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QUASICONFORMAL METRIC AND ITS APPLICATION TO QUASIREGULAR MAPPINGS

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The quasiconformal metric introduced by Kuusalo [5] seems to me useful for studying the n-dimensional quasiregular mappings but has not ever been fully utilized in these connections except what are found in V.M. Gol'dstein-S.K. Vodop'yanov [2] and H. Tanaka [14].

In this paper we shed light on some features of quasiconformal metrics on subdomains of $\bar{R}^n$ and apply those to quasiregular mappings to obtain several important properties of them, among others, a characterization for quasiregularity which comes to a generalization of the result in O. Martio, S. Rickman and J. Väisälä [6, Theorem 7.1]. Most of the statements in the sequel remain to hold in $\bar{R}^n$, but we often confine ourselves to R^n in order to avoid inessential complexities in technique.

1. Notations and terminologies

R^n $(n\geq 2)$: the n-dimensional euclidean space.
$\bar{R}^n$: the one point compactification of R^n.
m_α: the α-dimensional Hausdorff measure.
$m=m_n$: the n-dimensional Lebesgue measure.
q: the spherical metric.

For a point $x\in R^n$, the coordinates of x are denoted by $x^1, \cdots, x^n$ and $|x|$ is the euclidean norm.

Let E be a subset of $\bar{R}^n$, then $\bar{E}$, ∂E, E^c denote the closure, the boundary, the complement of E respectively, all taken with respect to $\bar{R}^n$.

Given two sets $E, F\subset R^n$, $d(E, F)$ is the euclidean distance between E and F, $d(E)$ is the euclidean diameter of E and $E\setminus F$ is the set-theoretical difference.

Suppose given a non-empty compact proper subset E of $\bar{R}^n$ and an open set $G\subset\bar{R}^n$, including E, then we call the pair (E, G) a condenser and we may define the (conformal) capacity cap(E, G) as the (conformal) modulus of the family of all paths connecting E and ∂G in G (cf. [3]). If $E=\phi$ or $\partial G=\phi$, then we set cap$(E, G)=0$.

A compact proper subset E of $\bar{R}^n$ is said of capacity zero if $\mathrm{cap}(E, G)=0$ for some open set $G\subset\bar{R}^n$ such that $E\subset G$ and $\bar{G}\neq\bar{R}^n$, otherwise of positive capacity. A subset E of $\bar{R}^n$ is of capacity zero if and only if all compact subsets of E are of capacity zero, or else E is of positive capacity. We refer to [6], [10] for the properties of the capacities.

2. Quasiconformal metrics

Let G be a domain in $\bar{R}^n$. Given two points $x, y\in G$, the *quasiconformal distance* $c_G(x, y)$ between x and y, relative to G, is defined by

$$c_G(x, y) = \inf \mathrm{cap}(E, G)\,,$$

where the infimum is taken over all continua E in G, which contain both x and y. It is easy to see that c_G is a pseudometric and a conformal invariant. According to [5] we call c_G a *quasiconformal metric.*

From the definition of quasiconformal metrics and the properties of condenser capacities follows immediately the following

Proposition 1. *Let G, G' be domains in $\bar{R}^n$ such that $G\subset G'$. Then*

$$c_G(x, y)\geq c_{G'}(x, y)$$

for any two points $x, y\in G$.

Proposition 2. *Let G be a domain in $\bar{R}^n$ and let F be a set closed relative to G, which is of capacity zero. Then*

$$c_{G\backslash F}(x, y) = c_G(x, y)$$

for any two points $x, y\in G\backslash F$.

Remark 1. Note that $G\backslash F$ is also a domain since F is of $(n-1)$-dimensional Hausdorff measure zero ([10, Corollary 1 of Theorem 8], [4, Corollary 1 of Theorem IV 4 and Theorem VII 3]).

Proof of Proposition 2. Let $x, y\in G\backslash F$ and let E be an arbitrary continuum in G, which contains both x and y. Select a non-increasing sequence $\{D_j\}_1^\infty$ of subdomains of G such that each D_j is relatively compact in G and $\bigcap_1^\infty \bar{D}_j=E$. Then for each j, we can find a path γ_j joining x with y in $D_j\backslash F$ since $D_j\backslash F$ is a domain (Remark 1) and $x, y\in D_j\backslash F$.

