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# ON ALGEBRAS OF SECOND LOCAL TYPE, I

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Throughout this paper, $A$ denotes a (left and right) artinian ring with identity 1, $J$ its Jacobson radical and all modules are (unital and) finitely generated.

Let $n$ be any natural number. Then we say that $A$ is of *right n-th local type* in case for every indecomposable right $A$-module $M$, the $n$-th top $\mathrm{top}^n M := M/MJ^n$ of $M$ is indecomposable. (Note that if $\mathrm{top}^n M$ is indecomposable, then so is $M$ since $A$ is artinian and $M$ is finitely generated.) Hence for such a ring $A$, the question of indecomposability of right $A$-modules can be reduced to the corresponding problem of right $A/J^n$-modules. In [11] H. Tachikawa has studied the case $n=1$ and obtained a necessary and sufficient condition for algebras (by algebra we always mean a finite dimensional algebra over a field $k$) to be of this type. Further the representation theory of algebras with square-zero radical is well known [5], [6], [7]. So in this paper, we examine the case $n=2$ and give some necessary conditions for rings with selfduality to be of this type. Further in particular for $QF$ (=quasi-Frobenius) rings, we give necessary and sufficient conditions to be of this type. More precisely, we show the following two theorems:

**Theorem 1.** *Let $A$ be a ring with selfduality which is of right 2nd local type and $e$ any primitive idempotent in $A$. Then*

(1) *$J^2e$ is a uniserial waist in $Ae$ if $J^2e \neq 0$* (see section 2 for definition of a waist),

(2) *$eJ^m$ is a direct sum of local modules for every $m \geq 2$,*

(3) *for each local direct summand $L$ of $eJ^2$, $LJ^2$ is uniserial (thus $eJ^4$ is a direct sum of uniserial modules).*

*Further if $A$ is an algebra, we have*

(4) *$Ae$ is uniserial if $h(Ae) \geq 5$.*

*In particular if the base field $k$ is, in addition, an algebraically closed field, then*

(5) *$Ae$ is uniserial if $h(Ae) \geq 4$,*

*and then*

(6) *$eJ^2$ is a direct sum of uniserial modules.*

**Theorem 2.** *Let $A$ be a QF ring. Then the following statements are equivalent:*

(1) *$A$ is of right 2nd local type.*

(2) *$A$ is of right 2nd colocal type* (see section 1 for definition).

(3) *For any primitive idempotent $e$ in $A$, $eA$ is uniserial if $h(eA)\geqslant 4$.*

(4) *$A/J^t$ is QF for every $t\geqslant 3$.*

(5) *For each $M_A$ indecomposable with $h(M)\geqslant 3$, there is a primitive idempotent $e$ in $A$ such that $M\cong eA/eJ^{h(M)}$.*

(6) *$A=A_1\times A_2$ for some QF rings $A_1$ and $A_2$ such that $A_1$ has cube-zero radical and $A_2$ is a serial ring.*

*Furthermore, each of these conditions are equivalent to the corresponding left side version.*

In the theorems above $h(M)$ denotes the height ($=$Loewy length) of $M$, namely $h(M):=\min\{n\in \boldsymbol{N}\cup\{0\}\mid MJ^n=0\}$. We remark that Theorem 1 (5) and (6) remain valid also in the case where $k$ is a splitting field for $A$.

In section 1, we introduce the basic tools used in the following sections. Section 2 is devoted to the structure of an indecomposable projective left module and in section 3, we examine the structure of an indecomposable projective right module mainly using the technique of Sumioka [10]. In section 4, we give the proof of Theorem 2. Finally in section 5, we give some examples.

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## 1. Preliminaries

**1.1.** Throughout the paper, we write homomorphisms on the opposite side to scalar multiplications, and for homomorphisms $p$: $K\to L$ and $q$: $L\to M$ of left $A$-modules and for a decomposition $D$: $L=\bigoplus_{i=1}^{n} L_i$ of $L$, $(p, D)=(p_i)_{n=1}^{i}$ and $(D, q)=(q_i)^T{}_{i=1}^{n}$ are matrix expressions of $p$ and $q$ relative to $D$, respectively (for homomorphisms of right $A$-modules, we write as $(p, D)=(p_i)^{T}{}_{i=1}^{n}$ and $(D, q)$ $=(q_i)_{i=1}^{n}$). In addition to the definition of right $n$-th local type for $n$ any natural number, we define the dual notion: $A$ is called to be of *left $n$-th colocal type* in case for every indecomposable left $A$-module $M$, the $n$-th socle $\mathrm{soc}^n M:=$ (the right annihilator of $J^n$ in $M$) of $M$ is indecomposable. It should be noted that if $A$ has a selfduality, then $A$ is of right $n$-th local type iff $A$ is of left $n$-th colocal type. Further noting that the composition lengths of the projective covers (over $A$) of all indecomposable right $A/J^n$-modules have a bound if $A/J^n$ is of finite representation type (i.e. it has only finitely many isomorphism classes of indecomposable right modules), we see easily that when $A$ is of right $n$-th local type, $A$ is of finite representation type iff so is $A/J^n$ (See

Auslander [3]).

Since the property to be of $n$-th local (colocal) type is Morita invariant, we may assume that $A$ is a basic ring. We put $\mathrm{pi}(A):=\{e_1, \cdots, e_p\}$ to be a basic set of primitive idempotents of $A$.

DEFINITION 1.2 ([2]). Let $D$: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of a right $A$-module $L$ and $p$: $K\to L$ be a homomorphism, and $j$ in $\{1, \cdots, n\}$. Then the pair $(p, D)$ (or simply $p$: $K\to\bigoplus_{i=1}^{n} L_i$) is called *j-fusible* in case there is a homomorphism $q$: $\bigoplus_{i\neq j} L_i\to L_j$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{(p_i)_{i\neq j}} & \bigoplus_{i\neq j} L_i \\ \| & & \downarrow q \\ K & \xrightarrow{\;p_j\;} & L_j \end{array}$$

commutes where $(p, D)=(p_i)^{T}{}_{i=1}^{n}$. The pair $(p, D)$ is called *fusible* in case $(p, D)$ is $j$-fusible for some $j=1, \cdots, n$. Finally $(p, D)$ is called *infusible* in case $(p, D)$ is not fusible.

