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ON ALGEBRAS OF SECOND LOCAL TYPE, II

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This is a continuation of the previous paper with the same title which will be referred to as [I]. Throughout the paper, A denotes a (left and right) artinian ring with identity 1, J its Jacobson radical and unless otherwise stated, all modules are (unital and) finitely generated.

Let n be any natural number. Then we say that A is of *right n-th local* (resp. *colocal*) *type* in case for every indecomposable right A-module M, the n-th top $\text{top}^n M := M/MJ^n$ (resp. the n-th socle $\text{soc}^n M :=$ the left annihilator of J^n in M) of M is indecomposable.

In this paper, we first examine an artinian ring which is of both left and right n-th local type (in this case the artinian ring is said to be of *two-sided n-th local type* or simply *n-th local type*) and give some necessary and sufficient conditions to be of this type, in particular for an algebra, we characterize this type by a structure of A (2.5). Note that this type of rings include the class of serial rings ([4]). Next, we come back to the case $n=2$ and restrict our interest to the case where A is an algebra over an algebraically closed field k, and give some further necessary conditions for A to be of right 2nd local type (3.4). (It is shown in [I, Example 2] that the necessary conditions stated in [I, Theorem 1] are not sufficient for A to be of right 2nd local type.) These conditions contain the list of all possible "shapes" of indecomposable projective *right* A-modules. (That of indecomposable projective *left* A-modules follows directly from [I, Theorem 1].) As an application, we give some necessary and sufficient conditions for a left serial algebra over an algebraically closed field to be of right 2nd local type (4.1). It should be noted that by [I, Theorem 1], an algebra over an algebraically closed field which is of right 2nd local type is left serial if every indecomposable projective left A-module P is of height≥ 4 (i.e. $J^3P \neq 0$). We remark that these theorems remain valid also in the case where the base field k is a splitting field for A. The last section is devoted to some examples.

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1. Preliminaries

The notation and terminology used throughout this paper is the same as that used in [I]. We may and will assume that A is a basic ring. For the convenience of readers, we quote some propositions from [I] and [6] which are frequently used in the sequel.

Theorem 1.1 ([I, Theorem 1]). *Let A be a ring with selfduality which is of right 2nd local type and e in* pi(A). *Then*

(1) *J^2e is a uniserial waist in Ae if $J^2e\neq 0$,*

(2) *eJ^m is a direct sum of local modules for every $m\geq 2$,*

(3) *for each local direct summand L of eJ^2, LJ^2 is uniserial (thus eJ^4 is a direct sum of uniserial modules). Further if A is an algebra, we have*

(4) *Ae is uniserial if $h(Ae)\geq 5$.*

In particular if the base field k is, in addition, an algebraically closed field, then

(5) *Ae is uniserial if $h(Ae)\geq 4$,*

and then

(6) *eJ^2 is a direct sum of uniserial modules.* //

We denote by 1_M the identity map of M for any A-module M.

Lemma 1.2 ([6, Lemma 1.1]). *Let M_1, M_2 and T be submodules of a left A-module M such that $M=M_1+M_2$ and $T=M_1\cap M_2$. If T' is a submodule of T and φ: $M_1\to M_2$ is an extension of $1_{T'}$, then putting $M_1':=(M_1)(1_{M_1}-\varphi)$ the following hold.*

(1) $M=M_1'+M_2$.

(2) $M_1'\cap M_2=(T)(1_T-\varphi)$.

(3) *The epimorphism $(1_{M_1}-\varphi)$: $M_1\to M_1'$ induces epimorphisms $M_1/T'\to M_1'$ and $T/T'\to M_1'\cap M_2$, in particular $|M_1'\cap M_2|\leq|T|-|T'|$.* //

Lemma 1.3 ([6, Lemma 1.2]). *Let M_1, M_2 and T be submodules of a left A-module M such that $M=M_1+M_2$ and $T=M_1\cap M_2$. Then*

(1) *1_T is extendable to a homomorphism $M_1\to M_2$ iff $M=M_1'\oplus M_2$ for some submodule M_1' of M.*

(2) *1_T is not extendable to any homomorphism $U\to M_2$ for any submodule U of M_1 with $T\subsetneqq U$ iff* soc$M=$socM_2. //

2. Artinian rings of n-th local type

In this section, we give some necessary and sufficient conditions for an artinian ring A to be of n-th local type for any natural number n.

Lemma 2.1. *Let n be any natural number and (E): $0\to S\overset{\alpha}{\to}L_1\oplus L_2\overset{\beta}{\to}M$*

$\to 0$ *be an exact sequence of right A-modules such that* $\alpha_1 S \lneqq L_1$ *and* $\alpha_2 S \leq L_2 J^n \neq L_2$ *where* $\alpha=(\alpha_1, \alpha_2)^T$. *Then* $\mathrm{top}^n M$ *is decomposable.*

Proof. The sequence (E) induces the following exact sequence:

$$0 \to \pi\alpha(S) \xrightarrow{\alpha'} \mathrm{top}^n L_1 \oplus \mathrm{top}^n L_2 \xrightarrow{\beta'} \mathrm{top}^n M \to 0$$

where π: $L_1 \oplus L_2 \to \mathrm{top}^n L_1 \oplus \mathrm{top}^n L_2$ is the canonical projection. Also, we have $\mathrm{Im}\,\alpha' \lneqq \mathrm{top}^n L_1$ by the assumption. Hence $\mathrm{top}^n M \cong ((\mathrm{top}^n L_1)/\mathrm{Im}\,\alpha') \oplus \mathrm{top}^n L_2$ is decomposable. //

Lemma 2.2. *Let* $(\alpha, D)=(\alpha_i)_{i=1}^{T\ n}$: $eA/eJ \to \bigoplus_{i=1}^{n} f_i A/I_i$ *be a homomorphism where* e, f_i *are in* $\mathrm{pi}(A)$, $I_i \lneqq f_i J$ *and* α_i *is a left multiplication by an element* u_i *in* $f_i Je$ *for each* $i=1, \cdots, n$. *Then* (α, D) *is j-fusible* ($j=1, \cdots, n$) *iff there are some* a_i *in* $f_j A f_i$ *for each* $i \neq j$ *and there is some* b *in* $I_j e$ *such that* $u_j = \sum_{i \neq j} a_i u_i + b$ *and* $a_i I_i \leq I_j$ *for each* $i \neq j$. *In particular when* (α, D) *is j-fusible, we have* $Au_j \leq \sum_{i \neq j} Au_i$ *if* $I_j e \leq J u_j$; *and* $Au_j \leq \sum_{i \neq j} Au_i + Ie$ *if* $I_j = f_j I$ *for some ideal* I *of* A.

Proof. (α, D) is j-fusible iff we have a commutative diagram

$$\begin{array}{ccc} eA/eJ & \xrightarrow{(\alpha_i)_{i\neq j}^T} & \bigoplus_{i \neq j} f_i A/I_i \\ \| & & \downarrow (\phi_i)_{i \neq j} \\ eA/eJ & \xrightarrow{\alpha_j} & f_j A/I_j \end{array}$$

for some homomorphism ϕ_i: $f_i A/I_i \to f_j A/I_j$ which are left multiplications by some elements a_i in $f_j A f_i$ for all $i \neq j$ iff $u_j = \sum_{i \neq j} a_i u_i + b$ and $a_i I_i \leq I_j$ for some a_i in $f_j A f_i$ and b in I_j (consequently b in $I_j e$) only if $u_j \in \sum_{i \neq j} Au_i + I_j e$ only if $Au_j \leq \sum_{i \neq j} Au_i + Ie$ (if $I_j = f_j I$ for some ideal I of A).

