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ON THE SET OF REGULAR BOUNDARY POINTS

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Introduction

Let X be a $\mathcal{P}$-harmonic space with a countable base in the sense of the axiomatics of Constantinescu and Cornea [3], U an open set of X and U_{reg} the set of regular boundary points of U. If X is a connected Brelot space, it is known that U_{reg} is dense on $\partial\bar{U}$ (see e.g. Hervé [4], Ikegami [6]). This is not valid for more general hamonic spaces. We prove two results related to this question. Assuming that the space has a base of regular sets, we obtain a necessary condition (by means of absorbent sets) for the case that U_{reg} is not dense on $\partial\bar{U}$.

1. Preliminaries

Let X be a $\mathcal{P}$-harmonic space with a countable base in the sense of Constantinescu and Cornea [3] and U an open set of X. We denote the set of regular (resp. irregular) points of ∂U by U_{reg} (resp. U_{ir}). If U is relatively compact and $M\subset\partial U$ with $\mu_x^U(M)=0$ for all $x\in U$, M is called *negligible*. Since X has a countable base, if M is negligible, $\bar{H}_{\chi_M}^U(x)=\mu_x^U(M)=0$ for all $x\in U$ (cf. [2, Satz 4.1.7]).

REMARK 1.1. Let $y\in\partial U$. A strictly positive hyperharmonic function u defined on the intersection of U and an open neighbourhood V of y is called a barrier at y if

$$\lim_{U\cap V\ni z\to y} u(z)=0\,.$$

Then $y\in U_{reg}$ if and only if there exists a barrier at y. This follows from [3, Proposition 2.4.7], [3, Theorem 6.3.3] and [3, Proposition 7.2.2]. Thus $y\in U_{reg}$ implies that for every open subset U' of U with $y\in\partial U'$, we have $y\in U'_{reg}$.

A relatively compact open set U is called a *Keldyš set*, if U_{ir} is negligible [8, Proposition 2].

The following result was proved by Lukeš and Netuka [9, Theorem 3]: Let U be an open set of X. If K is an arbitrary compact set of U, there is a Keldyš set V with $K\subset V\subset\bar{V}\subset U$.

Lemma 1.2. *Let U be an open set of X and $M\subset\partial U$ with $\bar{H}^U_{\chi_M}=0$. Let U' be an open subset of U. Then $\bar{H}^{U'}_{\chi_M\cap\partial U'}=0$.*

Proof. Cf. [3, Proposition 2.4.4].

In the sequel we shall need the following two well-known minimum principles.

Theorem 1.3. *Let U be relatively compact. Let $M\subset\partial U$ be a negligible set. For every lower bounded hyperharmonic function u on U, if*

$$\liminf_{x\to z} u(x)\geq 0$$

for all $z\in\partial U\setminus M$, then $u\geq 0$.

Proof. This has been proved in [2, Satz 4.4.6]. The same proof carries over into the present situation.

Let U be relatively compact and $\mathcal{F}_U$ the set of finite, continuous functions on $\bar{U}$ whose restrictions to U are hyperharmonic. A point $x\in\bar{U}$ is called *extremal* if ε_x is the only measure μ on $\bar{U}$ such that

$$\int u\,d\mu\leq u(x)$$

for all $u\in\mathcal{F}_U$. Then any extremal point is a regular point of ∂U (cf. [2, Satz 4.4.1], [3, Exercise 2.4.7]).

Theorem 1.4. *Let U be relatively compact. Any $u\in\mathcal{F}_U$ is positive if it is positive at any extremal point.*

Proof. The proof is a modification of [1, Satz 33]. We have to use [3, Lemma 2, p. 26].

In the following lemma we denote by $S(p)$ the smallest closed set outside which a potential p is harmonic. Let G be a relatively compact open set. The set of potentials p on X, for which $\emptyset\neq S(p)\subset\bar{G}$, is denoted by $\mathcal{P}_G$; $\mathcal{P}_G\neq\emptyset$ by [3, Proposition 2.3.1].