From the properties of condenser capacities we obtain

$$\begin{aligned} c_{G\backslash F}(x, y) &\leq \mathrm{cap}(|\gamma_j|, G\backslash F)\\ &=\mathrm{cap}(|\gamma_j|, G)\\ &\leq \mathrm{cap}(\bar{D}_j, G)\,, \end{aligned}$$

where $|\gamma_j|$ is the locus of γ_j.

Letting $j\to\infty$, since $\lim_{j\to\infty}\operatorname{cap}(\bar{D}_j, G)=\operatorname{cap}(E, G)$ ([6, Lemma 5.7]), we have

$$c_{G\setminus F}(x, y)\leq \operatorname{cap}(E, G),$$

from which it follows that

$$c_{G\setminus F}(x, y)\leq c_G(x, y).$$

The reverse inequality is derived from Proposition 1. q.e.d.

Theorem 1 (cf. [5, Theorem 2]). *Let G be a domain in $\bar{R}^n$. Then either c_G is a metric or c_G equals identically to zero according as G^c is of positive capacity or not. Furthermore whenever c_G is a metric, the topology induced by c_G is equivalent to the one induced by q and the identity mapping of G is the uniformly continuous mapping of the metric space (G, c_G) onto the metric space (G, q).*

Proof. If G^c is of capacity zero, then $\operatorname{cap}(E, G)=0$ for all continua E in G, hence $c_G(x, y)=0$ for all $x, y\in G$.

If G^c is of positive capacity, then [7, Lemma 3.11] proves that c_G is a metric and the identity mapping of G is the uniformly continuous mapping of (G, c_G) onto (G, q). Now for every $x\in G$ with $x\neq\infty$ and all $y\in\{y\in R^n: |x-y|< d(x, \partial G)\}$, we have

$$\begin{aligned} c_G(x, y) &\leq \operatorname{cap}(\bar{B}^n(x, |y-x|), B^n(x, d(x, \partial G)) \\ &= \omega_{n-1}\left(\log\frac{d(x, \partial G)}{|y-x|}\right)^{1-n}, \end{aligned}$$

where $B^n(x, r)=\{\tilde{x}\in R^n: |\tilde{x}-x|<r\}$ and ω_{n-1} is the area of the unit $(n-1)$-sphere. Suppose $\infty\in G$. If we set $\phi(x)=\frac{x}{|x|}$, then since ϕ is conformal, we have

$$c_G(\infty, y) = c_{\phi(G)}(0, \phi(y))\leq\omega_{n-1}\left(\log\frac{|\phi(y)|}{r}\right)^{1-n}$$

for all $y\in B^n(r)^c$, where $B^n(r)$ is a ball with the center 0 such that $B^n(r)^c\subset G$. These inequalities imply that the topology induced by c_G is weaker than the one induced by q, which completes the proof.

Here we refer to two estimates of quasiconformal metrics from below. From [6, Lemma 5.9] we have the following

Proposition 3. *Let G be a domain in R^n with $m(G)<\infty$. Then*

$$c_G(x, y)^{n-1}\geq b_n\frac{|x-y|^n}{m(G)}$$

for all $x, y\in G$, *where* b_n *is the constant in* [6, Lemma 5.9].

Proposition 4. *If* G *is a domain in* R^n *with a continuum* $C\subset\partial G$, *then*

$$c_G(x, y)\geq 2^{-1}c_n \log\left[1+\frac{\min\{|x-y|^2, d(C)^2\}}{2\min\{d(x, C)^2, d(y, C)^2\}}\right]$$

for all $x, y\in G$, *where* c_n *is the constant in* [16, Theorem 10.12].