In case $I_j e \leq J u_j$, we have $Au_j \leq \sum_{i \neq j} Au_i$ since $Au_j \leq \sum_{i \neq j} Au_i + Ju_j$ and Ju_j is small in Au_j. //

Lemma 2.3 (cf. [1, Theorem 3.2]). *Let n be any natural number. Then the condition*

(2R) $\alpha=(\alpha_1, \alpha_2)^T$: $S \to L_1 \oplus L_2$ *is fusible if* S *is a simple right A-module,* L_i *are local right A-modules and* $\alpha_1 S \leq L_1 J$, $\alpha_2 S \leq L_2 J^n$.
implies the following

(2R)$'$ *Let* $(\alpha, D):=(\alpha_1, \alpha_2)^T$: $T \to L_1 \oplus L_2$ *be a homomorphism of right A-modules such that* L_1 *is local,* L_2 *is local and colocal of height*$>n$ *and* $h(L_1) \leq h(L_2)$; *and* $\alpha_i T \leq L_i J$ *for each* $i=1$, 2 *and* α_1 *is monic. Then* (α, D) *is 2-fusible.*

Proof. Assume that (2R) holds and hypothesis of (2R)$'$ is satisfied. Then noting that α is monic since α_1 is, we have an exact sequence

$$(E) \quad 0 \to T \overset{\alpha}{\to} L_1 \oplus L_2 \xrightarrow{\beta=(\beta_1, \beta_2)} M \to 0$$

which does not split since $\alpha T \leq (L_1 \oplus L_2)J$. We have only to show that $\beta_2 L_2$ is a direct summand of M by [I, Proposition 1.2]. Note that β_2 is monic since α_1 is. Then putting $M_1 := \beta_1(L_1)$, $M_2 := \beta_2 L_2$ and $U := \beta_2\alpha_2(T)$, we have $M = M_1 + M_2$ and $U = M_1 \cap M_2$. Also, $h(M_2) > n$, M_i are local modules and M_2 is, in addition, colocal. Further $U \leq M_1 J$ and $h(M_1) \leq h(M_2)$. Take any simple submodule $S_A \leq U$ and consider the map $\varphi = (\varphi_1, \varphi_2)^T : S \to M_1 \oplus M_2$ where each φ_i is the inclusion map $S \to M_i$. Then $\varphi_1 S \leq U \leq M_1 J$ and $\varphi_2 S \leq \mathrm{soc} M_2 \leq M_2 J^n$ since M_2 is colocal of height $> n$. Hence $\varphi : S \to M_1 \oplus M_2$ is fusible by (2R). If it is 1-fusible, we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi_2} & M_2 \\ \| & & \downarrow \psi \\ S & \xrightarrow{\varphi_1} & M_1 \end{array}$$

for some $\psi : M_2 \to M_1$. Then ψ is monic since M_2 is colocal. Therefore ψ is an isomorphism since $h(M_1) \leq h(M_2)$ and both M_1 and M_2 are local. Accordingly, we may assume that $\varphi : S \to M_1 \oplus M_2$ is 2-fusible. Thus there is a homomorphism $\varphi : M_1 \to M_2$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi_1} & M_1 \\ \| & & \downarrow \psi \\ S & \xrightarrow{\varphi_2} & M_2 \end{array}$$

is commutative. Then $M = M_1 + M_2 = M_1' + M_2$ where $M_1' := (1_{M_1} - \psi)\ (M_1)$ by (1.2). Also, $|M_1' \cap M_2| < |U|$, M_1' is local, $M_1' \cap M_2 \leq M_1' J$ and $h(M_1') \leq h(M_2)$. Hence iterating this argument, we obtain $M = M_1' \oplus M_2$ for some $0 \neq M_1' \leq M$. Thus $\beta_2 L_2 = M_2$ is a direct summand of M. //

2.4. Here, we do not assume that every module is finitely generated.

DEFINITION. Let A be a ring and M a right (or left) A-module. Then M is called to be a *highest* module in case $h(M) = h(A_A)$ $(= h({}_A A))$.

Proposition 2.4.1. *The following statements for a ring A are equivalent*:

(1) *Every highest right A-module is local iff it is colocal.*

(2) *Every highest indecomposable projective right A-module is injective.*

(3) *Every highest indecomposable injective right A-module is projective.*

(4) *Every highest indecomposable right A-module is projective and injective.*

Proof. (1)$\Rightarrow$(2). Let M_A be highest indecomposable projective and E_A an injective hull of M. Then since M is local, M is colocal by (1) and hence E is colocal. Then again by (1), E is local. Therefore $M=E$ since $M\leq E$ and $h(M)=h(E)$.

(2)$\Rightarrow$(4). Let M_A be highest indecomposable and π: $\bigoplus_{i\in I} P_i \to M$ a projective cover of M with each P_i indecomposable. Then π canonically yields the epimorphism π': $\bigoplus_{i\in I} P_i/K_i \to M$ where $K_i := \mathrm{Ker}\,(\pi\,|\,P_i)$. Since M is highest, P_i/K_i is highest for some i in I. Accordingly, P_i is highest and $K_i=0$ by (2). So P_i is isomorphic to a direct summand of M since P_i is injective. Hence $P_i \simeq M$ since M is indecomposable. As a consequence, M is projective as well as injective by (2).

(3)$\Rightarrow$(4). Let M_A be highest indecomposable and σ: $M \to \bigoplus_{i\in I} E_i$ an injective hull of M with each E_i indecomposable. Put $M_i := \mathrm{Im}\,\sigma_i$ for all i in I where $\sigma=(\sigma_i)^T_{i\in I}$. Then σ induces the monomorphism σ': $M \to \bigoplus_{i\in I} M_i$. Since M is highest, M_i is highest for some $i\in I$. And, E_i is highest. Then by (3), E_i is projective, in particular, local. Hence $E_i=M_i$ and ρ_i: $M \to E_i$ is an epimorphism. Thus E_i is isomorphic to a direct summand of M since E_i is projective. Therefore $M\simeq E_i$ since M is indecomposable. As a consequence, M is injective and projective.

The implications (4)$\Rightarrow$(i) are obvious for $i=1$, 2 and 3. //

Proposition 2.4.2. *Let A be a ring with selfduality satisfying the following condition*:

(*) (R) *Every highest indecomposable projective right A-module is colocal*; *and*

(L) *Every highest indecomposable projective left A-module is colocal.*

Then all the conditions (1)–(4) *in* (2.4.1) *and their left side versions hold.*

Proof. (*) implies that every highest local right (left) A-module is projective and hence colocal. Then by selfduality of A, (*) implies (1) in (2.4.1) and its left side version. //

REMARK. In case A is an algebra, (2.4.1) remains valid also under the assumption that every module in consideration is finitely generated. For, the injective hull of every simple right A-module is finitely generated in this case.

Theorem 2.5. *Let A be a ring and n any natural number. Then the following statements are equivalent*:

(1) *A is of n-th local type, i.e.*

(R) *A is of right n-th local type*; *and*

(L) *A is of left n-th local type.*

(2) (R) $\alpha=(\alpha_1, \alpha_2)^T$: $S\to L_1\oplus L_2$ *is fusible if S is a simple right A-module, L_i are local right A-modules and* $\alpha_1 S\leq L_1 J$, $\alpha_2 S\leq L_2 J^n$; *and*

(L) *The left side version of* (2R).

(3) i) (R) *For each e in* pi(A), eJ^n *is a uniserial waist in eA if* $eJ^n\neq 0$; *and*

(L) *The left side version of* (3–iR).

ii) (R) *Every monomorphism* $S\to L_2 J^n$ *is extendable to a homomorphism* $L_1\to L_2$ *where L_i are local right A-modules with* $1<h(L_1)\leq n$ *and S is a simple submodule of* L_1; *and*

(L) *The left side version of* (3–iiR).

(4) (R) *Every indecomposable right A-module is local if it is of height*$>n$; *and*

(L) *The left side version of* (4R).

In particular, if A is an algebra, then the above conditions are equivalent to (3–i).

Proof. We show the following implications: (1)$\Rightarrow$(2)$\Rightarrow$(3–i), (2, 3–i)$\Rightarrow$(3–ii), (3)$\Rightarrow$(2), (2, 3)$\Rightarrow$(4)$\Rightarrow$(1) and in case A is an algebra, we show (3–i)$\Rightarrow$(4). By left-right symmetry, we have only to show (1R)$\Rightarrow$(2R)$\Rightarrow$(3–iL), (2R, 3–iR)$\Rightarrow$(3–iiR), (3–i, 3–iiR)$\Rightarrow$(2R) and (2R, 3R)$\Rightarrow$(4R), finally in case A is an algebra, (3–i)$\Rightarrow$(4R). (Note the implication (4R)$\Rightarrow$(1R) is trivial.)

(1R)$\Rightarrow$(2R). By (2.1), top^n(Cok α) is decomposable, thus Cok α is decomposable by (1R). Hence α: $S\to L_1\oplus L_2$ is fusible by [I, Proposition 1.3].