Lemma 1.5. *Let W and G be open relatively compact sets of X with $G\subset\bar{G}\subset W$. For every potential $p\in\mathcal{P}_G$ we denote*

$$A_p=\{z\in W\,|\,\hat{R}_p^{X\setminus W}(z)=p(z)\}\ .$$

Then there exists a $p\in\mathcal{P}_G$ such that $G\subset W\setminus A_p$.

Proof. Let p_0 be a finite strict potential on X. Then $W\subset\{z\in X\,|\,\hat{R}_{p_0}^{X\setminus W}(z)<p_0(z)\}$ by [3, Proposition 7.2.2]. Let $p=\hat{R}_{p_0}^{\bar{G}}$; p is a potential and $p\in\mathcal{P}_G$. Since $\hat{R}_p^{X\setminus W}\leq\hat{R}_{p_0}^{X\setminus W}$, for every $x\in G$

$$\hat{R}_p^{X\setminus W}(x) \leq \hat{R}_{p_0}^{X\setminus W}(x) < p_0(x) = p(x),$$

and $x \in W \setminus A_p$.

2. On the set of regular points

Let U be an open set of X. We shall investigate the conditions under which the set $\partial\bar{U} \setminus \overline{U_{reg}}$ may be nonempty.

Theorem 2.1. *Let U be a Keldyš set. Every $x \in \partial\bar{U} \setminus \overline{U_{reg}}$ has an open neighbourhood V with $\partial U \cap \bar{V} \subset \partial U \setminus \overline{U_{reg}}$ such that $\bar{U} \cap V$ is a nontrivial absorbent set of V. Moreover, $\bar{U} \setminus \overline{U_{reg}}$ is an absorbent set of $X \setminus \overline{U_{reg}}$.*

Proof. Let V be a Keldyš set, $V \ni x$ such that $\partial U \cap \bar{V} \subset \partial U \setminus \overline{U_{reg}}$. Obviously we can assume that V is connected (Lemma 1.2).

We have $V \setminus \bar{U} \neq \emptyset$ by the assumption $x \in \partial\bar{U}$. Let G be an open set with $G \subset \bar{G} \subset V \setminus \bar{U}$. We consider the set of potentials $\mathcal{P}_G$ (see p. 276).

First, let there exist a G, $\bar{G} \subset V \setminus \bar{U}$, and a $p \in \mathcal{P}_G$ with

$$(p - \hat{R}_p^{X\setminus V}) \mid \bar{U} \cap V \neq 0. \tag{2.1}$$

The function $u: = p - \hat{R}_p^{X\setminus V}$ is positive and harmonic on $U \cap V$, continuous on $\partial U \cap V$ and bounded on $\overline{U \cap V}$. Also, u does not vanish identically on $U \cap V$ and has the limit zero at every regular boundary point of V. Further,

$$\bar{H}^{U \cap V}_{\chi_{U_{ir} \cap \partial(U \cap V)}} = 0, \quad \bar{H}^{U \cap V}_{\chi_{V_{ir} \cap \partial(U \cap V)}} = 0,$$

by Lemma 1.2. Thus the set $U_{ir} \cup V_{ir}$ is negligible on $\partial(U \cap V)$. Since $\partial U \cap \bar{V} \subset \partial U \setminus \overline{U_{reg}}$, everywhere else on $\partial(U \cap V)$, u has the limit zero. Then Theorem 1.3 gives $u = 0$ on $U \cap V$, a contradiction.

Thus, for every G such that $\bar{G} \subset V \setminus \bar{U}$, and every $p \in \mathcal{P}_G$, the function $p - \hat{R}_p^{X\setminus V}$ equals zero on $\bar{U} \cap V$.

Let $y \in V \setminus \bar{U}$ be arbitrary and G an open set with $y \in G \subset \bar{G} \subset V \setminus \bar{U}$. Then by Lemma 1.5 there is a potential p_y such that $G \subset V \setminus A_{p_y} = \{z \in V \mid \hat{R}_{p_y}^{X\setminus V} < p_y(z)\}$. Thus

$$\bigcap_{y \in V \setminus \bar{U}} A_{p_y} = \bar{U} \cap V$$

is an absorbent set of V.