Proof. Let E be an arbitrary continuum in G, containing x, y. Select two points $x_1\in E$, $x_2\in C$ with $|x_1-x_2|=d(E, C)$ and let x_0 be the midpoint of the line segment joining x_1 with x_2. Then we see easily that both E and C meet $S^{n-1}(x_0, r)=\partial B^n(x_0,r)$ for each r, $r_1<r<r_2$, where $r_1=2^{-1}d(E, C)$, $r_2=2^{-1}\sqrt{d(E, C)^2+2\delta^2}$ and $\delta=2^{-1}\min\{d(E), d(C)\}$. Hence if we let Γ be the family of all paths connecting E and C in $B^n(x_0, r_2)\backslash\bar{B}^n(x_0, r_1)$, then using [16, Theorem 10.12], we obtain the following estimate of the modulus $M(\Gamma)$.

$$\begin{aligned}M(\Gamma)&\geq c_n\log\frac{r_2}{r_1}\\&= 2^{-1}c_n \log\left[1+\frac{2\delta^2}{d(E, C)^2}\right]\\&\geq 2^{-1}c_n \log\left[1+\frac{\min\{|x-y|^2, d(C)^2\}}{2\min\{d(x, C)^2, d(y, C)^2\}}\right].\end{aligned}$$

Since Γ is minorized by the family $\tilde{\Gamma}$ of all paths connecting E and ∂G in G, we have

$$\begin{aligned}\mathrm{cap}(E, G) = M(\tilde{\Gamma})&\geq M(\Gamma)\\&\geq 2^{-1}c_n \log\left[1+\frac{\min\{|x-y|^2, d(C)^2\}}{2\min\{d(x, C)^2, d(y, C)^2\}}\right],\end{aligned}$$

from which the required inequality follows. q.e.d.

Corollary 1. *Suppose that* G *is a domain in* $\bar{R}^n$, *all of whose boundary components contain at least two points. Then* c_G *is a metric and the set* $\{y\in G: c_G(x, y)\leq r\}$ *is compact for any* $x\in G$ *and any* $r>0$. *Therefore* (G, c_G) *is a complete metric space.*

Example 1. $c_{R^n}=0$ for all $x, y\in R^n$.

Example 2. If G is a bounded domain in R^n, then c_G is a metric since G^c is of positive capacity. Moreover (G, c_G) is a complete metric space whenever ∂G is a continuum.

Example 3. It is known by Gehring that

$$c_{B^n}(0, x) = \operatorname{cap}(J(|x|), B^n)$$

for all $x \in B^n$, where B^n is the unit ball and $J(|x|) = \{y \in B^n: 0 \le y^1 \le |x|, y^2 = \cdots = y^n = 0\}$. From this relation we have

$$\max\left\{c_n \log\frac{1+|x|}{1-|x|},\ \omega_{n-1}\left(\log\frac{\lambda_n}{|x|}\right)^{1-n}\right\} \le c_{B^n}(0, x) \le \omega_{n-1}\left(\log\frac{1}{|x|}\right)^{1-n},$$

where λ_n is a constant depending only on n and ω_{n-1} is the area of the unit sphere.

3. Quasiregular mappings

In the following the notation f: $G \to R^n$ always implies that G is a domain in R^n and f is a continuous mapping of G into R^n, unless otherwise stated.

Given f: $G \to R^n$, we employ the following notations:

$L(x, f, r) = \sup\{|f(y) - f(x)| : |y - x| = r\}$ for $x \in R^n$ and $r > 0$;

$L(x, f) = \limsup_{r \to 0} \dfrac{L(x, f, r)}{r}$;

$J(x, f) = \sup \limsup_{j \to \infty} \dfrac{m(f(A_j))}{m(A_j)}$,

where the supremum is taken over all regular sequences of closed sets tending to x in the sense explained in [13];

$N(y, f, A)$ is the cardinal number of $\{x \in A: f(x) = y\}$ for any $y \in R^n$ and any $A \subset G$;

$N(f, A) = \sup\{N(y, f, A): y \in R^n\}$ for any $A \subset G$;

Given an arbitrary relatively compact subdomain D in G and any $y \notin f(\partial D)$, $\mu(y, f, D)$ denotes the topological index in the sense stated in [9] (cf. [6], [10]);

$f'(x)$ denotes the Jacobian matrix whenever all partial derivatives exist at x;

$|f'(x)| = \sup\{|f'(x)h| : h \in R^n, |h| = 1\}$.