(2R)$\Rightarrow$(3–iL). It is clear that $J^n e$ is uniserial by the proof of [I, Proposition 2.1]. Suppose $J^n e\neq 0$. Then $J^n e=Au_2$ for some $0\neq u_2$ in $f_2 J^n e\setminus f_2 J^{n+1}e$ where f_2 is in pi(A). Let $u_1\in f_1 J^m e\setminus f_1 J^{m+1}e$ be any element where $f_1\in$pi(A) and $1\leq m$. Then by (2R), the map $\alpha=(\alpha_1, \alpha_2)^T$: $eA/eJ\to(f_1A/f_1J^{m+1})\oplus(f_2A/f_2J^{n+1})$ is fusible where α_i are the left multiplications by u_i since $\alpha_1(eA/eJ)\leq f_1J/f_1J^{m+1}$ and $\alpha_2(eA/eJ)\leq f_2J^n/f_2J^{n+1}$. In case it is 2-fusible, we have $J^n e=Au_2\leq Au_1$ by (2.2) since $J^{n+1}e=Ju_2$. In case α is 1-fusible, we get $Au_1\leq Au_2+J^{m+1}e=J^n e+J^{m+1}e$ by (2.2). If $m<n$, then $Au_1\leq J^{m+1}e$ and $u_1\in f_1J^{m+1}e$, a contradiction. Hence $n\leq m$ and $Au_1\leq J^n e$. As a consequence, we obtain $J^n e\leq Au_1$ or $Au_1\leq J^n e$. Now let ${}_AX$ be any submodule of Ae. We show $J^n e\leq X$ if $X\not\leq J^n e$. Obviously, we may assume $X\leq J^n e$. Suppose $X\not\leq J^n e$. Then there is an element x in $X\setminus J^n e$. Here we may assume that $x=fx$ for some f in pi(A) since if $fx\in J^n e$ for all f in pi(A), then $x\in J^n e$, a contradiction. Then by the above, we obtain $J^n e\leq Ax$ since $Ax\not\leq J^n e$. Hence $J^n e\leq Ax\leq X$. Thus $J^n e$ is a waist in Ae.

(2R, 3–iR)$\Rightarrow$(3–iiR). Let L_1 and L_2 be local right A-modules with $1<h(L_1)\leq n$, S a simple submodule of L_1 and α_2: $S\to L_2J^n$ any monomorphism. Note that in this case, $n<h(L_2)$ since $0\neq\alpha_2 S\leq L_2 J^n$. By (2R), we have that $(\alpha, D):=(\alpha_1, \alpha_2)^T$: $S\to L_1\oplus L_2$ is fusible where α_1: $S\to L_1$ is the inclusion map.

Assume that (α, D) is 1-fusible. Then noting that L_2 is colocal, L_2 is embedded into L_1. But this is impossible since $h(L_1)\leq n<h(L_2)$. Hence (α, D) is 2-fusible. Thus α_2 is extendable to a homomorphism $L_1\to L_2$.

(3–i, 3–iiR)⇒(2R). We may assume that $S=eA/eJ$, $L_1=fA/X$, $L_2=gA/gJ^{r+1}$ (by 3–iR) for some e, f and g in pi(A), $n\leq r$ and $X\leq fA$, further α_1 and α_2 are nonzero maps given by left multiplications by some $u\in fJ^m e\setminus fJ^{m+1}e\neq\phi$ (for $1\leq m$) and $v\in gJ^r e\setminus gJ^{r+1}\neq\phi$, respectively.

In case $m<n$. We show $h(L_1)\leq n$. Suppose $h:=h(L_1)>n$. Then $0\neq fJ^{h-1}\leq fJ^n$ and fJ^{h-1} is a uniserial waist in fA by (3–iR). $h(fA/X)=h$ yields $fJ^{h-1}\nleq X$ and hence $X\lneqq fJ^{h-1}$, that is $X=fJ^s$ for some $s\geq h$. Hence $X=fJ^h$ since $h(fA/X)=h$. Therefore $u\in fJ^{h-1}e\setminus fJ^h e$ since $0\neq\alpha_1 S$ is simple. Thus $m=h-1\geq n$, a contradiction. (Note that m is uniquely determined by u.) As a consequence, α: $S\to L_1\oplus L_2$ is 2-fusible by (3–iiR).

In case $n\leq m$. $Av=J^r e$ is a uniserial waist in Ae by (3–iL). Hence $Au\leq Av$ or $Av\leq Au$. Note since $n\leq m$, it holds that $Au=J^m e$ and $uA=fJ^m$ by (3–i). Then $u\notin X\geq uJ$ implies that $fJ^{m+1}\leq X$ and $fJ^m\nleq X$. Hence $X=fJ^{m+1}=uJ$ by (3–iR). If $Au\leq Av$, then $u=av$ for some a in fAg and $agJ^{r+1}=avJ=uJ=X$ since $vA=gJ^r$ by (3–iR). Hence α: $S\to L_1\oplus L_2$ is 1-fusible by (2.2). Next if $Av\leq Au$, then $v=au$ for some $a\in gAf$ and $aX=auJ=vJ=gJ^{r+1}$. Hence α: $S\to L_1\oplus L_2$ is 2-fusible by (2.2).

(2R, 3R)⇒(4R). By (2.3), (2R)′ in (2.3) holds. Let M be any right A-module of height$>n$. We show that M is decomposed into local right A-modules of height$>n$ and indecomposable right A-modules of height$\leq n$ by induction on $m:=|\text{top } M|$.

We may assume that $2\leq m$ since it is obvious in case $m=1$. Let $M=\sum_{i=1}^{m}L_i$ be an irredundant sum of local modules L_i. By the hypothesis of induction, we have $M=L_1+(\bigoplus_{i=2}^{r}M_i)$ for some $r\leq m$ such that M_i is a local module of height $>n$ or an indecomposable module of height$\leq n$. We may assume that $h(L_1)\leq h(M_i)$ for each $i=2, \cdots, m$. Put $T:=L_1\cap(\bigoplus_{i=2}^{r}M_i)$. Again, we may assume $T\neq 0$. Putting α_1: $T\to L_1$ and θ: $T\to\bigoplus_{i=2}^{r}M_i$ the inclusion maps, $\alpha_j:=\pi_j\theta$ where π_j: $\bigoplus_{i=2}^{r}M_i\to M_j$ is the canonical projection for each $j=2, \cdots, r$ and $\alpha:=(\alpha_i)_{i=1}^{T}{}^{r}$, we have an exact sequence:

$$(E)\quad 0\to T\overset{\alpha}{\to}L_1\oplus(\bigoplus_{i=2}^{r}M_i)\overset{\beta}{\to}M\to 0\,.$$

By the hypothesis of induction, we have only to show that M is decomposable. To this end, it is sufficient to show that α: $T\to L_1\oplus M_2\oplus\cdots\oplus M_r$ is fusible. Note that α_1 is monic. Since $n<h(M)$, we have $n<h(M_i)$ for some $i=2, \cdots, r$,

say $i=r$. Then M_r is local and colocal by (3–i). Further since the sum $M=\sum_{i=1}^{m}L_i$ is irredundant, we have $T\leq L_1J$ and hence $\pi_rT\leq M_rJ$. Accordingly, $(\alpha_1, \alpha_r)^T$: $T\to L_1\oplus M_r$ is 2-fusible by (2R)'. Thus there is a homomorphism γ: $L_1\to M_r$ such that $\alpha_r=\gamma\alpha_1$. Hence α: $T\to L_1\oplus M_2\oplus\cdots\oplus M_r$ is r-fusible. In fact, putting $\delta:=(\gamma, 0, 0, \cdots, 0)$: $L_1\oplus M_2\oplus\cdots\oplus M_{r-1}\to M_r$, we have $\delta(\alpha_i)^T_{i\neq r}=\gamma\alpha_1=\alpha_r$.

(3–i)$\Rightarrow$(4R) in case A is an algebra. We may assume that $n<h(A_A)$. Let M be any indecomposable right A-module of height $h>n$. Note that A/J^h satisfies (*) in (2.4.2) and has selfduality. Then applying (2.4.2) to the ring A/J^h, we obtain that $M_{(A/J^h)}$ is projective, that is, $M_A\cong eA/eJ^h$ for some e in pi(A). Thus M is local. //

REMARK. Theorem 2.5 is a generalization of Nakayama [4, Theorem 17].

3. Further necessary conditions

3.1. Throughout this section, the base field k is algebraically closed when A is assumed to be an algebra. Here, we investigate further necessary conditions for an algebra A to be of right 2nd local type and determine all the "shapes" of indecomposable projective right A-modules.

Lemma 3.1.1. *Let A be an artinian ring, C and L be right A-submodules of a right A-module M and let $C\leq MJ^h$ for a natural number h. Then $(C+L)/L\leq(M/L)J^h$. In particular, $h(M)=h(M/L)$ iff $MJ^{h(M)-1}\not\leq L$.*

Proof. Clear. //

Lemma 3.1.2. *Let A be an artinian ring of right 2nd local type, L_1, L_2 be local right A-modules and S a simple right A-module. Then any monomorphism $\alpha=(\alpha_1, \alpha_2)^T$: $S\to L_1\oplus L_2$ is fusible if $\alpha_1S\leq L_1J$ and $\alpha_2S\leq L_2J^2$.*

Proof. Clear from the implication (1R)$\Rightarrow$(2R) in Theorem 2.5. //

REMARK. In the above, if further $\alpha_1S\leq L_1J^2$ holds, then the conclusion remains valid under a weaker assumption that the L_i are indecomposable by [1, Proposition 2.5.a].