**Corollary 1.2.1** ([2, Corollary 1.4]). *Let $K_i\lneqq L_i$ for each $i=1, 2$ and $h$: $K_1\to K_2$ be an isomorphism. Define $p_1=k_1$, $p_2=k_2h$ where $k_i$: $K_i\to L_i$ is the inclusion map for each $i$. Then $h$ or $h^{-1}$ is extendable to a homomorphism $L_1\to L_2$ or $L_2\to L_1$, respectively iff $p$: $K_1\to L_1\oplus L_2$ is fusible.* //

**Proposition 1.2.2** ([2, Proposition 1.1]). *Consider an exact sequence $K\xrightarrow{p}L\xrightarrow{q}M\to 0$ of right $A$-modules and let $D$: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of $L$, $(p, D)=(p_i)^{T}{}_{i=1}^{n}$, $(D, q)=(q_i)_{i=1}^{n}$ and $j$ in $\{1, \cdots, n\}$. Then the following statements are equivalent*:

(1) *$(p, D)$ is $j$-fusible.*

(2) *There is a homomorphism $r=(r_i)_{i=1}^{n}$: $\bigoplus_{i=1}^{n} L_i\to X$ such that $rp=0$ and $r_j$ is an isomorphism.*

(3) *$q_j$ is a split monomorphism.*

Proof. See [2]. //

REMARK. In [2] the fusible maps were defined by the condition (2) above.

**Proposition 1.3** *Let $0\to K\xrightarrow{p}L\xrightarrow{q}M\to 0$ be a nonsplit exact sequence of right $A$-modules and $D$: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of $L$ $(n\geqslant 2)$. Then we have*

(1) *if $M$ is indecomposable, then $(p, D)$ is infusible,*

(2) *if $K$ is simple, each $L_i$ is local and $(p, D)$ is infusible, then $M$ is indecomposable.*

Proof. See [1] or [2]. //

**1.4.** Let $I$ be a two-sided ideal of $A$ and $e$ and $f$ in pi($A$). Then we have the canonical isomorphisms $\mathrm{Hom}_A(fA, eA/eI) \xrightarrow{\sim} eAf/eIf \xrightarrow{\sim} \mathrm{Hom}_A(Ae, Af/If)$. We denote by $p^*$ the image of every $p$ in $\mathrm{Hom}_A(fA, eA/eI)$ or the inverse image of every $p$ in $\mathrm{Hom}_A(Ae, Af/If)$ under the composition of these isomorphisms.

**Proposition 1.4.1** *Let $e, f_1, \cdots, f_n$ be in* pi($A$), *$l>m$, $j$ in $\{1, \cdots, n\}$ and $p=(p_i)_{i=1}^n: \bigoplus_{i=1}^n f_iA \to eJ^m/eJ^l$ be a homomorphism. Then the following statements are equivalent*:

(1) $p(f_jA) \leqslant \sum_{i \neq j} p(f_iA)$.

(2) *$p^*: Ae/J^{l-m}e \to \bigoplus_{i=1}^n Af_i/J^lf_i$ is $j$-fusible, where $p^*$ is the map induced by the homomorphism $(p_i^*)_{i=1}^n$.*

Proof. There is some $u_i$ in $eJ^mf_i$ such that each $p_i^*$ is the left multiplication by $u_i$. Then $p$ has the property stated in (1) iff $(u_jf_jA+eJ^l)/eJ^l \leqslant (\sum_{i \neq j} u_if_iA + eJ^l)/eJ^l$

iff $u_jA \leqslant \sum_{i \neq j} u_iA + eJ^l$

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some $a_i$ in $f_iA$ and $b$ in $eJ^l$

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some $a_i$ in $f_iAf_j$ and $b$ in $eJ^lf_j$

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some $a_i$ in $f_iAf_j$ and $b$ in $J^lf_j$

iff $p^*$ is $j$-fusible. //

In future $p^*$ shall always mean the above induced homomorphism when the domain of $p$ is of the form as above.

**Corollary 1.4.2.** *Under the same situation as above but $l=m+1$, the following are equivalent*:

(1) *$\bar{p}: \bigoplus_{i=1}^n f_iA/f_iJ \to eJ^m/eJ^{m+1}$ (the induced map) is a monomorphism.*

(2) *$p^*: Ae/Je \to \bigoplus_{i=1}^n Af_i/J^{m+1}f_i$ is infusible.*

*In particular if $p: \bigoplus_{i=1}^n f_iA \to eJ^m$ is a projective cover of $eJ^m$, then $p^*: Ae/Je \to \bigoplus_{i=1}^n Af_i/J^{m+1}f_i$ is infusible.* //

**Corollary 1.4.3.** *Let $p: \bigoplus_{i=1}^n f_iA \to eJ^m$ be a projective cover of $eJ^m$ and* $0 \to$

$Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^{n} Af_i/J^{m+1}f_i \to M \to 0$ *be an exact sequence. Then M is indecomposable.*

Proof. Clear from (1.4.2) and (1.3). //

## 2. Structure of an indecomposable projective left module

For an $A$-module $M$, we put $|M| :=$ the composition length of $M$.

**Proposition 2.1.** *Let $A$ be of right $n$-th local type, $n$ any natural number and $e$ in* pi$(A)$. *Then $J^n e$ is uniserial.*

Proof. It is sufficient to prove that $|J^m e/J^{m+1}e| \leqslant 1$ for every $m \geqslant n$. Suppose $|J^m e/J^{m+1}e| \geqslant 2$ for some $m \geqslant n$. Then we have a homomorphism $p: Af_1 \oplus Af_2 \to J^m e/J^{m+1}e$; $f_1, f_2$ in pi$(A)$ such that the induced map $\bar{p}$: $(Af_1/Jf_1) \oplus (Af_2/Jf_2) \to J^m e/J^{m+1}e$ is a monomorphism. Putting $L = (f_1 A/f_1 J^{m+1}) \oplus (f_2 A/f_2 J^{m+1})$, we have an exact sequence $0 \to eA/eJ \xrightarrow{p^*} L \to M \to 0$ where $M$ is indecomposable by (1.4.2) and (1.3). But since $p^*(eA/eJ) \leqslant LJ^m \leqslant LJ^n$, top$^n M \cong$ top$^n L$ is decomposable. This is a contradiction. //

DEFINITION 2.2 ([4]). Let ${}_A L \leqslant {}_A M$. Then $L$ is called to be a *waist* in $M$ in case $0 \neq L \neq M$ and for each ${}_A N \leqslant {}_A M$, it holds that $L \leqslant N$ or $N \leqslant L$.

