regular element in  $G$  has a conjugate in  $L$ .) Thus we may assume  $G = LO_p(G)$ . Let  $F$  be the Frattini subgroup of  $O_p(G)$  and set  $\bar{G} = G/F$ . Then  $O_{p'}(\bar{G}) = 1$ , so that  $l(B_0(\bar{G})) = p\text{-reg } cl(\bar{G}) = cl(L) = p\text{-reg } cl(G) = l(B_0(G))$ . By (2.1),  $m(G) \geq m(\bar{G})$ . Hence we may assume  $F = 1$ , so that  $G$  is  $p$ -primitive.

### 3. Sufficiently large primes

In this section we shall show the two theorems mentioned in Introduction. For this purpose the following Theorem (3.1) is essential. In the proof we use Jordan's theorem which states that there exists a function  $J$  defined on positive integers such that if  $L$  is a finite complex linear group of degree  $n$ ,

then  $L$  contains a normal abelian subgroup  $A$  with its index  $[L: A] < J(n)$ . The author owes to Gow [10] the idea of using Jordan's theorem in such a situation.

Let  $p$  be a prime and let  $W$  be an elementary abelian  $p$ -group. We will always regard such a  $W$  as a vector space over  $GF(p)$ , the field of  $p$  elements. For a linear transformation  $t$  on  $W$ , we denote the rank of  $t$  by  $rk_W(t)$ . If a group  $K$  acts on  $W$ , i.e.  $W$  is a  $GF(p)K$ -module, then let  $\text{Ker } W$  denote  $\{x \in K \mid xw = w \text{ for all } w \in W\}$ .

Let  $G = LV$  be a  $p$ -primitive group, where  $L$  and  $V$  have the same meaning for  $G$  as in § 2. It is convenient to rewrite  $m(G)$ :

$$m(G) = \frac{1}{|L|} \sum_{x \in L} p^{rk_V(x-1)}.$$

This follows from the following three facts: (i) every  $p$ -regular element in  $G$  has a conjugate in  $L$ ; (ii) for each  $x \in L$ ,  $C_G(x) = C_L(x)C_V(x)$ ; (iii) any two elements in  $L$  are conjugate in  $G$  if and only if they are conjugate in  $L$ .

We note that for such a group  $l(B_0(G)) = cl(L)$ .

**Theorem 3.1.** *Let  $n$  be a positive integer. Then there exists a constant  $b_n$  depending only on  $n$  such that the following statement is true:*

*Let  $G = LV$  be a  $p$ -primitive group. Here  $L$  and  $V$  have the same meaning as above. Let  $\dim_{GF(p)} V = n$ . Then, if  $p > b_n$ , we have that  $m(G) \geq cl(L)$ .*

*Proof.* If  $n = 1$ , it is easy to verify that we always have the inequality. Hence we may set  $b_1 = 1$ . We will complete the proof by induction on  $n$ . Since  $L$  is a  $p'$ -group, we may apply Jordan's theorem mentioned above to conclude that  $L$  contains a normal abelian subgroup  $A$  with  $[L: A] < J(n)$ . We fix a maximal normal abelian subgroup  $A$  with this property and set  $\bar{l} = |L/A|$ ,  $a = |A|$ . We divide the proof into several steps. For simplicity let us call a  $p$ -primitive group  $G = LV$  of type  $(ML)$  if it satisfies the inequality:  $m(G) \geq cl(L)$ .

Step 1. There exists a constant  $c_n$  depending only on  $n$  such that if  $p > c_n$ , then  $G$  is of type  $(ML)$  or  $m(G) > \frac{2}{3} \frac{p^n}{\bar{l}}$ .

*Proof.* Suppose  $m(G) \leq \frac{2}{3} \frac{p^n}{\bar{l}}$ . We distinguish two cases.

Case a.  $V$  is an irreducible  $GF(p)L$ -module.