Lemma 3.1.3. *Let A be an algebra of right 2nd local type, M a quasi-projective local right A-module and L_1 and L_2 be simple right A-submodules of M such that $3\leq h(M/L_2)=h(M/L_2)$ $(:=h)$. If there exist simple right A-modules $S_i\leq(M/L_i)J^{h-1}$ such that $S_1\cong S_2$, then we have $L_1=L_2$.*

Proof. Let S be a simple right A-module and α_i: $S\to S_i$ be isomorphisms. Then by (3.1.2), $(\alpha_1, \alpha_2)^T$: $S\to M/L_1\oplus M/L_2$ is fusible, say 2-fusible. Thus we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha_1} & M/L_1 \\ \| & & \downarrow \beta \\ S & \xrightarrow{\alpha_2} & M/L_2 \end{array}$$

for some homomorphism β: $M/L_2 \to M/L_2$. $S_1 \nleq \operatorname{Ker} \beta$ yields $h(\operatorname{Coim} \beta)=h$ by (3.1.1). Then β is an epimorphism since M/L_2 is local of height h. Accordingly, β is an isomorphism since $|M/L_1|=|M/L_2|$. Quasi-projectivity of M implies that β is liftable to an automorphism γ of M. And, the projective cover of M is of the form eA for some e in pi(A) since M is local. Therefore by projectivity of eA, γ is liftable to an automorphism δ of eA which is a left multiplication by an element t in $eAe\backslash eJe$. Then the restriction isomorphism γ': $L_1 \to L_2$ of γ is given by the left multiplication by $\bar{t}:=t+eJe$ in eAe/eJe. But since we assume that the base field k is algebraically closed, γ' is also given by a right multiplication by an element t' in k. As a consequence, $L_2=L_1t'$ $=L_1$. //

Proposition 3.1.4. *Let A be an algebra of right 2nd local type. Then for every quasi-projective local right A-module M of height ≥ 3, we have* $|\operatorname{soc} M| \leq 2$. *Further if e is in* pi(A), $3 \leq t$ *and* $eJ^{t-1}/eJ^t=S_1 \oplus S_2$ *with each S_i simple, then* $S_1 \ncong S_2$.

Proof. Put $h:=h(M) \geq 3$. Then there exists a simple right A-module $S \leq MJ^{h-1}$. Assume that $|\operatorname{soc} M| \geq 3$. Then there are simple right A-modules S_1 and S_2 such that $S \oplus S_1 \oplus S_2$ is a direct summand of soc M. Put $M_i:=M/S_i$ for each i. Then $h(M_1)=h(M_2)=h(M) \geq 3$ by (3.1.1). Also, $M_1J^{h-1} \geq (S \oplus S_1)/S_1 \cong (S \oplus S_2)/S_2 \leq M_2J^{h-1}$. Hence by (3.1.3), we have $S_1=S_2$, a contradiction. Therefore we must have $|\operatorname{soc} M| \leq 2$. Next, suppose $eJ^{t-1}/eJ^t=S_1 \oplus S_2$ with each S_i simple but $S_1 \cong S_2$. Then putting $M_i:=M/S_i$ for each $i=1$, 2 where $M:=eA/eJ^t$ is quasi-projective local, similarly we have $h(M_1)=h(M_2)$ $=t \geq 3$ and $M_1J^{t-1} \geq (S_1 \oplus S_2)/S_1 \cong (S_1 \oplus S_2)/S_2 \leq M_2J^{t-1}$. Then $S_1=S_2$ by (3.1.3), a contradiction. //

Corollary 3.1.5. *Let A be an algebra of right 2nd local type and e in* pi(A). *Then eJ^2 is a direct sum of at most two uniserial modules.*

Proof. Clear from (1.1; 6) and (3.1.4).

REMARK. [I, Example 2] was a counter example of sufficiency of the necessary conditions stated in (1.1) for A to be of right 2nd local type. This example does not satisfy the condition stated in (3.1.4).

3.2 In case eJ^2 is uniserial.

Proposition 3.2.1. *Let A be an algebra of right 2nd local type, e in* pi(A). *Suppose eJ^2 is uniserial and* $|\mathrm{soc}(eA)|=1$. *Then eA is uniserial if $h(eA)\geq 4$.*

Proof. Let $D:=\mathrm{Hom}_k(?, k)$ be the usual selfduality of A. Then since eJ is colocal and of height≥ 3, $D(eJ)$ is local and of height≥ 3. Hence $D(eJ)$ is colocal by (1.1), thus eJ is local. Therefore eA is uniserial since eJ^2 is uniserial. //

Lemma 3.2.2. *Let A be a ring, M_A a module and S_A a submodule of M. Then S is a semisimple derect summand of M iff $MJ\cap S=0$. In particular,* soc M $\leq MJ$ *iff M has no simple direct summand.*

Proof. ($\Rightarrow$). If $M=S\oplus X$ for some $X_A\leq M_A$ with S semisimple, then $MJ=XJ$ and $S\cap MJ=S\cap XJ=0$.

($\Leftarrow$). Suppose that $MJ\cap S=0$. Then $S\cong(S+MJ)/MJ$ and $M/MJ=(S+MJ)/MJ\oplus X/MJ$ for some X_A with $MJ\leq X\leq M$. Therefore S is semisimple and $M=S+MJ+X=S+X$ since MJ is small in M. Further $(S+MJ)\cap X\leq MJ$ implies that $S\cap X\leq MJ$ and $S\cap X=S\cap(S\cap X)\leq S\cap MJ=0$. Thus $M=S\oplus X$. //

Proposition 3.2.3. *Let A be an algebra of right 2nd local type, e in* pi(A). *Suppose that eJ^2 is uniserial and* $|\mathrm{soc}(eA)|=2$. *Then $eJ=X\oplus Y$ for some X_A and $Y_A\leq eJ$ such that X is simple; and Y is a uniserial module (in case $h(eA)\geq 4$) or a colocal module of height 2 (in case $h(eA)=3$).*

Proof. By assumption, we have $\mathrm{soc}(eJ)=\mathrm{soc}(eA)\not\leq eJ^2$. Hence by (3.2.2), it follows from $|\mathrm{soc}(eJ)|=2$ that $eJ=X\oplus Y$ with X simple and Y colocal.

In case $h(eA)=3$, $h(Y)=h(eJ)=2$ since X is simple. Thus Y is colocal and of height 2.

In case $h(eA)\geq 4$. Since X is simple, $eJ^2=YJ$. Thus YJ is uniserial. Further $h(Y)\geq 3$ since $0\neq eJ^3=YJ^2$. Hence Y is local by the same argument as in the proof of (3.2.1). Therefore Y is uniserial. //

3.3. In case eJ^2 is a direct sum of two uniserial modules.

Lemma 3.3.1. *Let A be a ring, M a right A-module and L a right A-submodule of M. Then the following statements are equivalent:*

(1) $LJ=L\cap MJ$.

(2) *$M=L+X$ and* $|\mathrm{top}\, M|=|\mathrm{top}\, L|+|\mathrm{top}\, X|$ *for some $X_A\leq M$.*

(3) *Every sum $L=\sum_{i=1}^{m}L_i$ with each L_i local and* $m=|\mathrm{top}\, L|$ *can be extended to a sum $M=\sum_{i=1}^{m}L_i+\sum_{i=1}^{n}X_i$ with each X_i local and* $m+n=|\mathrm{top}\, M|$.

In particular, if both L and M are modules of height 2 and without simple *direct summands, then all the conditions* (1)–(3) *hold.*

Proof. Let π: $M \rightarrow$top M be the canonical projection.

(1)$\Rightarrow$(2). We have top $M=\pi(L)\oplus\pi(X')$ for some X'_A with $MJ\leq X'\leq M$. Let p: $P\rightarrow\pi(X')$ be a projective cover of $\pi(X')$. Then $p=\pi q$ for a homomorphism q: $P\rightarrow X'$. Put $X:=\operatorname{Im} q$. Then $M=L+X'=L+X+MJ=L+X$. Also, $|\pi(X')|=|\text{top } P|=|\text{top } X|$ for Ker $q\leq$Ker $p=PJ$. Further $|\pi(L)|=|\text{top } L|$ by (1). Hence $|\text{top } M|=|\text{top } L|+|\text{top } X|$.