**Proposition 2.2.1.** *Let $A$ be a ring with selfduality which is of right 2nd local type and $e$ in* pi$(A)$. *Then $J^2 e$ is a waist in $Ae$ if $J^2 e \neq 0$.*

Proof. Deduced from the following three lemmas for an artinian ring $A$:

**Lemma 2.2.2** ([9, Lemma 1.2]). *Let ${}_A M$ be nonsimple indecomposable. Then* soc$(JM)$=soc $M$.

Proof. Let $S$ be any simple submodule of $M$ and $X$ be any proper submodule of $M$. If $S+X=M$ then $S$ is not contained in $X$. Thus $S \cap X = 0$. Hence $S = M$, a contradiction. Therefore $S$ is small in $M$ i.e. $S \leqslant JM$. Hence soc $M \leqslant JM$ and soc $M$=soc$(JM)$. //

**Lemma 2.2.3.** *Let ${}_A M$ be local and* soc$^2 M$ *indecomposable. Then* soc $(J^2 M)$=soc $M$ *if $J^2 M \neq 0$.*

Proof. Clear from (2.2.2) nothing that $JM$ is nonsimple indecomposable since $J^2 M \neq 0$ and soc$^2 M \leqslant JM$. //

**Lemma 2.2.4.** *Let $A$ be a ring of left 2nd colocal type, ${}_A M$ be local and $J^2 M$ be a nonzero uniserial module. Then $J^2 M$ is a waist in $M$.*

Proof. Suppose that $J^2 M$ is not a waist in $M$. Then for some $X \leqslant M$,

$J^2M \nleqslant X$ and $X \nleqslant J^2M$. And, $J^2M \cap X = J^tM$ for some $t \geqslant 3$. Hence $M/J^t \geqslant (J^2M/J^tM) \oplus (X/J^tM)$ where $J^2M/J^tM \neq 0$ and $X/J^tM \neq 0$. On the other hand since $\mathrm{soc}^2(M/J^tM)$ is indecomposable and $J^2(M/J^tM) \neq 0$, we have that $\mathrm{soc}(M/J^tM) = \mathrm{soc}(J^2M/J^tM)$ is simple by (2.2.3). This is a consradiction. //

We get Theorem 1 (1) from Propositions 2.1 and 2.2.1.

**Corollary 2.2.5.** *Let $A$ be a ring with selfduality which is of right 2nd local type, $e$ in* pi$(A)$ *and $h = h(Ae)$. Then we have $\mathrm{soc}^{h-t}(Ae) = J^te$ for every $t = 0, \cdots, h$.*

Proof. It is clear from Theorem 1 (1) in case $t \geqslant 2$. The other cases ($t = 0$, 1) are trivial.

**Lemma 2.3.1.** *Let ${}_AL_1$ and ${}_AL_2$ be local of height $\geqslant 3$ such that for each $i = 1$, 2, $\mathrm{soc}^3L_i$ is uniserial and $J^2e_i$ is a uniserial waist in $Ae_i$ where $Ae_i$ is the projective cover of $\mathrm{soc}^3L_i$. Suppose that ${}_AK$ is simple and there exists an isomorphism $p_i$: $K \to \mathrm{soc}\, L_i$ for each $i = 1$, 2. Consider an exact sequence:*

$$0 \to K \xrightarrow{p=(p_1,\, p_2)} L_1 \oplus L_2 \xrightarrow{q=\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} M \to 0\,.$$

*Then $\mathrm{soc}^2M$ is decomposable if $p$: $K \to \mathrm{soc}^2L_1 \oplus \mathrm{soc}^2L_2$ is fusible.*

Proof. Assume that $p$: $K \to \mathrm{soc}^2L_1 \oplus \mathrm{soc}^2L_2$ is fusible, say 2-fusible. Then we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \mathrm{soc}^2L_1 \\ \| & & \downarrow r \\ K & \xrightarrow{p_2} & \mathrm{soc}^2L_2\,. \end{array}$$

And, $M \geqslant (\mathrm{soc}^2L_1)q_1 + L_2q_2 = U \oplus L_2q_2$ where $U = (\mathrm{soc}^2L_1)\,(q_1 - rq_2) \neq 0$. Now for each $x$ in $\mathrm{soc}^2M$, $x = l_1q_1 + l_2q_2$ for some $(l_1, l_2)$ in $L_1 \oplus L_2$. Since $ux = 0$ for each $u$ in $J^2$, we have $ul_1q_1 = -ul_2q_2 \in L_1q_1 \cap L_2q_2 = Kp_1q_1$ ($=: S$). Hence $J^2l_1q_1 = J^2l_2q_2 \leqslant S$ where $S$ is simple. In particular, $\mathrm{soc}^2M \leqslant \mathrm{soc}^3L_1q_1 + \mathrm{soc}^3L_2q_2$.

i) In case for each $x$ in $\mathrm{soc}^2M$, there are $l_1$, $l_2$ with $x = l_1q_1 + l_2q_2$ such that $J^2l_1q_1 = J^2l_2q_2 = 0$. Then we have $J^2l_1 = 0$ for $q_1$ is monic. Thus $l_1$ is in $\mathrm{soc}^2L_1$ and $x$ is in $U \oplus L_2q_2$. Therefore $\mathrm{soc}^2M \leqslant U \oplus L_2q_2$. Hence $\mathrm{soc}^2M$ is decomposable.

ii) In case for some $x$ in $\mathrm{soc}^2M$, there are $l_1$, $l_2$ with $x = l_1q_1 + l_2q_2$ such that $J^2l_1q_1 = J^2l_2q_2 = S$. We may assume that $x = ex$ for some $e$ in pi$(A)$. Since $S$ is simple and $q_i$ are monic, $J^3l_1 = J^3l_2 = 0$. Thus $l_i$ is in $\mathrm{soc}^3L_i \backslash \mathrm{soc}^2L_i$ for each $i$. Also, we may assume that $l_i = el_i$ for each $i$ since $x = ex$. Further we have $\mathrm{soc}^3L_i = Ael_i$ for each $i = 1$, 2 since $\mathrm{soc}^3L_i$ are uniserial. Hence we may assume that $e=e_1=e_2$. Define a homomorphism $s$: $\mathrm{soc}^3 L_1 \to \mathrm{soc}^3 L_2$ by $ael_1 \mapsto ael_2$ for each $a$ in $A$. Then $s$ is well-defined. In fact, if $t$ is in $Ae$ and $tl_1=0$, then $t$ is in $\mathrm{Ann}_{Ae}(l_1)$, the annihilator of $l_1$ in $Ae$. On the other hand, by the fact that $J^2 el_1 \neq 0$, we see $\mathrm{Ann}_{Ae}(l_1)$ does not contain $J^2 e$ which is a uniserial waist in $Ae$. Hence $\mathrm{Ann}_{Ae}(l_1)$ is contained in $J^3 e$ and $t$ is in $J^3 e$. Thus $tl_2$ is in $J^3 l_2=0$.

Further the diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \mathrm{soc}^3 L_1 \\ \| & & \downarrow s \\ K & \xrightarrow{p_2} & \mathrm{soc}^3 L_2 \end{array}$$

is commutative. For, $J^2(l_1, l_2)$ $(\neq 0)$ is contained in the simple module $\mathrm{Im}\, p$ since $J^2(l_1, l_2)q=0$. Hence $J^2(l_1, l_2)=\mathrm{Im}\, p$. Let $c$ be a nonzero element in $K$. Then $K=Ac$ and $cp=(ul_1, ul_2)$ for some $u$ in $J^2$. Therefore $c(p_1 s)=ul_1 s=ul_2$ $=cp_2$. Thus $p_1 s=p_2$.