According to [6] we say that f is *quasiregular* if f is ACL^n and $|f'(x)|^n \le K \det f'(x)$ a.e. in G for some constant $K \ge 1$. We refer to [6], [10] for the basic properties of quasiregular mappings. Here we quote only the following fundamental facts.

If f: $G \to R^n$ is a non-constant quasiregular mapping, then f is sense-preserving, discrete and open, and hence $f(G)$ is a domain. "f is *sense-preserving*" means that $\mu(y, f, D) > 0$ for every relatively compact subdomain D in G and for all $y \in f(D) \setminus f(\partial D)$. Let (E, D) be an arbitrary condenser in G, i.e. $D \subset G$, then the inequality

$$\operatorname{cap}(f(E), f(D)) \le K_I(f) \operatorname{cap}(E, D)$$

holds and further

$$\operatorname{cap}(E, D) \le K_O(f) N(f, D) \operatorname{cap}(f(E), f(D))$$

also holds if D is a *normal domain* for f, that is, D is a relatively compact subdomain of G and $f(\partial D)=\partial f(D)$, where $K_I(f)$, $K_O(f)$ are the inner, the outer dilatation of f respectively. From the above capacity inequalities we obtain easily the following

Theorem 2. *Let f: $G\to R^n$ be a quasiregular mapping. Then*

$$c_{D'}(f(x), f(y))\leq K_I(f)\ c_D(x, y)$$

for any two domains $D\subset G$, $D'\supset f(D)$ and for all $x, y\in D$. Further if f is not constant and D is a normal domain for f, then

$$\inf\{c_D(x, \tilde{y}): \tilde{y}\in f^{-1}(f(y))\}\leq K_O(f)N(f, D)c_{f(D)}(f(x), f(y))$$

for any $x, y\in D$.

Remark 2. Let f be a mapping of a domain D into a domain D'. Suppose that there exists a constant $K>0$ with the property:

$$(*)\qquad c_{D'}(f(x), f(y))\leq Kc_D(x, y)\qquad \text{for all } x, y\in D\,.$$

If $c_{D'}$ is a metric, then f is continuous. Furthermore if c_D, $c_{D'}$ are metrics, then f is a uniformly continuous mapping of (D, c_D) into $(D', c_{D'})$ and hence f is also a uniformly continuous mapping of (D, c_D) into $(\bar{R}^n, q)$ (Theorem 1).

The condition (*) assures the quasiregularity for mappings under some assumptions. To see this, we need some preliminaries.

Given f: $G\to R^n$, we say, according to [9], that f is *locally of bounded variation in the Banach seuse* (briefly, locally BVB in G) if $\int_{R^n}N(y, f, D)dm(y)<\infty$ for every relatively compact subdomain D of G.

Suppose that f: $G\to R^n$ is locally BVB and that D is a relatively compact subdomain of G. Set

$$\Phi_i(E, D)=\int_{R^n}N(y, f, D\cap P_i^{-1}(E))dm(y)$$

for each i, $1\leq i\leq n$, and for Borel sets E in $P_i(D)$, where P_i is the orthogonal projection of R^n onto $R_i^{n-1}=\{x\in R^n\colon x^i=0\}$. Then $\Phi_i(E, D)$ is a countably additive set function of Borel sets in $P_i(D)$. The (symmetrical) derivative $\Phi_i'(z, D)$ of $\Phi_i(E, D)$, i.e.

$$\Phi_i'(z, D)=\lim_{r\to 0}\frac{\Phi_i(B^{n-1}(z, r), D)}{m_{n-1}(B^{n-1}(z, r))}$$

exists and is finite m_{n-1}-a.e. in $P_i(D)$.

Lemma 1 (cf. [6, Lemma 2.17]). *Let $f: G \to R^n$ be locally BVB. If there exists a constant $c>0$ such that*

$$(\sharp) \qquad [\sum_1^k d(f(\Delta_j))]^n \leq c\Phi_i'(z, Q)[\sum_1^k m_1(\Delta_j)]^{n-1}$$

for each relatively compact open n-interval Q in G, each i, $1\leq i\leq n$, a.e. $z\in P_i(Q)$ and any disjoint finite sequence $\{\Delta_1, \cdots, \Delta_k\}$ of closed subintervals of $Q\cap P_i^{-1}(z)$, then f is ACL^n.