We denote by  $V_A$  the restriction of  $V$  to  $A$ . Let  $V_A = W_1 \oplus W_2 \oplus \dots \oplus W_t$ , where  $W_i$  is an irreducible  $GF(p)A$ -module for each  $i$ . By Clifford's theorem the orders of  $\text{Ker } W_i$  ( $i = 1, \dots, t$ ) are the same, and we denote it by  $a_1$ . Since  $A/\text{Ker } W_i$  acts fixed point freely on  $W_i$ , setting  $A' = \text{Ker } W_1 \cup \dots \cup \text{Ker } W_t$ , we have

$$\begin{aligned}
 m(G) &> \frac{1}{|L|} \sum_{x \in A-A'} p^{rk_V(x-1)} \\
 &= \frac{p^n}{|L|} (|A| - |A'|) \geq \frac{p^n}{|L|} (a - ta_1) = \frac{p^n}{l} \left( 1 - t \frac{a_1}{a} \right).
 \end{aligned}$$

This, together with the assumption on  $m(G)$ , implies that  $a/a_1 < 3t$ . On the other hand,  $A$  is isomorphic to a subgroup of  $A/\text{Ker } W_1 \times \cdots \times A/\text{Ker } W_t$ , so that  $a \leq (a/a_1)^t$ . Therefore  $|L| < aJ(n) < (3t)^t J(n) \leq (3n)^n J(n)$ . Hence, if  $p > (3n)^n J(n)$ , then  $m(G) \geq |L|^{-1} \{1 + p(|L| - 1)\} \geq |L|^{-1} \{1 + (|L| + 1)(|L| - 1)\} = |L|$ , which implies that  $G$  is of type  $(ML)$ .

Case b.  $V$  is a reducible  $\text{GF}(p)L$ -module.

Let  $V_A = W_1 \oplus \cdots \oplus W_t$ , where  $W_i$  is an irreducible  $\text{GF}(p)A$ -module. Changing the notation, if necessary, we may assume  $|\text{Ker } W_1| = \max\{|\text{Ker } W_i| \mid i=1, \dots, t\}$ . Let us decompose  $V$  so that  $V = V_1 \oplus V_2$ , where  $V_1$  is an irreducible  $\text{GF}(p)L$ -module such that  $(V_1)_A$  contains  $W_1$  as a submodule and where  $V_2$  is a non-zero  $\text{GF}(p)L$ -module. Set  $a_1 = |\text{Ker } W_1|$ . Then as in Case a, we obtain  $a/a_1 < 3t$ . Let  $L_i = \text{Ker } V_i (i=1, 2)$ . Let  $s$  be the number of the irreducible  $\text{GF}(p)A$ -modules appearing in a decomposition of  $(V_1)_A$  into direct sum of irreducible ones. The same argument as in Case a implies  $|A/A \cap L_1| \leq (a/a_1)^s < (3t)^s$ . Noting  $|L_2| = |L_2 L_1 / L_1| \leq |L/L_1| = |L/AL_1| |AL_1/A| \leq |L/A| |A/A \cap L_1|$ , we obtain

(1)  $|L_2| < J(n)(3n)^n$ .

Let  $\{t_i\}$  be a set of coset representatives of  $L_2$  in  $L$ . It is easy to verify that

$$m(G) = \frac{1}{|L|} \sum_i p^{rk_{V_2}(t_i-1)} \sum_{x \in L_2} p^{rk_{V_1}(x-t_i^{-1})}.$$

For a fixed  $t_i$  there exists at most one  $x \in L_2$  such that  $p^{rk_{V_1}(x-t_i^{-1})} = 1$ , since  $L_1 \cap L_2 = 1$ . Therefore

$$m(G) \geq \frac{1}{|L|} \sum_i p^{rk_{V_2}(t_i-1)} \{1 + p(|L_2| - 1)\}.$$

Hence,

(2) if  $p > |L_2|$ , then  $m(G) \geq m(L/L_2 \cdot V_2) |L_2|$ .

Set  $n_2 = \dim_{\text{GF}(p)} V_2$ . By induction hypothesis there exists a constant  $b_{n_2}$  such that

(3) if  $p > b_{n_2}$ , then  $m(L/L_2 \cdot V_2) \geq cl(L/L_2)$ .

On the other hand, we always have  $cl(L/L_2) |L_2| \geq cl(L)$  (see [13], Lemma 1). This, together with (1), (2), and (3), implies that if  $p > \max\{J(n)(3n)^n, b_{n_2}\}$ , then  $m(G) \geq cl(L)$ .

Hence it suffices to set  $c_n = \max\{J(n)(3n)^n, b_1, \dots, b_{n-1}\}$ .

Step 2. There exists a constant  $d_n$  depending only on  $n$  such that if  $p > d_n$ , then  $G$  is of type  $(ML)$  or  $a > p^{n-1/3}$ .