Then putting $V:=(\mathrm{soc}^3 L_1)\,(q_1-sq_2)$, the same argument as in i) shows that $\mathrm{soc}^2 M \leqslant V \oplus L_2 q_2$ and $\mathrm{soc}^2 M$ is decomposable. //

**Proposition 2.3.2.** *Let $A$ be a ring with selfduality which is of right 2nd local type and ${}_A L_1$, ${}_A L_2$ be local of height $\geqslant 3$ such that $\mathrm{soc}^3 L_i$ are uniserial and $|L_1| \leqslant |L_2|$. Then for every isomorphism $r$: $\mathrm{soc} L_1 \to \mathrm{soc} L_2$, $r$ is extendable to a monomorphism $L_1 \to L_2$ if $r$ is extendable to a homomorphism $\mathrm{soc}^2 L_1 \to \mathrm{soc}^2 L_2$.*

Proof. Put $K=\mathrm{soc}\, L_1$, $p_1=$identity map of $\mathrm{soc}\, L_1$ and $p_2=r$. Consider an exact sequence $0 \to K \xrightarrow{p=(p_1,\, p_2)} L_1 \oplus L_2 \xrightarrow{q} M \to 0$. If $r$ is extendable to a homomorphism $\mathrm{soc}^2 L_1 \to \mathrm{soc}^2 L_2$, then $p$: $K \to \mathrm{soc}^2 L_1 \oplus \mathrm{soc}^2 L_2$ is fusible. Hence by (2.3.1), $\mathrm{soc}^2 M$ is decomposable, thus $M$ is decomposable. Therefore $p$: $K \to L_1 \oplus L_2$ is fusible by (1.3). Hence by (1.2.1), $r$ is extendable to a homomorphism $q$: $L_1 \to L_2$ since $|L_1| \leqslant |L_2|$ where $q$ is monic since $\mathrm{soc}\, L_1$ is simple. //

**2.4.** Throughout the rest of this section, $A$ is a ring with selfduality which is of right 2nd local type. Here, we examine indecomposable projective left $A$-modules of height $\geqslant 4$.

**Proposition 2.4.1.** *Let $e$ and $f$ be in $\mathrm{pi}(A)$ and $fJe/fJ^2 e \neq 0$. Then $Af$ is uniserial if $h(Ae) \geqslant 4$.*

Proof. Take some $u$ in $fJe \backslash fJ^2 e$ and define $p$: $Af \to Je$ by the right multiplication by $u$. Then $\mathrm{Ker}\, p \leqslant J^2 f$ or $\mathrm{Ker}\, p \geqslant J^2 f$ since $J^2 f$ is a waist in $Af$ (if $J^2 f \neq 0$). Assume that $\mathrm{Ker}\, p \geqslant J^2 f$. Then $h(\mathrm{Im}\, p) \leqslant 2$ since $\mathrm{Im}\, p \cong Af/\mathrm{Ker}\, p$ is an epimorph of $Af/J^2 f$. Hence $\mathrm{Im}\, p \leqslant \mathrm{soc}^2(Ae) \leqslant J^2 e$ for $h\,(Ae) \geqslant 4$ and $\mathrm{soc}^2$ $(Ae)=J^{h(Ae)-2}e$. But by the definition of $p$ we have $\mathrm{Im}\, p \not\leqslant J^2 e$, a contradiction.

Accordingly, $\operatorname{Ker} p \leqslant J^2 f$. Then $\operatorname{Ker} p = J^t f$ for some $t \geqslant 2$ and $Af/J^t f$ is embedded into $Je$. Therefore $|Jf/J^2 f|=1$ since $Jf/J^t f$ is embedded into $J^2 e$ which is uniserial. Hence $Af$ is uniserial. //

**Proposition 2.4.2.** *Assume that e is in* pi$(A)$, $h(Ae) \geqslant 4$ *and Ae is not uniserial. Then*

(1) *all simple submodules of* $Je/J^2 e$ *are pairwise isomorphic, and*

(2) $J^2 e/J^3 e \cong J^3 e/J^4 e$.

Proof. Let $u\colon \bigoplus_{i=1}^{n} Af_i \to Je/J^4 e$ be a projective cover of $Je/J^4 e$. Then $n \geqslant 2$ since $Ae$ is not uniserial. Putting $L_i := (Af_i)u$, we have $L_i \cap L_j = J^2 e/J^4 e$, $L_i \gneqq J^2 e/J^4 e$ for each $i \neq j$ in $\{1, \cdots, n\}$. By (2.4.1), each $L_i$ is uniserial and $h(L_i)=3$. Further $\operatorname{soc} L_i = J^3 e/J^4 e$ is simple and $\operatorname{soc}^2 L_i = J^2 e/J^4 e$ for each $i=1, \cdots, n$.

(1) For any $i \neq j$ in $\{1, \cdots, n\}$, the identity map $p\colon \operatorname{soc} L_i \to \operatorname{soc} L_j$ is extendable to a homomorphism $\operatorname{soc}^2 L_i \to \operatorname{soc}^2 L_j$ since $L_i \cap L_j = J^2 e/J^4 e = \operatorname{soc}^2 L_i = \operatorname{soc}^2 L_j$. Hence by (2.3.2), $p$ is extendable to an isomorphism $L_i \to L_j$. Thus all simple submodules of $Je/J^2 e$ are pairwise isomorphic.

(2) Putting $p_i\colon J^2 e/J^4 e \to L_i$ and $q_i\colon L_i \to L_1 + L_2$ to be inclusion maps for $i=1, 2$, we have an exact sequence

$$0 \to J^2 e/J^4 e \xrightarrow{(p_1,\, p_2)} L_1 \oplus L_2 \xrightarrow{\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} L_1 + L_2 \to 0$$

where $L_1 + L_2$ is colocal. Hence the identity map $r\colon \operatorname{soc}^2 L_1 \to \operatorname{soc}^2 L_2$ is not extendable to any isomorphism $L_1 \to L_2$. On the other hand, the identity map $p\colon \operatorname{soc} L_1 \to \operatorname{soc} L_2$ is extendable to an isomorphism $s\colon L_1 \to L_2$ since $r|(\operatorname{soc} L_1) = p$. As a consequence, $s|(\operatorname{soc}^2 L_1) \neq r$. But if $J^2 e/J^3 e \ncong J^3 e/J^4 e$, then the restriction map

$$\operatorname{Hom}_A(\operatorname{soc}^2 L_1, \operatorname{soc}^2 L_2) \to \operatorname{Hom}_A(\operatorname{soc} L_1, \operatorname{soc} L_2)$$

is an injection. This implies that $s|(\operatorname{soc}^2 L_1) = r$ since both $s|(\operatorname{soc}^2 L_1)$ and $r$ are extensions of $p$. This is a contradiction. //