Proof. The proof is much the same as that of [6, Lemma 2.17].

It is easy to see that f is ACL. To prove that f is ACL^n, since the situation is the same in any case, it is sufficient to show that $\left|\frac{\partial f}{\partial x^n}\right|^n$ is integrable on each relatively compact open n-interval Q in G.

Suppose $Q=Q_0\times J$, where Q_0 is an open $(n-1)$-interval in R^{n-1} and J is an open 1-interval in R^1. Set

$$g(z, u)=\left|\frac{\partial f}{\partial x^n}(z, u)\right|, \quad g_j(z, u)=\frac{j}{2}\int_{-1/j}^{1/j}|g(z, u+t)|\,dt$$

for each positive integer j with $0<\frac{1}{j}<d(Q, \partial G)$, whenever these make sense. Then we see, as in [6], that g, g_j are all measureble in Q and

$$(1) \qquad g_j(z, u)\to g(z, u) \qquad \text{a.e. in } Q_0$$

for a.e. $u\in J$.

Now given each $u\in J$ and each j, we set

$$F_{u,j}(E)=\Phi_n\Big(E, Q_0\times\Big(u-\frac{1}{j}, u+\frac{1}{j}\Big)\Big)$$

for Borel sets E in Q_0. Since $F_{u,j}'(z)<\infty$ a.e. in Q_0 the condition $(\sharp)$ implies that $f(z, t)$ is absolutely continuous on $\left[u-\frac{1}{j}, u+\frac{1}{j}\right]$ as the function of t and the nth power of its total variation is not greater than $cF_{u,j}'(z)\left(\frac{2}{j}\right)^{n-1}$ for a.e. $z\in Q_0$. Hence we obtain

$$g_j(z, u)^n\leq c\frac{j}{2}F_{u,j}'(z)$$

a.e. in Q_0. Integrating over Q_0

$$(2) \qquad \int_{Q_0} g_j(z, u)^n dm_{n-1}(z)\leq c\frac{j}{2}\int_{Q_0}F_{u,j}'(z)dm_{n-1}(z)$$

$$\leq c\frac{j}{2}F_{u,j}(Q_0)$$
$$=c\frac{j}{2}\int_{R^n}N\left(y, f, Q_0\times\left(u-\frac{1}{j}, u+\frac{1}{j}\right)\right)dm(y)$$

for each $u\in J$.

If we let

$$\Psi(E)=\int_{R^n}N(y, f, Q_0\times E)dm(y)$$

for Borel sets $E\subset J$, then Ψ is countably additive for Borel sets in J and hence the derivative $\Psi'(u)$ of Ψ exists and is finite a.e. in J. For $u\in J$ such that (1) holds and $\Psi'(u)$ exists, Fatou's lemma and (2) yield

$$\int_{Q_0}g(z, u)^n dm_{n-1}(z)\leq\liminf_{j\to\infty}\int_{Q_0}g_j(z, u)^n dm_{n-1}(z)$$
$$\leq c\lim_{j\to\infty}\left[\frac{j}{2}\Psi\left(\left(u-\frac{1}{j}, u+\frac{1}{j}\right)\right)\right]$$
$$=c\Psi'(u).$$

Integrating over J, we have

$$\int_Q g(x)^n dm(x)\leq c\int_J\Psi'(u)dm_1(u)$$
$$\leq c\Psi(J)$$
$$=c\int_{R^n}N(y, f, Q)dm(y)<\infty,$$

which completes the proof.

Lemma 2. *Given f: $G\to R^n$, if there exists a constant $K>0$ such that the property* $(*)$ *is satisfied for any two domains $D\subset G$, $D'\supset f(D)$, then*

$$L(x, f)^n\leq\tilde{K}J(x, f)$$

for all $x\in G$, where $\tilde{K}$ is a constant depending only on n, K.

Proof. Given $x\in G$ and r, $0<r<\frac{1}{2}d(x, \partial G)$, choose $y\in G$ such that $|x-y|=r$ and $|f(x)-f(y)|=L(x, f, r)$. Let J_r be the line segment joining x with y and set $D_r=\{z\in R^n: d(z, J_r)<r\}$.