Proof. Suppose  $G$  is not of type  $(ML)$ . Then by Step 1,  $cl(L) > m(G) > \frac{2}{3} \frac{p^n}{l}$ , if  $p > c_n$ . Since  $cl(L) \leq |L| < J(n)a$ , we obtain  $J(n)^2 a > \frac{2}{3} p^n$ . Hence, if  $p > \left\{ \frac{3}{2} J(n)^2 \right\}^3$ , then  $a > p^{n-1/3}$ . Thus it suffices to set  $d_n = \max \left\{ c_n, \frac{27}{8} J(n)^6 \right\}$ .

Let us decompose  $V_A$  as follows:

$$V_A = W_{11} \oplus W_{12} \oplus \dots \oplus W_{1t_1} \oplus \dots \oplus W_{m1} \oplus \dots \oplus W_{mt_m},$$

where  $W_{ij}$  is an irreducible  $GF(p)A$ -module of dimension  $d_i$ , for each  $i, j$ , and where  $d_i \neq d_j$ , if  $i \neq j$ .

It is easy to see that we may identify  $W_{ij}$  with  $GF(p^{d_i})$  in such a way that every element in  $A$  acts on  $W_{ij} = GF(p^{d_i})$  as multiplication by an element in  $GF(p^{d_i})^\times$ , where  $GF(p^{d_i})^\times$  denotes the multiplicative group of  $GF(p^{d_i})$ . Therefore, corresponding to the above decomposition of  $V_A$ ,

$$A \subseteq \left\{ \left( \begin{array}{cccc} \lambda_{11} & & & \\ & \dots & & \\ & & \lambda_{1t_1} & \\ & & & \dots \\ & & & & \lambda_{m1} & \\ & & & & & \dots \\ & & & & & & \lambda_{mt_m} \end{array} \right) \mid \left. \begin{array}{l} \lambda_{ij} \in GF(p^{d_i})^\times \\ i = 1, \dots, m \\ j = 1, \dots, t_i \end{array} \right\}$$

We will denote by  $T$  the group in the right hand side. For every  $x \in T$  and every  $i, j$ , we denote by  $\lambda_{ij}(x)$  “ $W_{ij}$ -component” of  $x$ ; that is,  $x$  acts on  $W_{ij} = GF(p^{d_i})$  as multiplication by  $\lambda_{ij}(x)$ . For every subgroup  $B$  of  $T$ , we set  $B_{ij} = \{x \in B \mid \lambda_{kl}(x) = 1, \text{ if } (k, l) \neq (i, j)\}$ , for each  $i, j$ .

It is easy to verify the following (4) and (5). Here  $GL(V)$  denotes the general linear group of  $V$ .  $N_{GL(V)}$  (resp.  $C_{GL(V)}$ ) denotes the normalizer (resp. centralizer) in  $GL(V)$ .

$$(4) \quad N_{GL(V)}(T) = \left\{ \left( \begin{array}{cccc} s_1 & & & \\ & \dots & & \\ & & s_m & \end{array} \right) \mid \left. \begin{array}{l} s_i \in S_{t_i} \\ i = 1, \dots, m \end{array} \right\} \cdot \left\{ \left( \begin{array}{cccc} \sigma_{11} & & & \\ & \dots & & \\ & & \sigma_{1t_1} & \\ & & & \dots \\ & & & & \sigma_{m1} & \\ & & & & & \dots \\ & & & & & & \sigma_{mt_m} \end{array} \right) \mid \left. \begin{array}{l} \sigma_{ij} \in Gal(p^{d_i}) \\ i = 1, \dots, m \\ j = 1, \dots, t_i \end{array} \right\} \cdot T,$$

where  $S_{t_i}$  denotes the group of the permutation matrices of degree  $d_i t_i$  obtained

by replacing entries 0, 1 of the permutation matrices of degree  $t_i$  by zero matrix, identity matrix of degree  $d_i$  respectively; and where  $\text{Gal}(p^{d_i})$  denotes the group of linear transformations on  $W_{ij} = \text{GF}(p^{d_i})$  induced by the Galois group of  $\text{GF}(p^{d_i})$  over  $\text{GF}(p)$ .

(5)  $C_{GL(V)}(T) = T$ .

In addition we have the following.

(6) Let  $B$  be a subgroup of  $T$  such that  $|B| > p^{n-1/2}$ . Then  $N_{GL(V)}(B) \subseteq N_{GL(V)}(T)$ , and  $C_{GL(V)}(B) = T$ .