**Proposition 2.4.3.** *Assume that e, f and g are in* pi$(A)$, $h(Ae) \geqslant 5$, *Ae is not uniserial,* $fJe/fJ^2 e \neq 0$ *and* $J^2 e/J^3 e \cong Ag/Jg$. *Then* $fAf/fJf \cong gAg/gJg$ *as rings.*

Proof. There exists a submodule $L$ of $Je/J^4 e$ such that $L$ is uniserial of height 3 and top $L \cong Af/Jf$, top $JL \cong Ag/Jg$. We identify these isomorphic modules. Further $Af$ and $Ag$ are both uniserial by (2.4.1) and the fact that $h(Ae) \geqslant 5$ and also $h(Af) \geqslant 4$. Then we can define a homomorphism $t\colon \operatorname{End}_A(Af/Jf) \to \operatorname{End}_A(Ag/Jg)$ by $t(p) := \overline{(q\,|\,Jf/J^3 f)}$ for each $p$ in $\operatorname{End}_A(Af/Jf)$ where $p$ is induced by some $q$ in $\operatorname{End}_A(Af/J^3 f)$ and $\bar{r}$ is the map in $\operatorname{End}_A(Jf/J^2 f)$ induced by $r$ for every $r$ in $\operatorname{End}_A(Jf/J^3 f)$. (We identified $\operatorname{End}_A(Jf/J^2 f) = \operatorname{End}_A(Ag/Jg)$.)

Then $t$ is well-defined and injective since for each $q$ in $\mathrm{End}_A(Af/J^3f)$, $(Af/J^3f)q \leqslant Jf/J^3f$ iff $(Jf/J^3f)q \leqslant J^2f/J^3f$ (See [10, section 3]). Further by (2.3.2), every automorphism $p$ of soc $L$ is extendable to an automorphism of $L$ if $p$ is extendable to an automorphism of $\mathrm{soc}^2 L$. Thus $t$ is surjective. (Note that both $Af/J^3f$ and $Jf/J^3f$ are quasi-projective since we have $Jf/J^3f \cong Ag/J^2g$ from the fact that $Ag$ is uniserial.) Hence $fAf/fJf \cong \mathrm{End}_A(Af/Jf) \cong \mathrm{End}_A(Ag/Jg) \cong gAg/gJg$ as rings.

REMARK. In the above, if $A$ is a $k$-algebra, then the isomorphism defined as above is a $k$-algebra isomorphism.

**2.4.4.** Proof of Theorem 1 (4) and (5). Assume that $A$ is an algebra and suppose that $Ae$ is not uniserial, and $h(Ae) \geqslant 4$. Let $p\colon \bigoplus_{i=1}^{n} P_i \to Je/J^3e$ be a projective cover of $Je/J^3e$ where each ${}_AP_i$ is indecomposable. Then $n \geqslant 2$. By (2.4.2), there is an $f$ in $\mathrm{pi}(A)$ such that every $P_i$ is isomorphic to $Af$. And, $J^2e/J^3e \cong Ag/Jg$ for some $g$ in $\mathrm{pi}(A)$. If we put $L_i := (P_i)p$ for $i=1, 2$, then $L_i \cong Af/J^2f$, $J^2e/J^3e \lneqq L_i \leqslant Je/J^3e$, $L_1 \cap L_2 = J^2e/J^3e$ and top $L_i \cong Af/Jf$ for each $i=1, 2$. Since we have an exact sequence

$$0 \to J^2e/J^3e \to L_1 \oplus L_2 \to L_1 + L_2 \to 0$$

where $J^2e/J^3e \cong Ag/Jg$, $L_1 \oplus L_2 \cong (Af/J^2)^{(2)}$ and $L_1 + L_2$ is colocal, there exists an infusible homomorphism $Ag/Jg \to (Af/J^2f)^{(2)}$ by (1.3; 1). Therefore $(fA/fJ)^{(2)}$ is isomorphic to a direct summand of $gJ/gJ^2$ by (1.4.2). Hence $\dim (gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. If $h(Ae) \geqslant 5$ or $k$ is algebraically closed, then by (2.4.3), $d := \dim_{gAg/gJg}(gJf/gJ^2f) = \dim (gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. Hence $(Ag/Jg)^{(d)}$ is isomorphic to a direct summand of $Jf/J^2f$ and $d \geqslant 2$. Thus $|Jf/J^2f| \geqslant 2$. This contradicts the uniseriality of $Af$. Hence $Ae$ must be uniserial. //

## 3. Structure of an indecomposable projective right module

**Lemma 3.1.** *Let* $0 \to K \xrightarrow{p} L \xrightarrow{q} M \to 0$ *be an exact sequence of left $A$-modules such that $K$ is simple,* $D\colon L = \bigoplus_{i=1}^{n} L_i$ *is a decomposition of $L$* $(n \geqslant 2)$ *and for each* $i=1, \cdots, n$, $L_i = Ae_i/I_i$ *for some* $e_i$ *in* $\mathrm{pi}(A)$ *and* $J^{m+1}e_i \leqslant I_i \lneqq J^m e_i$ $(m \geqslant 1)$. *Then* $JM = \mathrm{soc}^m M$ *if* $(p, D)$ *is infusible.*

Proof. Put $l_i := e_i + I_i$, $\bar{l}_i = l_i + JL$, $m_i := l_i q$, $\bar{m}_i := m_i + JM$ and $m_i' := m_i + \mathrm{soc}^m M$. Then we have $\bigoplus_{i=1}^{n} A\bar{l}_i = L/JL \cong M/JM = \bigoplus_{i=1}^{n} A\bar{m}_i$ where each $A\bar{m}_i$ is simple. It follows from $h(M) \leqslant m+1$ that $JM \leqslant \mathrm{soc}^m M$. Assume that $JM \lneqq \mathrm{soc}^m M$. Then we show that $(p, D)$ is fusible. (Clearly, we may assume that each $p_i \neq 0$ i.e. each $p_i$ is a monomorphism where $(p, D) = (p_i)_{i=1}^{n}$.) By assumption the sum $M/\mathrm{soc}^m M=\sum_{i=1}^{n} Am'_i$ is redundant i.e. $Am'_j \leqslant \sum_{i\neq j} Am'_i$ for some $j$, say $j=1$. So $m'_1=\sum_{i\neq 1} -a_i m'_i$ for some $a_i$ in $A$. By putting $a_1=1$, we have $\sum_{i=1}^{n} a_i m_i \in \mathrm{soc}^m M$ and $J^m(a_i l_i)_{i=1}^{n}\cdot q=0$. Thus $J^m(a_i l_i)_{i=1}^{n} \leqslant \mathrm{Im}\, p$. Further putting $e:=e_1$ we may assume that $a_i=ea_i$ for each $i\neq 1$. Put $l:=(a_i l_i)_{i\neq 1}$. Then we have $l_1\in L_1$, $l\in \bigoplus_{i\neq 1} L_i$, $l_1=el_1$, $l=el$ and $J^m(l_1, l)\leqslant \mathrm{Im}\, p$. On the other hand, it holds that $J^m(l_1, l)\neq 0$ since we have $J^m l_1\neq 0$ by the assumption $I_i \lneqq J^m e_i$. Accordingly, $J^m(l_1, l)=\mathrm{Im}\, p$ since $\mathrm{Im}\, p$ is simple. Define a map $r\colon L_1\to \bigoplus_{i\neq 1} L_i$ by $xl_1 \mapsto xl$ for each $xl_1\in L_1$. Then $r$ is well-defined. In fact, if $xl_1=0$, then $xe\in I_1\leqslant J^m$ and then $xe(l_1, l)\in \mathrm{Im}\, p$. Thus $xe(l_1, l)=sp$ for some $s$ in $K$. Therefore $sp_1=xel_1=xl_1=0$ and $s(p_i)_{i\neq 1}=xel$. But since $p_1$ is a monomorphism, we have $s=0$ and $xl=xel=0$. Further by the similar argument as in (2.3.1), $p_1 r=(p_i)_{i\neq 1}$ i.e. $(p, D)$ is fusible. //