If D' is an arbitrary domain containing $f(D_r)$, then the condition $(*)$ and Proposition 3 yield

$$\frac{L(x, f, r)^n}{r^n} = \frac{|f(x)-f(y)|^n}{r^n}$$
$$\leq \frac{K^{n-1}}{b_n} \frac{m(D')}{r^n} c_{D_r}(x, y)^{n-1}$$
$$= \frac{K^{n-1}}{b_n} c_{D_r}(x, y)^{n-1} \frac{m(D_r)}{r^n} \frac{m(D')}{m(D_r)}.$$

It is easy to see that both $c_{D_r}(x, y)$ and $\frac{m(D_r)}{r^n}$ are constant for all x, r and y which are taken as above. Set $\tilde{K}=\frac{K^{n-1}}{b_n} c_{D_r}(x, y) \frac{m(D_r)}{r^n}$ and if we bring D' arbitrarily close to $f(D_r)$, then we have

$$\frac{L(x, f, r)^n}{r^n} \leq \tilde{K} \frac{m(f(D_r))}{m(D_r)} \leq \tilde{K} \frac{m(f(\bar{D}_r))}{m(\bar{D}_r)}.$$

Obviously, $\tilde{K}$ depends only on n, K.

Letting $r \to 0$, we obtain

$$L(x, f)^n \leq \tilde{K} J(x, f).$$

q.e.d.

Theorem 3. *Suppose that f: $G \to R^n$ is as in Lemma 2. If f is sense-preserving and locally BVB, then f is quasiregular.*

Proof. First of all we assert that f is ACL^n. To do so, we have only to show that there exists a constant $c>0$ with the property in Lemma 1. Let Q be an arbitrary open n-interval with $\bar{Q} \subset G$. Fix i, $1 \leq i \leq n$, and let $z \in P_i(Q)$ with $\Phi_i'(z, Q) < \infty$. Given any disjoint finite sequence $\{\Delta_1, \cdots, \Delta_k\}$ of closed subintervals of $P_i^{-1}(z) \cap Q$, set $D_{j,r} = \{x \in R^n$: $d(x, \Delta_j) < r\}$ for each j, $1 \leq j \leq k$, and for $r>0$. Let $D'_{j,r}$ be an arbitrary domain containing $f(D_{j,r})$ whenever $D_{j,r} \subset G$.

Suppose that r is so small as the following properties hold: $D_{j,r} \subset Q$ for each j, $1 \leq j \leq k$; $D_{1,r}, \cdots, D_{k,r}$ are disjoint; $r \leq n m_1(\Delta_j)$ for all j, $1 \leq j \leq k$. Then owing to the manner in which r was chosen we have

$$c_{D_{j,r}}(x, y) \leq \frac{m(D_{j,r})}{r^n} \leq \frac{2\omega_{n-1} m_1(\Delta_j)}{r}$$

for each j ($j=1, \cdots, k$) and all $x, y \in \Delta_j$.

On the other hand Proposition 3 yields

$$c_{D'_{j,r}}(f(x), f(y))^{n-1} \geq b_n \frac{|f(x)-f(y)|^n}{m(D'_{j,r})}.$$

By these two inequalities and the condition (*) we obtain

$$|f(x)-f(y)| \leq c_1 r^{(1-n)/n} m(D'_{j,r})^{1/n} m_1(\Delta_j)^{(n-1)/n}$$

for all $x, y \in \Delta_j$ ($j=1, \cdots, k$), where c_1 is a constant depending only on n, K. It follows from this inequality that

$$d(f(\Delta_j)) \leq c_1 r^{(1-n)/n} m(f(D_{j,r}))^{1/n} m_1(\Delta_j)^{(n-1)/n}$$

for each j ($j=1, \cdots, k$).