Proof of (6). Let  $B'$  (resp.  $T'$ ) be the subalgebra, of the full matrix algebra (over  $\text{GF}(p)$ ) of degree  $n$ , generated by  $B$  (resp.  $T$ ). In the definition of  $T$ , if we relax the condition that  $\lambda_{ij} \in \text{GF}(p^{d_i})^\times$  for each  $i, j$  to the one that  $\lambda_{ij} \in \text{GF}(p^{d_i})$  for each  $i, j$ , then we obtain  $T'$ . Since  $|B'| > p^{n-1/2}$ , we see  $n \leq \dim_{\text{GF}(p)} B' \leq \dim_{\text{GF}(p)} T' = n$ . Hence  $B' = T'$ . Therefore  $N_{GL(V)}(B)$  normalizes  $T'$ , so that it normalizes the set of invertible elements of  $T'$ , that is,  $T$ . Similar argument applies to the centralizer, and by (5) we obtain the second assertion.

Step 3. Assume that  $p > d_n$  and that  $G$  is not of type  $(ML)$ . Then the following holds.

- (a)  $L \subseteq N_{GL(V)}(T)$ ,
- (b)  $A = L \cap T$ , and
- (c)  $cl(L) < \frac{a}{\bar{l}} + J(n)p^{n-1/2}$ .

Proof. By Step 2, we can apply (6) to  $B = A$  to obtain (a). By (a),  $L \cap T$  is a normal abelian subgroup of  $L$  containing  $A$ , so that the maximal nature of  $A$  implies (b). It is easily seen that

$$cl(L) = \frac{1}{|L|} \sum_{x \in L} |C_L(x)| = \frac{1}{|L|} \sum_{y \in L} |C_A(y)| + \frac{1}{|L|} \sum_{x \in L-A} |C_L(x)|.$$

Therefore

$$cl(L) = \frac{1}{|L|} \sum_{y \in A} |C_A(y)| + \frac{1}{|L|} \sum_{y \in L-A} |C_A(y)| + \frac{1}{|L|} \sum_{x \in L-A} |C_L(x)|.$$

If  $|C_A(y)| > p^{n-1/2}$  for some  $y \in L - A$ , then applying (6) to  $B = C_A(y)$ , we see  $y \in T$ , contradicting (b). Hence

$$\text{the second sum} \leq (1 - \bar{l}^{-1})p^{n-1/2}.$$

Similarly

$$\text{the third sum} \leq \frac{1}{|L|} \sum_{x \in L-A} |C_A(x)| \frac{|L|}{|A|} < (\bar{l} - 1)p^{n-1/2}.$$

Noting  $\bar{l} < J(n)$ , we obtain the result.



Step 4. There exists a constant  $e_n$  depending only on  $n$  such that if  $p > e_n$ , then  $G$  is of type  $(ML)$  or  $A=T$ .

Proof. Suppose that  $p > d_n$  and that  $G$  is not of type  $(ML)$ . Then Step 1 and Step 3 (c) imply that

$$J(n)p^{n-1/2} + \frac{a}{\bar{l}} > \frac{2}{3} \frac{p^n}{\bar{l}}.$$

For each  $i, j$  ( $i=1, \dots, m; j=1, \dots, t_i$ ), let  $A_{ij}$  be the subgroup of  $A \subseteq T$ . Set  $|A_{ij}| = (p^{d_i} - 1)/u_{ij}$ ,  $u_{ij}$  being an integer. We see

$$a = |A_{ij}| |A/A_{ij}| < \frac{p^{d_i} - 1}{u_{ij}} p^{n-d_i} < \frac{p^n}{u_{ij}}.$$

It will follow from the above inequalities that if  $u_{ij} \geq 2$  for some  $i, j$  then  $p < 36J(n)^4$ . Hence, if  $p > 36J(n)^4$ , then  $u_{ij} = 1$  for all  $i, j$ ; that is,  $A=T$ . Thus it suffices to set  $e_n = \max\{d_n, 36J(n)^4\}$ .

Now we can complete the proof of the theorem.

Step 5. There exists a constant  $b_n$  depending only on  $n$  such that if  $p > b_n$ , then  $G$  is of type  $(ML)$ .