**Proposition 3.2.** *Let $A$ be a ring with selfduality which is of right 2nd local type, $m\geqslant 2$, $e, f_1, \cdots, f_n (n\geqslant 2)$ in* pi($A$) *and $p\colon \bigoplus_{i=1}^{n} f_i A\to eJ^m/eJ^{m+1}$ be a projective cover of $eJ^m/eJ^{m+1}$. Then $p^*\colon Ae/Je\to \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is infusible.*

Proof. Let $0\to Ae/Je \xrightarrow{p^*} \bigoplus_{i+1}^{n} Af_i/J^{m+1}f_i \to M\to 0$ be an exact sequence. Then $M$ is indecomposable by (1.4.3). By (3.1), $JM=\mathrm{soc}^m M$. Accordingly, $JM$ is indecomposable since $JM\geqslant \mathrm{soc}^2 M$ and $\mathrm{soc}^2 M$ is indecomposable. Then from the exact sequence $0\to Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i\to JM\to 0$, we obtain that $p^*\colon Ae/Je \to \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is infusible by (1.3). //

**3.3.** Proof of Theorem 1 (2). Let $p\colon \bigoplus_{i=1}^{n} f_i A\to eJ^m$ be a projective cover of $eJ^m$ and $f_i$ in pi($A$) for each $i=1, \cdots, n$. If $n=1$, then the assertion is trivial. So we may assume that $n\geqslant 2$. There is some $u_i$ in $eJ^m f_i \backslash eJ^{m+1} f_i$ such that the $i$-th coordinate map of $p$ is the right multiplication by $u_i$ for each $i=1, \cdots, n$. Put $\bar{u}_i:=u_i+eJ^{m+1}$, $u'_i:=u_i+J^{m+1}f_i$ and $e':=e+Je$. Then $eJ^m=\sum_{i=1}^{n} u_i A$ where each $u_i A$ is local. Suppose that $eJ^m$ is not a direct sum of local modules. Then $\sum_{i=1}^{n} u_i a_i=0$ for some $a_i$ in $A$ and $u_j a_j\neq 0$ for some $j=1, \cdots, n$. We may assume that there is some $g$ in pi($A$) such that $u_j a_j g\neq 0$ and $a_i=f_i a_i g$ for each $i=1, \cdots, n$. Then it holds that $a_i$ is in $f_i Jg$ for each $i$. In fact, if $f_i\neq g$, then $a_i\in f_i Ag=f_i Jg$. And, in case $f_i=g$, we have $f_i Ag/f_i Jg=f_i Af_i/f_i Jf_i$ is a division ring. Furthermore, $\sum_{i=1}^{n} u_i a_i=0$ implies $\sum_{i=1}^{n} \bar{u}_i a_i=0$ and hence each $\bar{u}_i a_i=0$, since $\bar{u}_i A$ are independent. Then putting $\bar{a}_i:=a_i+f_i Jg$, we have that $\bar{u}_i\bar{a}_i$ is defined and is zero. Hence if $a_i$ is not in $f_i Jg$, then $\bar{u}_i=(\bar{u}_i\bar{a}_i)\bar{a}_i^{-1}=0$, a contradiction. Further $Au_i=J^m f_i$ since $J^m f_i$ is uniserial for $m\geqslant 2$. Therefore we may assume that $Au_ia_i\leqslant Au_na_n$ for each $i$ and $Au_na_n=J^sg$ for some $s\geqslant m+1\geqslant 3$. Define a homomorphism $q_i$: $Af_i/J^{m+1}f_i\to Ag/J^{s+1}g$by $x\mapsto xa_i$ for each $i=1, \cdots, n$. Then $q_n$ is a monomorphism since $\mathrm{soc}(Af_n/J^{m+1}f_n)=J^m f_n/J^{m+1}f_n$ is simple and is mapped by $q_n$ onto the simple module $J^sg/J^{s+1}g$. Further putting $q_i':=q_i|(Jf_i/J^{m+1}f_i)$, we have $\mathrm{Im}\ q_i'\leqslant\mathrm{soc}^m(Jg/J^{s+1}g)=J^{s+1-m}g/J^{s+1}g=\mathrm{Im}\ q_n'$ for each $i=1, \cdots, n$. Hence if we put $q_i'':=q_i': Jf_i/J^{m+1}f_i\to J^{s+1-m}g/J^{s+1}g$ and $q:=(q_i'')^T{}_{i=1}^{n}$, then $p^*$: $Ae/Je\to\bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is fusible since $e'p^*q=0$ and $q_n''$ is an isomorphism. This contradicts (3.2). Hence $eJ^m$ must be a direct sum of local modules.

**3.4.** Proof of Theorem 1 (3) and (6). Suppose that $|LJ^s/LJ^{s+1}|\geqslant 2$ for some $s\geqslant 1$. $LJ^s$ is a direct sum of local modules for $LJ^s$ is a direct summand of $eJ^{2+s}$. Further $L=vA$ for some $v$ in $eJ^2g\backslash eJ^3g$ and for some $g$ in $\mathrm{pi}(A)$. Hence $LJ^s=vJ^s=u_1A\oplus u_2A\oplus\cdots$ for some $u_i$ in $eJ^{2+s}f_i\backslash eJ^{3+s}f_i$ where $f_i$ are in $\mathrm{pi}(A)$. Then for each $i=1, 2$, there is some $a_i$ in $gJ^sf_i$ such that $u_i=va_i$. Define a map $p_i$: $Ag/J^3g\to Af_i/J^{s+3}f_i$ by $x\mapsto xa_i$ for each $i=1, 2$. Then $p_1$ and $p_2$ are both monomorphisms since putting $v':=v+J^3g$ and $u_i':=u_i+J^{s+3}f_i$, $\mathrm{soc}(Ag/J^3g)=J^2g/J^3g=Av'$ and $\mathrm{soc}(J^sf_i/J^{s+3}f_i)=J^{s+2}f_i/J^{s+3}f_i=Au_i'$ are simple modules and $(Av')p_i=Au_i'$ for each $i=1, 2$. In particular, $Ag$ is uniserial by Theorem 1 (1).