(2)$\Leftrightarrow$(3). Clear.

(2)$\Rightarrow$(1). Noting that $YJ\leq Y\cap MJ$ for every $Y_A\leq M$, we have $|\text{top } M| =|\pi(L)+\pi(X)|\leq|\pi(L)|+|\pi(X)|\leq|\text{top } L|+|\text{top } X|=|\text{top } M|$. Thus $|\pi(L)|=|\text{top } L|$ i.e. $|LJ|=|L\cap MJ|$. Hence $LJ=L\cap MJ$.

If both L and M are modules of height 2 and without simple direct summands, then $LJ=\operatorname{soc} L=L\cap\operatorname{soc} M=L\cap MJ$ by (3.2.2). Hence (1)–(3) hold. //

Proposition 3.3.2. *Let A be an algebra of right 2nd local type and e in* pi(A). *If eJ^2 is a direct sum of two uniserial right A-modules, then so is eJ.*

Proof. Assume that the hypothesis of the proposition is satisfied and put $P:=eA/eJ^3$. Then P is a quasi-projective local right A-module of height 3. By (3.1.4) and the assumption that eJ^2 is a direct sum of two uniserial modules, we have $2=|PJ^2|\leq|\operatorname{soc} PJ|=|\operatorname{soc} P|\leq 2$. Hence soc $PJ=PJ^2$ i.e. PJ has no simple direct summand by (3.2.2). In particular, if $PJ=\sum_{i=1}^{n}L_i$ is an irredundant sum of local modules L_i, then $h(L_i)=2$ for all i and every partial sum $\sum_{i\in I}L_i$ with $I\subseteq\{1, \cdots, n\}$ has no simple direct summand. We claim the following:

(a) Every colocal submodule of PJ is uniserial.

(b) Let $PJ=\sum_{i=1}^{n}L_i$ be any irredundant sum of local modules L_i. If L_i and L_j are colocal for some $i\neq j$ in $\{1, \cdots, n\}$, then soc $L_i\neq$soc L_j.

Proof of (a). Let L be a colocal submodule of PJ. If L is simple, then the assertion is trivial. So we may assume that $h(L)=2$. Put $S:=\operatorname{soc} L$ $(=LJ)$. Then $PJ^2\geq LJ=S$ and $PJ^2=S\oplus T$ for some simple module T_A by assumption. Also, $S\not\cong T$ by (3.1.4). This implies that P/S is quasi-projective. Hence $|\operatorname{soc} P/S|\leq 2$ by (3.1.4). It follows from $L\cap PJ^2=S$ that $2\geq |\operatorname{soc} P/S|\geq|(L/S)\oplus(PJ^2/S)|=|L/S|+1$. Hence L/S is simple i.e. L is uniserial.

Proof of (b). Suppose that soc $L_i=$soc $L_j=S$. Then since L_i+L_j has no simple direct summand, soc $(L_i+L_j)=(L_i+L_j)J=L_iJ+L_jJ=S$ is simple. Hence L_i+L_j is uniserial by (a). Thus $L_i=L_j$.

Now we come back to the proof of the proposition. We have soc $P=$soc

$(PJ)=PJ^2=S\oplus T$ for some simple modules S_A and T_A with $S\not\cong T$ by (3.1.4). We next show that PJ is a direct sum of two uniserial modules. Otherwise, by (b), PJ has a local submodule L of height 2 which is not colocal i.e. soc $L=S\oplus T$. Define injections α_1: $S\to L/T$ and α_2: $S\to P$ in the obvious way. Then by (3.1.2), $(\alpha, D):=(\alpha_1, \alpha_2)^T$: $S\to(L/T)\oplus P$ is fusible. If it is 1-fusible, then $\varphi\alpha_2=\alpha_1$ for some φ: $P\to L/T$ and $2=h(L/T)\geq h(\mathrm{Im}\,\varphi)=h(\mathrm{Coim}\,\varphi)=3$ by (3.1.1). This contradiction shows that (α, D) is 2-fusible i.e. $\varphi\alpha_1=\alpha_2$ for some φ: $L/T\to PJ$ (note that $\mathrm{Im}\,\psi\leq PJ$ for any homomorphism ψ: $L/T\to P$ since $h(L/T)=2$ and $\mathrm{soc}^2 P=PJ$) where φ is monic since φ does not vanish the simple socle of L/T. Accordingly, PJ has a uniserial submodule L_1 of height 2 and with soc $L_1=S$. Similarly, PJ has a uniserial submodule L_2 of height 2 and with soc $L_2=T$. By (3.3.1), there is an irredundant sum $PJ=\sum_{i=1}^{n}L_i$ of local modules L_i. $M\neq L_1\oplus L_2$ implies that $3\leq n$ and soc $L_i=S\oplus T$ for all $i\geq 3$ by (b). Applying the same argument as above to $L:=L_3$, we have $\varphi\alpha_1=\alpha_2$ for some φ: $L/T\to PJ$. Put $N:=\varphi(L/T)$. If $N\neq L_1$, then $N+L_1$ is not colocal by (a) and then soc $L_1\lneqq$ soc $(N+L_1)$. Hence by (1.3; 2), there is a map η: $N\to L_1$ such that $\eta\alpha_2=\theta_2$ where θ_2: $S\to L_1$ is the inclusion map. If $N=L_1$, putting $\eta=1_{L_1}$ we also have $\eta\alpha_2=\theta_2$. Let π: $L\to L/T$ be the canonical projection, θ_1: $S\to L$ the inclusion map and put $\lambda:=\eta\varphi\pi$. Then we have a commutative diagram:

$$\begin{array}{ccccccc} L & \xrightarrow{\pi} & L/T & \xrightarrow{\varphi} & N & \xrightarrow{\eta} & L_1 \\ {\scriptstyle\theta_1}\uparrow & & {\scriptstyle\alpha_1}\uparrow & & {\scriptstyle\alpha_2}\uparrow & & {\scriptstyle\theta_2}\uparrow \\ S & = & S & = & S & = & S\,. \end{array}$$

Hence $\lambda\theta_1=\theta_2$ and $L_3+L_1=L_3'\oplus L_1$ where $L_3':=(1_{L_3}-\lambda)\,(L_3)\cong L/S$. Also, soc $L_3'=T$ and $h(L_3')=2$. Therefore $PJ=(L_1+L_3)+L_2+\cdots+L_n=(L_1+L_3')+L_2+\cdots+L_n$. This contradicts (b). As a consequence, $PJ=L_1\oplus L_2$. Now we have that eJ is decomposable since $\mathrm{top}^2(eJ)=eJ/eJ^3=PJ$ is decomposable. Also, $|\mathrm{soc}\,(eJ)|=2$ since $2=|\mathrm{soc}\,(eJ^2)|\leq|\mathrm{soc}\,(eJ)|\leq|\mathrm{soc}\,(eA)|\leq 2$. Hence $eJ=X\oplus Y$ with both X and Y colocal. Then both XJ and YJ are uniserial since $eJ^2=XJ\oplus YJ$. Further $2=|\mathrm{top}\,PJ|=|eJ/eJ^2|=|\mathrm{top}\,X\oplus\mathrm{top}\,Y|$ yields that both X and Y are local thus uniserial. //

Summarizing the above propositions and (1.1), we obtain the following

Theorem 3.4. *Let A be an algebra (over an algebraically closed field) which is of right 2nd local type and let e be in* pi(A). *Then*

(L) (1) *J^2e is a uniserial waist in Ae if $J^2e\neq 0$.*

(2) *Ae is uniserial if $h(Ae)\geq 4$.*

(3) *Therefore the structure of Ae is one of the following:*

(L_1) *Ae is uniserial.*

(L_2) *$h(Ae)=2$ and Ae is not uniserial.*

(L_3) *$h(Ae)=3$ and Ae is colocal but not uniserial.*

(R) (1) *eJ^2 is a direct sum of at most two uniserial right A-modules.*

(2) *$|\mathrm{soc}\, L|\leq 2$ for each quasi-projective local right A-module L of height ≥ 3.*

(3) *If $eJ^{t-1}/eJ^t=S_1\oplus S_2$ with each S_i a simple right A-module, then $S_1\not\cong S_2$ for each $t\geq 3$.*

(4) *The structure of eA is one of the following:*

(R_1) *eA is uniserial.*

(R_2) *$h(eA)=2$, eA is not uniserial and $|\mathrm{soc}\,(eA)|\geq 3$.*

(R_3) *$h(eA)=3$, eA is colocal but not uniserial.*

(R_4) *$h(eA)=3$, $eJ=X\oplus Y$ where X_A is simple and Y_A is a colocal but not uniserial module of height 2.*

(R_5) *eJ is a direct sum of two (nonzero) uniserial right A-modules.* //

REMARK. (1) All these types actually appear in examples (see Example 1 in section 5).