Proof. Assume  $p > e_n$ . By Step 4, we may assume  $A=T$ . Let  $\{y_r | r=0, 1, \dots, \bar{l}-1\}$  be a set of coset representatives of  $T$  in  $L$  such that  $y_0 \in T$ . We can rewrite  $m(G)$ :

$$(7) \quad m(G) = \frac{1}{\bar{l}} m(TV) + \frac{1}{|L|} \sum_{r=1}^{\bar{l}-1} \sum_{x \in T} p^{rk_V(y_r - x)}.$$

We note that  $m$  is multiplicative: if  $X=Y \times Z$  is a direct product then  $m(X) = m(Y)m(Z)$ . This implies that  $m(TV) = \prod_{i,j} m(T_{ij}W_{ij})$ , since  $TV$  is the direct product of  $T_{ij}W_{ij}$  ( $i=1, \dots, m; j=1, \dots, t_i$ ). It is easily seen that  $m(T_{ij}W_{ij}) = |T_{ij}|$  for each  $i, j$ . Hence  $m(TV) = |T|$ . Therefore if  $L=T$ , then  $m(G) = |T| = cl(L)$ , implying that  $G$  is of type  $(ML)$ . Assume  $L \neq T$ . Fix  $y_r (r > 0)$ . We will estimate

$$\sum_{x \in T} p^{rk_V(y_r - x)}$$

from below. Since this quantity depends only on the coset  $y_r T$ , we may assume that  $y_r$  is contained in the product of the first two groups in the right hand side of (4). For simplicity, set  $y = y_r$ . For each  $i, y$  acts on the set  $\{W_{ij} | j=1, \dots, t_i\}$ . Let  $\{W_{ij} | j=1, \dots, t_i\} = O_{i1} \cup \dots \cup O_{iu_i}$  be the orbit decomposition. Corresponding to  $V = \sum W_{ij}$  (direct sum), let  $v_{ij}$  denote the  $W_{ij}$ -component of  $v \in V$ . We call an element  $x \in T$  regular if  $rk_V(y - x) = n$ ; non-regular,

otherwise. For each  $i$  ( $1 \leq i \leq m$ ), let  $N_i$  be the norm map from  $\text{GF}(p^{d_i})$  to  $\text{GF}(p)$ . For every  $O_{ik}$ , we set  $N_{ik}(x) = \prod N_i(\lambda_{ij}(x))$  for  $x \in T$ , where the product is over  $j$  such that  $W_{ij} \in O_{ik}$ .

Suppose that  $x \in T$  is non-regular. Then there exists a  $v \in V - \{0\}$  such that  $yv = xv$  and there exist  $i$  and  $k$  such that for every  $W_{ij}$  in  $O_{ik}$ ,  $v_{ij} \neq 0$ . We claim that the following  $(ik)$  holds.

$$(ik): N_{ik}(x) = 1.$$

Let us choose the notation so that  $O_{ik} = \{W_{i1}, \dots, W_{if}\}$ ,  $f = |O_{ik}|$ . Let  $v'_{ij}$  ( $1 \leq j \leq f$ ) be the  $W_{ij}$ -component of  $yv$ . It follows that for a suitable permutation  $\pi$  of  $\{1, \dots, f\}$ ,  $v'_{ij}$  and  $v_{ij\pi}$ , both in  $\text{GF}(p^{d_i})$ , are algebraically conjugate over  $\text{GF}(p)$  for every  $j$ . Therefore

$$\prod_{j=1}^f N_i(v'_{ij}) = \prod_{j=1}^f N_i(v_{ij}).$$

On the other hand, we have  $v'_{ij} = \lambda_{ij}(x)v_{ij}$  ( $1 \leq j \leq f$ ). Therefore

$$\prod_{j=1}^f N_i(v'_{ij}) = \prod_{j=1}^f N_i(\lambda_{ij}(x)) \prod_{j=1}^f N_i(v_{ij}).$$

The above equations imply the claim, since  $v_{ij} \neq 0$  for every  $j$  ( $1 \leq j \leq f$ ). Note that the number of elements  $(\mu_1, \dots, \mu_f) \in \text{GF}(p^{d_i})^\times \times \dots \times \text{GF}(p^{d_i})^\times$  ( $f$  times) such that  $\prod_{j=1}^f N_i(\mu_j) = 1$  equals  $(p-1)^{-1}(p^{d_i}-1)^f$ . Hence the number of elements  $x \in T$  which do not satisfy any of the conditions  $(ik)$  ( $1 \leq i \leq m$ ,  $1 \leq k \leq u_i$ ) equals

$$\prod_{i=1}^m \prod_{k=1}^{u_i} \{(p^{d_i}-1)^{|O_{ik}|} - (p-1)^{-1}(p^{d_i}-1)^{|O_{ik}|}\} = |T| \left(1 - \frac{1}{p-1}\right)^n.$$