Hence for every $x \in \partial\bar{U} \setminus \overline{U_{reg}}$ there is an open neighbourhood $V \subset X \setminus \overline{U_{reg}}$ such that $\bar{U} \cap V$ is an absorbent set of V. By the sheaf property of hyperharmonic functions, the function v which is 0 on $\bar{U} \setminus \overline{U_{reg}}$ and ∞ on $(X \setminus \overline{U_{reg}}) \setminus \bar{U}$ is hyperharmonic on $X \setminus \overline{U_{reg}}$. Thus $\bar{U} \setminus \overline{U_{reg}}$ is an absorbent set of $X \setminus \overline{U_{reg}}$. This still holds if $\partial\bar{U} \setminus \overline{U_{reg}} = \emptyset$.

REMARK 2.2. If $\partial\bar{U}\backslash\overline{U_{reg}}=\emptyset$, then $\bar{U}\backslash\overline{U_{reg}}$ is a union of some components of $X\backslash\overline{U_{reg}}$.

Theorem 2.3. *Let X have a base of regular sets and U an open set of X. Then all the assertions of Theorem 2.1 are valid.*

Proof. Let $x\in\partial\bar{U}\backslash\overline{U_{reg}}$ be arbitrary and the connected set V in the proof of Theorem 2.1 be regular [2, Satz 4.3.5].

We assume that there exist the set G and the potential p such that (2.1) holds. Then, the function u has the same properties as previously. Moreover, u is continuous on $\overline{U\cap V}$ and equals 0 at every point of ∂V. Since $\partial U\cap V\subset U_{ir}$, by the barrier criterion also $\partial U\cap V\subset(U\cap V)_{ir}$. Thus the set of regular, and hence of extremal boundary points is contained in ∂V. From Theorem 1.4 we obtain $u=0$ on $U\cap V$, a contradiction.

Everything else needed for the conclusion may be proved exactly as for Theorem 2.1.

The following result was obtained for Brelot spaces (cf. [4, Théorème 8.2], [6, Theorem 7]).

Corollary 2.4. *Let X be elliptic and U an open set of X. Then $\partial\bar{U}\backslash\overline{U_{reg}}=\emptyset$.*

Proof. X has a base of regular sets.

EXAMPLE 2.5. It is known that for the heat equation $\partial\bar{U}\backslash\overline{U_{reg}}$ may be nonempty. Let $X=\boldsymbol{R}^2$ and

$$U=(0,\ 1)\times(0,\ 1)\,.$$

Then $U_{reg}=([0,\ 1]\times\{0\})\cup(\{0\}\times[0,\ 1])\cup(\{1\}\times[0,\ 1])$, and $\bar{U}\backslash\overline{U_{reg}}$ is absorbent on $X\backslash\overline{U_{reg}}$, which may be seen directly. The same observation follows immediately by Theorem 2.3, and since U is a Keldyš set [7, p. 1501], also by Theorem 2.1.

EXAMPLE 2.6. Let X be the space of [3, Example 3.2.13] and

$$U=\{(x,\ y,\ 0)\in X\,|\,0<x^2+y^2<1\}\ .$$

Then $X\backslash U$ is thin at (0, 0, 0), and $\{(0,\ 0,\ 0)\}=\partial\bar{U}=U_{ir}$. Now $\bar{U}=\bar{U}\backslash\overline{U_{reg}}$ is an absorbent set of $X=X\backslash\overline{U_{reg}}$, which can be seen directly and by Theorem 2.3.

REMARK 2.7. If U is a Keldyš set, then for every $x\in U$, $\mathrm{supp}(\mu_x^U)\subset\overline{U_{reg}}$. Denoting

$$T:=\overline{\bigcup_{x\in U}\mathrm{supp}(\mu_x^U)}\,,$$

$T\subset\overline{U_{reg}}$. As $\overline{U_{reg}}\subset T$ always, $T=\overline{U_{reg}}$. It was proved in [5, Lemma 1.4] that $\bar{U}\backslash T$ is an absorbent set of $X\backslash T$. Writing $T=\overline{U_{reg}}$, this gives the assertion of Theorem 2.1. However, Theorem 2.3 cannot be obtained in this way, since $T=\overline{U_{reg}}$ does not always hold.

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