(2) In general, the conditions in (3.4) are not sufficient for algebras to be of right 2nd local type (see Example 4 in section 5).

4. Left serial algebras of right 2nd local type

Throughout this section, our ring A is a left serial ring and the base field k is algebraically closed when A is considered as an algebra. We know by (1.1) that most of algebras of right 2nd local type is left serial. So in this section, we examine the left serial case and show that in this case the necessary conditions obtained in section 3 are sufficient for an algebra A to be of right 2nd local type modifying the proof of Sumioka [6, Proposition 3.8]. Namely, we show:

Theorem 4.1. *Let A be a left serial algebra over an algebraically closed field k. Then the following statements are equivalent:*

(1) *A is of right 2nd local type.*

(2) *eJ is a direct sum of at most two uniserial modules if $h(eA)\geq 3$ for each e in* pi(A).

(3) *Every indecomposable right A-module is local if it is of height ≥ 3.*

REMARK. This theorem is similar to [8, Proposition 4.4].

4.2. We quote the following definitions and propositions concerning a left serial ring A from Sumioka [6].

Let A be a left serial ring, L a uniserial left A-module of length n and put $L_i:=\mathrm{soc}^i L$ and $D_i(L):=\mathrm{End}_A(\mathrm{top}\ L_i)$ for each $i=1, \cdots, n$. Then $D_i(L)$ are division rings. For $n\geq i\geq j\geq 1$, any element $\bar{\varphi}_i$ in $D_i(L)$ is induced by an endomorphism φ_i of L_i since L_i is quasi-projective, and φ_i induces an element $\bar{\varphi}_j$ in $D_j(L)$. We define a map λ_{ij}: $D_i(L)\rightarrow D_j(L)$ by $(\bar{\varphi}_i)\lambda_{ij}=\bar{\varphi}_j$. Then as easily seen, λ_{ij} are well-defined and ring monomorphisms with $\lambda_{ij}\lambda_{jl}=\lambda_{il}$ for all i, j and l with $n\geq i\geq j\geq l\geq 1$. Hence by the maps λ_{ij}, we can regard the sequence $D_1(L), D_2(L), \cdots, D_n(L)$ as a descending chain of division rings.

Lemma 4.2.1 ([6, Lemma 3.1]). *Let A be a left serial ring. Then the following conditions are equivalent for a uniserial module ${}_AL$ of length n and a natural number $r\leq n$:*

(1) $D_r(L)=D_n(L)$.

(2) *Every automorphism α of* soc L *is extendable to an automorphism of L if α is extendable to an automorphism of* $\mathrm{soc}^r L$. //

Let A be a left serial ring, S a simple left A-module and L a uniserial left A-module of length≥ 2. We denote by $c(S)$ the number of isomorphism classes of uniserial left A-modules of length 2 whose socles are isomorphic to S and put $m(L):=\dim D_1(L)_{D_2(L)}$. Then we have

Lemma 4.2.2 ([6, Lemma 3.3]). *Let A be a left serial ring and e in* pi(A). *Then $|eJ/eJ^2|\leq 2$ iff $c(\mathrm{soc}\ L)+m(L)\leq 3$ for every uniserial left A-module L of length≥ 2 and with* soc $L\cong Ae/Je$. //

Lemma 4.2.3 ([6, Lemma 4.3]). *Let A be a left serial ring, e in* pi(A) *and $r\geq 1$. Then the following conditions are equivalent*:

(1) *For any uniserial modules L_1 and L_2 such that* soc $L_1\cong Ae/Je$ *and $r\leq |L_1|\leq |L_2|$, any isomorphism α:* soc $L_1\rightarrow$ soc L_2 *is extendable to a homomorphism $L_1\rightarrow L_2$ if α is extendable to a homomorphism* $\mathrm{soc}^r L_1\rightarrow \mathrm{soc}^r L_2$.

(2) *eJ^{r-1} is a direct sum of uniserial modules.* //

DEFINITION. Let A be a left serial ring. Then we say that a simple left A-module S is of *V-type* in case $S\cong Ae/Je$ for some e in pi(A) such that eJ is a direct sum of at most two uniserial right A-modules.

Lemma 4.3. *Let A be a left serial algebra which satisfies the condition* (2) *in* (4.1) *and S a simple left A-module. If $S\cong$*soc *L for some left A-module L of height≥ 3, then S is of V-type.*

Proof. Let $S\cong Ae/Je$ where e is in pi(A). By the selfduality $D:=\mathrm{Hom}_k$ (?, k), we have $eA\cong D(E(Ae/Je))\cong D(E(L))$ where $E(-)$ denotes the injective hull of $(-)$. This implies that $h(eA)\geq h(L)\geq 3$. Hence the assertion follows from the condition (2) in (4.1). //

Lemma 4.4. *Let A be a left serial ring and let L_1 and L_2 be uniserial left*

A-modules such that $2\leq|L_1|\leq|L_2|$ and soc L_1 is of V-type. Then every isomorphism α: soc *$L_1\to$* soc *L_2 is extendable to a homomorphism $L_1\to L_2$ if α is extendable to a homomorphism* $\mathrm{soc}^2L_1\to\mathrm{soc}^2L_2$.

Proof. Clear from (4.2.3). //

Let A be a left serial algebra. Then since we assume that the base filed k is algebraically closed, we have $D_1(L)=D_2(L)=\cdots=D_n(L)$ $(=k)$ and hence $m(L)=1$ for every uniserial left A-module L. Following the terminology of [7] or [6], all simple left A-modules are of first kind in this case. This is equivalent to say that eJ/eJ^2 is (zero or) square-free (i.e. a direct sum of pairwise nonisomorphic simple right A-modules) for each e in pi(A).

Lemma 4.5 (Cf. [6, Lemma 3.4]). *Let A be a left serial algebra and let L_1 and L_2 be uniserial modules such that $2\leq|L_1|\leq|L_2|$ and $S:=$*soc *$L_1\cong$*soc *L_2 is of V-type. If* $\mathrm{soc}^2L_1\cong\mathrm{soc}^2L_2$, *then any isomorphism α:* soc *$L_1\to$* soc *L_2 is extendable to a monomorphism $L_1\to L_2$.*

Proof. Clear from (4.2.1) and (4.4). //

Lemma 4.6. *Let A be a left serial ring and L_i a uniserial left A-module of length≥ 2 and α_i: $S\to L_i$ a homomorphism for each $i=1,\cdots,n$ where S is a simple left A-module of V-type and $n\geq 3$. If $0\to S\xrightarrow{\alpha}\bigoplus_{i=1}^{n}L_i\xrightarrow{\beta}M\to 0$ is an exact sequence with $\alpha=(\alpha_i)_{i=1}^n$, then M is decomposable.*

Proof. This follows from (4.4) and the proof of [6, Lemma 3.5]. //

Lemma 4.7. *Let A be a left serial algebra satisfying the condition* (2) *in* (4.1) *and let M be a left A-module. Then M is indecomposable with* $|\mathrm{top}\ M|=2$ *iff $M=L_1+L_2$ for some uniserial left A-modules L_i with $2\leq|L_1|\leq|L_2|$ such that $S:=L_1\cap L_2$ is simple and the identity map 1_S of S is not extendable to any homomorphism* $\mathrm{soc}^2L_1\to\mathrm{soc}^2L_2$. *Moreover in this case M is colocal.*

Proof. $(\Leftarrow)$ and $S=$soc M follow immediately from (1.3).

$(\Rightarrow)$. It is clear that $h(M)\geq 2$. If $h(M)=2$, the assertion is obvious. Therefore we may assume that $h(M)\geq 3$. Then $M=L_1+L_2$ for some uniserial modules L_1 and L_2 such that $L_1\cap L_2\neq 0$ and $2\leq|L_1|\leq|L_2|\geq 3$. Thus (soc $L_1\cong$) soc L_2 is of V-type by (4.3). By (4.4), the rest of the proof is quite similar to that of [6, Proposition 3.6]. //

Corollary 4.7.1. *Let A be a left serial ring and M a colocal left A-module such that soc M is of V-type and* $|\mathrm{top}\ M|=2$. *Then $M=L_1+L_2$ for some uniserial left A-modules L_i with $2\leq|L_1|\leq|L_2|$ such that $S:=$*soc *$M=L_1\cap L_2$ and 1_S is not extendable to any homomorphism* $\mathrm{soc}^2L_1\to\mathrm{soc}^2L_2$. //

REMARK. In the above, we have $h(M)=h(L_2)=|L_2|$. So the number $s(M):=\min\{|L_1|, |L_2|\}=|M|-h(M)+1$ is uniquely determined by M. Further we define $s(L):=|L|$ for every uniserial left A-module L.