i) In case $s\geqslant 2$. By the above,

$$Av'\xrightarrow{(p_1,\, p_2)}(J^sf_1/J^{s+3}f_1)\oplus(J^sf_2/J^{s+3}f_2)$$

is fusible. Also, $\mathrm{soc}^3(Af_i/J^{s+3}f_i)=J^sf_i/J^{s+3}f_i$ is uniserial. Hence

$$Av'\xrightarrow{(p_1,\, p_2)}(Af_1/J^{s+3}f_1)\oplus(Af_2/J^{s+3}f_2)$$

is fusible by (2.3.2), say 2-fusible. Then for some $a$ in $f_1Af_2$, the diagram

$$\begin{array}{ccl} Av' & \xrightarrow{p_1} & Af_1/J^{s+3}f_1 \\ \big\downarrow & & \big\downarrow \text{right multiplication by } a \\ Av' & \xrightarrow{p_2} & Af_2/J^{s+3}f_2 \end{array}$$

is commutative. Therefore $u_2'=u_1'a$. Putting $\bar{u}_i:=u_i+eJ^{s+3}$ for each $i=1, 2$, we have $\bar{u}_2=\bar{u}_1a$ since $u_2$ is in $u_1a+eJ^{s+3}f_2$. Thus $\bar{u}_2A\leqslant\bar{u}_1A$. This contradicts the linear independency of $\bar{u}_1A$ and $\bar{u}_2A$.

ii) In case the base field $k$ is algebraically closed. It remains only the case $s=1$. Similarly, it holds that

$$Av'\xrightarrow{(p_1,\, p_2)}(Jf_1/J^4f_1)\oplus(Jf_2/J^4f_2)$$

is fusible. But since $0 \neq u_i \in eJ^3 f_i \leqslant J^3 f_i$ for each $i=1, 2$, $h(Af_i) \geqslant 4$ and then $Af_i/J^4 f_i$ is uniserial of length 4 and $Jf_i/J^4 f_i = \mathrm{soc}^3(Af_i/J^4 f_i)$ by Theorem 1 (5). Then

$$Av' \xrightarrow{(p_1, p_2)} (Af_1/J^4 f_1) \oplus (Af_2/J^4 f_2)$$

is fusible by (2.3.2). Hence by the same argument as in i) we have a contradiction. //

## 4. *QF* rings of right 2nd local type

**Lemma 4.1.** *Let $A$ be a QF ring and $e$ and $f$ be in* $\mathrm{pi}(A)$ *such that* $fJe/fJ^2e \neq 0$. *Then*

(a) *If $Je/J^2e$ is simple, then $h(Af) \geqslant h(Ae)$; and*

(b) *If $fJ/fJ^2$ is simple, then $h(eA) \geqslant h(fA)$.*

Proof. (a). It follows from the fact that $Je/J^2e$ is simple and $fJe/fJ^2e \neq 0$ that there is an epimorphism $p$: $Af \rightarrow Je$. If $p$ is a monomorphism, then $Je$ is injective and is a direct summand of $Ae$. Thus $Je=0$ for $Je$ is small in $Ae$. But this is impossible since $Je/J^2e$ is simple. Therefore $\mathrm{Ker}\, p \geqslant \mathrm{soc}\, Af = J^{h(Af)-1}f$ since $Af$ is colocal. Hence $h(Af) \geqslant h(Je)+1 = h(Ae)$.

(b) Similar. //

**4.2.** Proof of Theorem 2. Let $(x)'$ be the left side version of $(x)$ for each $x=1, 3$. We show the following implications: $(1) \Rightarrow (3)' \Leftrightarrow (3) \Rightarrow (6) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$. Note that $(2) \Leftrightarrow (1)'$ is clear since $A$ has a selfduality. Denote by $D$ the selfduality $\mathrm{Hom}_A(?, A)$ of $A$.

$(1) \Rightarrow (3)'$. Let $e$ be in $\mathrm{pi}(A)$ and $h := h(Ae) \geqslant 4$. Then $J^2e$ is a uniserial waist in $Ae$. Hence $\mathrm{soc}^2 eA = D(Ae/J^2e)$ is a waist in $eA = D(Ae)$ and $\mathrm{soc}^2 eA = eJ^{h-2}$ is a direct sum of local modules for $h-2 \geqslant 2$. But since $eJ^{h-2} \leqslant eA$ and $eA$ is colocal, $eJ^{h-2}$ is local. Hence $|Je/J^2e| = |\mathrm{soc}^2(eA)/\mathrm{soc}(eA)| = 1$ and $Ae$ is uniserial.

$(3)' \Leftrightarrow (3)$. Clear from the fact that both height and uniseriality are preserved by $D$.

$(3) \Rightarrow (6)$. By the equivalence $(3) \Leftrightarrow (3)'$ and left-right symmetry, it is sufficient to prove that under the assumption $(3)'$, if $A$ is an indecomposable ring and $J^3 \neq 0$, then $A$ is a left serial ring. Let $Q$ be the left quiver of $A$, namely the oriented graph with vertex set $\{1, \cdots, p\}$ where $\mathrm{pi}(A) = \{e_1, \cdots, e_p\}$ and with $n_{ji}$ arrows $i \rightarrow j$ iff $\dim_{(e_jAe_j/e_jJe_j)}(e_jJe_i/e_jJ^2e_i) = n_{ji}$. Note that $A$ is an indecomposable ring iff $Q$ is connected. It follows from $J^3 \neq 0$ that $h(Ae_i) \geqslant 4$ for some $i=1, \cdots, p$ and then $Ae_i$ is uniserial by $(3)'$. By 4.1 and the selfduality $D$, we have $h(Ae_j) \geqslant h(Ae_i)$ $(\geqslant 4)$ if either

(a) there is an arrow $i \rightarrow j$; or

(b) there is an arrow $j\to i$.

Hence $Ae_j$ is uniserial of height$\geqslant 4$ for any $j=1,\cdots,p$ by (4.1), (3)$'$ and the fact that $Q$ is connected. Thus $A$ is a left serial ring.

(6)$\Rightarrow$(4). Clear from the fact that for a serial ring $A$, $A$ is $QF$ iff the admissible sequence of $A$ is constant.