Since these elements are regular, we see that

$$\sum_{x \in T} p^{rk_V(y-x)} > |T| \left(1 - \frac{1}{p-1}\right)^n p^n.$$

Thus it follows from (7) that

$$m(G) > \frac{a}{l} + \frac{p^n}{l} \frac{1}{2^n}, \quad \text{if } p > 3.$$

This, together with Step 3 (c), implies that it suffices to set

$$b_n = \max \{e_n, 2^{2n} J(n)^4, 3\}.$$

The proof of Theorem (3.1) is completed.

REMARK 1. We may assume  $\{b_n\}$  is a non-decreasing sequence. In fact,

the sequence  $\{b_n\}$  constructed in the above proof satisfies this condition. In the rest of the paper we always assume this natural condition on  $\{b_n\}$ .

**Theorem 3.2.** *Let  $n$  be a positive integer and let  $b_n$  be as in Theorem (3.1) (see Remark 1). Then the following statement is true:*

*Let  $p$  be a prime larger than  $b_n$ . Let  $G$  be a  $p$ -solvable group with sectional  $p$ -rank  $\leq n$ . Then we have that  $m(G) \geq l(B_0(G))$ .*

*Proof.* Fix a prime  $p$  such that  $p > b_n$ . Let  $S$  be the class of  $p$ -solvable groups with sectional  $p$ -rank  $\leq n$ . Then  $S$  is subgroup and factor group closed, so that Proposition (2.2) implies that it suffices to consider  $p$ -primitive groups in  $S$ . Let  $G$  be a  $p$ -primitive group in  $S$  with  $p$ -rank  $n'$ . Then  $b_n \geq b_{n'}$ , since  $n \geq n'$ . Hence the inequality for  $G$  follows from Theorem (3.1).

**Theorem 3.3.** *Let  $n$  be a positive integer and let  $b_n$  be as in Theorem (3.1) (see Remark 1). Then the following statement is true:*

*Let  $p$  be a prime larger than  $b_n$ . Let  $G$  be a  $p$ -solvable group and let  $B$  be a  $p$ -block of  $G$  with a defect group  $D$ . Assume that the sectional rank of  $D$  equals  $n$ . Then we have that  $k(B) \leq |D|$ .*

*Proof.* Let  $p$ ,  $G$ ,  $B$ , and  $D$  be as in the above statement. We first consider the case where  $B$  is the principal  $p$ -block of  $G$ . Hence  $D$  is a  $p$ -Sylow subgroup of  $G$ . Let us recall the proof of Proposition (1.2) and use the notation there. By Theorem (3.2),  $(ML)$  is true for  $C_G(x_i)$ , for every  $i$ . Hence the proof of Proposition (1.2) implies the assertion.

For general blocks, we check [13], § 3. Then it will follow that it suffices to prove the following statement ([13], p. 37):

Under the assumption

$G$  has only trivial normal  $p'$ -subgroup and  $p$  is larger than  $b_n$ ,

it holds that

the number of conjugacy classes in  $G$  is not larger than  $|G|_p$ .

But the group  $G$  in this statement has only one  $p$ -block, so that the statement follows from the first paragraph.

**REMARK 2.** Notation is as in (3.3). Let  $d$  be the defect of  $B$ . Clearly  $d \geq n$ , so that  $b_d \geq b_n$ . Hence the statement in (3.3) remains true if we replace  $b_n$  by  $b_d$ . Therefore Theorem (3.3) sharpens Gow's observation, stated in [10], Introduction, that his bound on  $k(B)$  is in keeping with Brauer's conjecture as  $p$  becomes large.

**REMARK 3.** As shown in the proof of (3.1), we may set  $b_1=1$ . Using the classification of finite complex linear groups of degree 2 ([4], Theorem 26.1), we can verify, by a straightforward calculation which is similar in spirit to the proof of (3.1), that we may set  $b_2=1$ . Therefore (3.3) implies that  $(K)$

is true for a  $p$ -solvable group, if a defect group of the  $p$ -block is of sectional rank at most 2. This gives another proof of [13], Corollary.

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Added in proof. For a related result, see “*A note on the number of irreducible characters in a  $p$ -block with normal defect group*”, Proc. Japan Acad. **59A** (1983), 488–489.