Lemma 4.8. *Let A be a left serial algebra and let L be a uniserial left A-module of length ≥ 2 and M a colocal left A-module such that* soc M *is of V-type and* $|\text{top } M|=2$. *If $|L|\leq s(M)$, then any isomorphism α:* soc $L\to$soc M *is extendable to a homomorphism $L\to M$.*

Proof. This is a simple modification of the proof of the case (i) of [6, Lemma 3.7]. Put $S:=\text{soc } M$. By Corollary 4.7.1, $M=L_1+L_2$ for some uniserial left A-modules L_i with $2\leq|L_1|\leq|L_2|$ such that $L_1\cap L_2=S$ and 1_S is not extendable to any homomorphism $\text{soc}^2L_1\to\text{soc}^2L_2$. $|L|\leq s(M)$ yields $|L|\leq|L_1|\leq|L_2|$. Since 1_S is not extendable to any homomorphism $L_1\to L_2$, we have $\text{soc}^2L_1\not\cong\text{soc}^2L_2$ by (4.5) and the fact that S is of V-type. By (4.2.2), $c(S)\leq 2$ since $m(S)=1$. Accordingly, $\text{soc}^2L\cong\text{soc}^2L_1$ or $\text{soc}^2L\cong\text{soc}^2L_2$. Hence by (4.5), we infer that α is extendable to a homomorphism $L\to L_i$ for some $i=1, 2$ and thus to a homomorphism $L\to M$. //

Proposition 4.9. *Let A be a left serial ring, S a simple left A-module of V-type and L_i colocal left A-modules with $|\text{top } L_i|\leq 2$ for all $i=1, \cdots, n$, in particular let L_1 be uniserial and $|L_1|\leq s(L_i)$ for all $i=2, \cdots, n$. Assume that a sequence*

$$0\to S\xrightarrow{\alpha}\bigoplus_{i=1}^{n}L_i\xrightarrow{\beta}M\to 0$$

is exact. Then M is decomposable if $|\text{top }(\bigoplus_{i=1}^{n}L_i)|\geq 3$.

Proof. Put $\alpha:=(\alpha_i)_{i=1}^{n}$. Then we may assume that $\alpha_i\neq 0$ for all i. If $|\text{top } L_j|=2$ for some $j=2, \cdots, n$, then by (4.8), $(\alpha_1, \alpha_j)\colon S\to L_1\oplus L_j$ is 2-fusible. Thus $\alpha\colon S\to\bigoplus_{i=1}^{n}L_i$ is j-fusible by the same argument as in the proof of the implication (2R, 3R)$\Rightarrow$(4R) in Theorem 2.5. Therefore M is decomposable. So we may assume that L_i are uniserial for all $i=1, \cdots, n$. Then by (4.6), M is decomposable. //

Lemma 4.10. *Let A be a left serial ring and M an indecomposable left A-module of height* 2. *Then* soc M *is homogeneous* (*i.e. a direct sum of copies of one simple left A-module*).

Proof. Let $M=\sum_{i=1}^{n}L_i$ be an irredundant sum of uniserial left A-modules. Then since M is indecomposable of height 2, every partial sum $L=\sum_{i\in I}L_i$ with $I\subseteq\{1, \cdots, n\}$ has no simple direct summand and hence soc $L=JL$ by (3.2.2).

Assume that soc M is not homogeneous and put soc $M=\bigoplus_{i=1}^{m}S_i^{(m_i)}$ with each S_i simple and $S_i\not\cong S_j$ if $i\neq j$. Then $2\leq m$. Put $I_j:=\{i\in\{1,\cdots,n\}\mid \text{soc } L_i\cong S_j\}$ and $M_j:=\sum_{i\in I_j}L_i$ for each $j=1,\cdots,m$. Then for any $I\subseteq\{1,\cdots,m\}$, we have soc $(\sum_{j\in I}M_j)=\sum_{j\in I}JM_j=\sum_{j\in I}\sum_{i\in I_j}JL_i=\sum_{j\in I}\sum_{i\in I_j}\text{soc } L_i\cong\bigoplus_{j\in I}S_j^{(t_j)}$ for some natural numbers t_j. In particular, soc $M_1\cong S_1^{(t_1)}$ and soc $(\sum_{j\neq 1}M_j)\cong\bigoplus_{j\neq 1}S_j^{(t_j)}$. Hence $M_1\cap(\sum_{j\neq 1}M_j)=0$, that is, $M=M_1\oplus(\sum_{j\neq 1}M_j)$ is decomposable, a contradiction. //

The following corollary is of interest comparing with Kawada algebras ([3], [5]).

Corollary 4.11. *Let A be a left serial algebra of right 2nd local type and M an indecomposable right A-module. Then* top M *is homogeneous.*

Proof. Clear from (4.11) noting that top $M\cong D(\text{soc } DM)=D(\text{soc}(\text{soc}^2DM))$ and the fact that $\text{soc}^2DM\cong D\,\text{top}^2M$ is indecomposable left A-module of height 2 (we may assume that $h(M)\geq 2$ since the assertion is trivial in case $h(M)=1$) where D is a selfduality of A. //

Lemma 4.12. *Let A be a left serial ring and M an indecomposable left A-module of height* 2 *such that* soc $M\cong S^{(r)}$ *where S is a simple left A-module of V-type. Then M is colocal and* $|\text{top } M|\leq 2$.

Proof. It is sufficient to prove that M can be decomposed into colocal left A-modules M_i with $|\text{top } M_i|\leq 2$ for any left A-module M of height 2 such that soc $M\cong S^{(r)}$ where S is a simple left A-module of V-type. We prove this by induction on $n:=|\text{top } M|$.

If $n=1$ or 2, the assertion is clear from (1.3). So we may assume that $n\geq 3$. By the hypothesis of induction, we have only to show that M is decomposable. Hence we may assume that for any irredundant sum expression $M=\sum_{i=1}^{n}L_i$ (with each L_i uniserial) of M, $|L_i|=h(L_i)=2$ for all i. Let $M=\sum_{i=1}^{n}L_i$ be an irredundant sum of uniserial left A-modules L_i. Then again by the hypothesis of induction, $M=L_1+(\bigoplus_{i=2}^{m}M_i)$ for some colocal left A-modules M_j with $|\text{top } M_i|\leq 2$. Also, we have $2=|L_1|=s(M_i)$ for all $i=2,\cdots,m$. Putting $S:=L_1\cap(\bigoplus_{i=2}^{m}M_i)$ and π_j: $\bigoplus_{i=2}^{m}M_i\to M_j$ the canonical projections, we have an exact sequence

$$0\to S\overset{\alpha}{\to}L_1\oplus(\bigoplus_{i=2}^{m}M_i)\overset{\beta}{\to}M\to 0$$

where $\alpha=(\alpha_i)_{i=1}^{m}$, $\beta=(\beta_i)_{i=1}^{T\,m}$, $\alpha_1:=-1_S$, $\alpha_j:=1_S\pi_j$ for all $j=2,\cdots,m$ and β_i

are the inclusion maps. If $S=0$, then clearly M is decomposable. Thus we may assume that S is simple since L_1 is a uniserial left A-module of length 2. Then by (4.9), M is decomposable for S is of V-type. //

4.13. Proof of Theorem 4.1.

(3)⇒(1). Trivial.

(1)⇒(2). Noting that every colocal right A-module is uniserial since A is a left serial algebra, we see that the cases (R_3) and (R_4) in Theorem 3.4 do not occur. Hence the condition (2) holds.

(2)⇒(3). By selfduality of A, the condition (3) is equivalent to the following:

(3)′ Every indecomposable left A-module is colocal if it is of height≥ 3.