(4)$\Rightarrow$(5). Let $M_A$ be indecomposable of height $h\geqslant 3$. Then $A/J^h$ is $QF$ by (4). Let $0\to K\hookrightarrow \bigoplus_{i=1}^{m} P_i\to M\to 0$ be a projective cover of $M$ over $A/J^h$ with each $P_i$ indecomposable. Then $\mathrm{soc}(\bigoplus_{i=1}^{m} P_i)\nleqslant K$ implies that soc $P_i\nleqslant K$ for some $i=1,\cdots,m$ and then $P_i\cap K=0$ since $P_i$ is colocal. Hence $P_i$ is embedded into $M$. But since $P_i$ is injective, $P_i$ is isomorphic to a direct summand of $M$. Hence $P_i\cong M$ for $M$ is indecomposable. Further $P_i\cong eA/eJ^h$ for some $e$ in pi$(A)$.

(5)$\Rightarrow$(1). Clear. //

## 5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field $k$. (See Gabriel [8] for details concerning bounden quiver algebras.)

EXAMPLE 1. Let $A$ be the algebra defined by the following bounden quiver:

$$\alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2; \quad \beta\alpha=\alpha\gamma=0\,,\ \alpha^2=\gamma\beta\,,$$

namely, the algebra having $\{e_1, e_2, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\}$ as $k$-basis and with multiplication given by the following table:

| left \ right | $e_1$ | $e_2$ | $\alpha$ | $\beta$ | $\gamma$ | $\gamma\beta$ | $\beta\gamma$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | | $\alpha$ | | $\gamma$ | $\gamma\beta$ | |
| $e_2$ | | $e_2$ | | $\beta$ | | | $\beta\gamma$ |
| $\alpha$ | $\alpha$ | | $\gamma\beta$ | | | | |
| $\beta$ | $\beta$ | | | | $\beta\gamma$ | | |
| $\gamma$ | | $\gamma$ | | $\gamma\beta$ | | | |
| $\gamma\beta$ | $\gamma\beta$ | | | | | | |
| $\beta\gamma$ | | $\beta\gamma$ | | | | | |

(each blank is zero).

Then $A$ is weakly symmetric and hence $QF$. Further as easily seen, $A$ has cube-zero radical. Therefore $A$ is of right (and left) 2nd local type by Theorem 2. But since $A$ is not a serial ring, $A$ is neither of right (1st) local type nor of left (1st) local type.

EXAMPLE 2. Let $A$ be the algebra defined by the following quiver $Q$:

$$\begin{array}{ccccccc} & & 5 & & & & \\ & & \downarrow{\scriptstyle\delta} & & & & \\ 4 & \xrightarrow{\gamma} & 1 & \xleftarrow{\alpha} & 2 & \xleftarrow{\beta} & 3\,, \end{array}$$

namely, the algebra having $\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta\}$ as $k$-basis with multiplication given by the following table:

| left \ right | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha\beta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | | | | | $\alpha$ | | $\gamma$ | $\delta$ | $\alpha\beta$ |
| $e_2$ | | $e_2$ | | | | | $\beta$ | | | |
| $e_3$ | | | $e_3$ | | | | | | | |
| $e_4$ | | | | $e_4$ | | | | | | |
| $e_5$ | | | | | $e_5$ | | | | | |
| $\alpha$ | | $\alpha$ | | | | | $\alpha\beta$ | | | |
| $\beta$ | | | $\beta$ | | | | | | | |
| $\gamma$ | | | | $\gamma$ | | | | | | |
| $\delta$ | | | | | $\delta$ | | | | | |
| $\alpha\beta$ | | | $\alpha\beta$ | | | | | | | |

(each blank is zero).

Then as easily verified, $A$ satisfies all the conditions stated in Theorem 1. But it is not of right 2nd local type. For instance, let $M$ be the right $A$-module corresponding to the following $k$-representation of $Q^{op}$ (the opposite quiver of $Q$, with all arrows reversed)

$$\begin{array}{ccccccc} & & k & & & & \\ & & \uparrow{\scriptstyle(1,0)} & & & & \\ k & \xleftarrow[(0,1)]{} & k\oplus k & \xrightarrow[\begin{pmatrix}1&0\\0&1\end{pmatrix}]{} & k\oplus k & \xrightarrow[(1,1)]{} & k \end{array}$$

namely, the module having $\{m_1, m_1', m_2, m_2', m_3, m_4, m_5\}$ as $k$-basis and with right $A$-action given by the following table:

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha\beta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $m_1$ | $m_1$ | | | | | $m_2$ | | | $m_4$ | $m_3$ |
| $m_1'$ | $m_1'$ | | | | | $m_2'$ | | $m_4$ | | $m_3$ |
| $m_2$ | | $m_2$ | | | | | $m_3$ | | | |
| $m_2'$ | | $m_2'$ | | | | | $m_3$ | | | |
| $m_3$ | | | $m_3$ | | | | | | | |
| $m_4$ | | | | $m_4$ | | | | | | |
| $m_5$ | | | | | $m_5$ | | | | | |

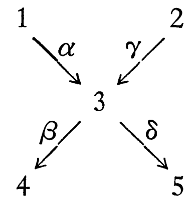
(each blank is zero).

Then $M$ is indecomposable but $\mathrm{top}^2M$ is decomposable:

$$\mathrm{top}^2M = \left[\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ k & \underset{1}{\leftarrow} & k & \underset{1}{\rightarrow} & k & \rightarrow & 0 \end{array}\right] \oplus \left[\begin{array}{ccccccc} & & k & & & & \\ & & \uparrow{\scriptstyle 1} & & & & \\ 0 & \leftarrow & k & \underset{1}{\rightarrow} & k & \rightarrow & 0 \end{array}\right].$$

Hence the conditions stated in Theorem 1 are not sufficient for algebras (even if $k$ is algebraically closed) to be of right 2nd local type.

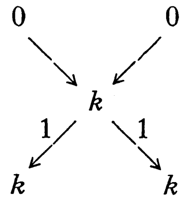
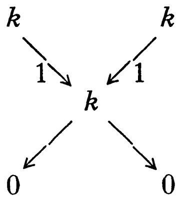
EXAMPLE 3. Let $A$ be the algebra defined by the following bounded quiver:




; $\beta\alpha = \delta\gamma = 0$.

Then we can see that $A$ has just 13 indecomposable left modules (up to isomorphism), all of which have indecomposable second tops and second socles since the indecomposable left $A$-modules of height $\geqslant 3$ are both projective and injective. Hence $A$ is of right and left 2nd local type.[1] But it is neither of right (1st) local type nor of left (1st) local type. For instance, let $M_1$ and $M_2$ be the left $A$-modules corresponding to the following $k$-representations of the bounden quiver:

1) In Part II of this series of papers, we shall give some necessary and sufficient conditions for artinian rings to be of right and left $n$-th local type for any natural number $n$. Using this result, it is clear that the algebra defined in Example 3 is of right and left 2nd local type.

 and  , respectively.

Then $M_1$ and $M_2$ are indecomposable but $M_1$ is not colocal and $M_2$ is not local.

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