We show the implication (2)⇒(3)′. Assume the condition (2) holds. Let M be any left A-module of height≥ 3. Then in order to verify the condition (3)′, it is sufficient to show that M is decomposed into (a): colocal left A-modules M_i of height≥ 3 with $|\text{top } M_i|\leq 2$; and (b): indecomposable left A-modules of height≤ 2. We prove this by induction on $n:=|\text{top } M|$. It is clear in case $n=1$ or 2 by (4.7). Now assume $n\geq 3$. Let $M=\sum_{i=1}^{n} L_i$ be an irredundant sum of local left A-modules L_i. Then by the hypothesis of induction, $M=L_1+(\bigoplus_{i=2}^{s} M_i)$ such that M_i is of (a)-type of the above for $i=2$, $\cdots$, r and is of (b)-type for $i=r+1, \cdots, s$. We may assume that $|L_1|\leq s(M_i)$ for each $i=2, \cdots, r$ if $r=s$; and $|L_1|\leq 2$ if $r<s$. Note that $1<r$ since $h(M)\geq 3$. Again, by the hypothesis of induction, we have only to show that M is decomposable. Therefore we may assume that $T:=L_1\cap(\bigoplus_{i=2}^{s} M_i)$ is not zero and $|L_1|\geq 2$. Let $\pi_j\colon \bigoplus_{i=2}^{s} M_i\to M_j$ be the canonical projection for each $j=2, \cdots, s$. Then we have an exact sequence

$$(E)\quad 0\to T\overset{\alpha}{\to} L_1\oplus(\bigoplus_{i=2}^{s} M_i)\overset{\beta}{\to} M\to 0$$

where $\alpha=(\alpha_i)_{i=1}^{s}$, $\beta=(\beta_i)_{i=1}^{T\ s}$, $\alpha_1=-1_T$, $\alpha_j=1_T\pi_j$ for each $j=2, \cdots, s$ and β_i are the inclusion maps. We divide the argument into two cases: (i) $|L_1|\geq 3$; and (ii) $|L_1|=2$.

(i) In case $|L_1|\geq 3$. It follows $r=s$ and soc L_1 is of V-type by (4.3). If T is not simple, α_j is extendable to a homomorphism $L_1\to M_j$ for each $j=2$, $\cdots$, s by (4.4) and (4.8) since $\text{soc}^2 L_1\leq T$. Then there is a uniserial left A-modlue L_1' such that $|L_1'|<|L|$, $M=L_1'+(\bigoplus_{i=2}^{s} M_i)$ and $|L_1'\cap(\bigoplus_{i=1}^{s} M_i)|<|T|$ by (1.2). Iterating this argument, we come to the case (ii) or the case (i) with T simple. Hence we may assume that $T=\text{soc } L_1$ is simple of V-type since $|L_1|$

≥ 3. Then by (4.9), M is decomposable for $n \geq 3$.

(ii) In case $|L_1|=2$. Clearly, $T=\text{soc } L_1$ is simple. If $\alpha_j=0$ for some $j=2, \cdots, s$ in (E), then M_j is a direct summand of M. Hence we may assume that $\alpha_j \neq 0$ for any $j=2, \cdots, s$. Then $T \cong \text{soc } M_i$ for every $i=2, \cdots, r$ and T is of V-type since $h(M_i) \geq 3$. If $r=s$, then by (4.9), M is decomposable. Thus we may assume that $r<s$. Since for each $i=r+1, \cdots, s$, $\alpha_i \neq 0$ and soc M_i is homogeneous by (4.10), we have soc $M_i \cong T^{(r_i)}$ for some r_i. Then by (4.12), M_i is colocal and $|\text{top } M_i| \leq 2$ since T is of V-type. Hence M is decomposable by (4.9). //

REMARK. (1) The implication (2)$\Rightarrow$(3) in (4.1) is still true in the case where the base field k is not algebraically closed by [7] or [6, Lemma 3.7].

(2) Considering the results of [I, Theorem 2], (2.5) and (4.1), it is of interest to characterize those algebras having the following property for any fixed natural number n:

Every indecomposable right module is local if it is of height$>n$.

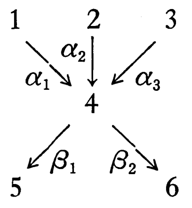
5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field k as in [I]. (See [2] for details concerning bounden quiver algebras.) For a vertex i of a bounden quiver, we denote by e_i the primitive idempotent corresponding to the vertex i.

EXAMPLE 1. Algebras of right 2nd local type which have an e in pi(A) of type (L_i) for some $1 \leq i \leq 3$ or of type (R_j) for some $1 \leq j \leq 5$ in Theorem 3.4.

(L_1) and (R_1): Take serial rings.

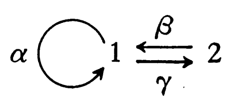
(L_2) and (R_2): Let A be the algebra defined by the following bounden quiver



; $\beta_j \alpha_i = 0$ for all i, j.

Then clearly, A is of right 2nd local type since $J^2=0$. Also, Ae_4 is of type (L_2) and e_4A is of type (R_2).

(L_3) and (R_3): Take the algebra A defined in [I, Example 1], namely



; $\beta\alpha = \alpha\gamma = 0, \alpha^2 = \gamma\beta$.

Then as verified in [I], A is of right 2nd local type and Ae_1 is of type (L_3) and e_1A is of type (R_3).

(R_4): Let A be the algebra defined by the following

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & \beta\swarrow & & \nwarrow\delta & & \\ 2 & \xrightarrow{\alpha} & 1 & & & & 5 \\ & & \gamma\nwarrow & & \swarrow\varepsilon & & \\ & & & 4 & & & \end{array} \; ; \quad \beta\delta=\gamma\varepsilon\,.$$

Then computing the Auslander-Reiten quiver (see [2]) of A, we see that A has the following property:

Every indecomposable right A-module is local if it is of height≥ 3.

In particular, A is of right 2nd local type and e_1A is of type (R_4).

(R_5): Let A be the following quiver algebra of type A_5

$$1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5\,.$$

Then by [8, Proposition 4.4] or [6], A is of right (1st) local type and hence 2nd local type. Also, e_3A is of type (R_5).

Example 2. A left serial algebra of right 2nd local type which is not of right (1st) local type.

Let A be the following bounden quiver algebra

$$\begin{array}{ccccccc} & & 5 & & & & \\ & & \delta\downarrow & & & & \\ 1 & \xrightarrow{\alpha} & 2 & \xrightarrow{\beta} & 3 & \xleftarrow{\gamma} & 4 \\ & & \varepsilon\uparrow & & & & \\ & & 6 & & & & \end{array} \; ; \quad \beta\delta=\beta\varepsilon=0\,.$$

Then A is left serial and also by Theorem 4.1, A is of right 2nd local type. In fact, e_iA is simple for each i=1, 4, 5, 6; and e_2A is of height 2 and e_3A is of height 3 having type (R_5). But by [8, Proposition 4.4], A is not of right (1st) local type since e_2A is of type (R_2).

Example 3. An algebra of right 2nd local type having an indecomposable right A-module of height≥ 3 that is not local.

Let A be the algebra defined by the following quiver of type A_5

$$1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5\,.$$

Then as easily seen, A is of right 2nd local type and the following indecomposable right A-module is not local but it is of height 3:

$$k \xrightarrow{1} k \xleftarrow{1} k \xrightarrow{1} k \xrightarrow{1} k\,.$$

EXAMPLE 4. The conditions in Theorem 3.4 are not sufficient for algebras to be of right 2nd local type in general.

Let A be the following quiver algebra of type D_6

$$\begin{array}{ccccccccc} & & 6 & & & & & & \\ & & \uparrow & & & & & & \\ 1 & \leftarrow & 2 & \rightarrow & 3 & \leftarrow & 4 & \leftarrow & 5\,. \end{array}$$

Then as easily verified, A satisfies all the conditions in (3.4). But it is not of right 2nd local type. For instance, let M be the right A-module corresponding to the following k-representation of $\underset{\sim}{Q}^{op}$

$$\begin{array}{ccccccccc} & & k & & & & & & \\ & & \downarrow {\scriptstyle \begin{bmatrix}0\\1\end{bmatrix}} & & & & & & \\ k & \xrightarrow[\begin{bmatrix}1\\0\end{bmatrix}]{} & k\oplus k & \xleftarrow[\begin{bmatrix}1&0\\0&1\end{bmatrix}]{} & k\oplus k & \xrightarrow[\begin{bmatrix}1&0\\0&1\end{bmatrix}]{} & k\oplus k & \xrightarrow[(1,1)]{} & k\,. \end{array}$$

Then M is indecomposable but $\mathrm{top}^2 M$ is decomposable:

$$\begin{array}{rl} \mathrm{top}^2 M = & \left(\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ k & \xrightarrow[1]{} & k & \xleftarrow[1]{} & k & \xrightarrow[1]{} & k & \rightarrow & 0 \end{array}\right) \\ & \\ \oplus & \left(\begin{array}{ccccccccc} & & k & & & & & & \\ & & \downarrow 1 & & & & & & \\ 0 & \rightarrow & k & \xleftarrow[1]{} & k & \xrightarrow[1]{} & k & \rightarrow & 0 \end{array}\right). \end{array}$$

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