Summing over $1 \leq j \leq k$ and using Hölder's inequality, we have

$$\{\sum_1^k d(f(\Delta_j))\}^n \leq c \frac{\sum_1^k m(f(D_{j,r}))}{m_{n-1}(B^{n-1}(z, r))} \{\sum_1^k m_1(\Delta_j)\}^{n-1},$$

where c depends only on n, K. Now

$$\begin{aligned}
\sum_1^k m(f(D_{j,r})) &= \sum_1^k \int_{f(D_{j,r})} 1 dm \\
&\leq \sum_1^k \int_{f(D_{j,r})} N(y, f, D_{j,r}) dm(y) \\
&= \int_{R^n} N(y, f, \bigcup_1^k D_{j,r}) dm(y) \\
&\leq \int_{R^n} N(y, f, Q \cap P_i^{-1}(B^{n-1}(z, r))) dm(y) \\
&= \Phi_i(B^{n-1}(z, r), Q).
\end{aligned}$$

Hence

$$\{\sum_1^k d(f(\Delta_j))\}^n \leq c \frac{\Phi_i(B^{n-1}(z, r), Q)}{m_{n-1}(B^{n-1}(z, r))} \{\sum_1^k m_1(\Delta_j)\}^{n-1}.$$

Thus letting $r \to 0$, we obtain

$$\{\sum_1^k d(f(\Delta_j))\} \leq c \Phi'_i(z, Q) \{\sum_1^k m_1(\Delta_j)\}^{n-1},$$

from which it follows that f is ACL^n (Lemma 1).

Since f is continuous and sense-preserving, f is monotone in the sense that if D is an arbitrary relatively compact subdomain of G, then the unbounded connected component of $f(\partial D)^c$ contains no point of $f(D)$. Hence all components of f are monotone functions in the sense of Lebesgue. It is known that a monotone continuous ACL^n-function is differentiable almost everywhere in the domain of the function ([11]). So f is differentiable a.e. in G, from which it follows that $L(x,f)=|f'(x)|$ and $J(x,f)=\det f'(x)$ (as f is sense-preserving), a.e. in G. Consequently Lemma 2 implies that

$$|f'(x)|^n \leq \tilde{K} \det f'(x)$$

a.e. in G, where $\tilde{K}$ depends only on n, K, which concludes the proof.

REMARK 3. If $f: G \to R^n$ is sense-preserving, discrete and open, then f is locally BVB in G, since $N(f, A) < \infty$ for every relatively compact subset A of G ([6, Lemma 2.12]). Hence the above Theorem 3 generalizes a part of the Theorem 7.1 in [6].

As applications of the preceding results we prove alternatively the several known properties of quasiregular mappings.

Theorem 4. *Let f: $G \to R^n$ be a non-constant quasiregular mapping. If G^c is of capacity zero, then $f(G)^c$ is also of capacity zero.*

Proof. On account of Theorem 2,

$$c_{f(G)}(f(x), f(y)) \leq K_I(f) c_G(x, y)$$

holds for any x, $y \in G$. The right-hand side of this inequality is always zero since c_G is identically equal to zero (Theorem 1). Hence $c_{f(G)}(f(x), f(y)) = 0$ for all $x, y \in G$. Therefore if $c_{f(G)}$ is a metric, that is, $f(G)^c$ is of positive capacity, then f is constant, which comes to a contradiction. Thus $f(G)^c$ is of capacity zero. q.e.d.

Theorem 5 ([7, Theorem 3.17]). *Let G, G' be domains in $\bar{R}^n$ and let $K \geq 1$ be a constant. Suppose that G'^c is of positive capacity. Then a family of quasiregular mappings f of G into G' such that $K_I(f) \leq K$ is equicontinuous if we consider G' as a metric space with the metric q.*

Proof. If G^c is of capacity zero, then all mappings belonging to the family in the theorem are constant and hence the theorem is trivial. Suppose that G^c is of positive capacity. Given $x \in G$ and $\varepsilon > 0$, choose $\eta > 0$ such that $c_{G'}(\tilde{x}, \tilde{y}) < \eta$ implies $q(\tilde{x}, \tilde{y}) < \varepsilon$. If U is a neighbourhood of x such that $c_G(x, y) < \frac{\eta}{K}$ for all $y \in U$, then $q(f(x), f(y)) < \varepsilon$ for any f belonging to the family under consideration and for all $y \in U$. q.e.d.

Theorem 6 ([7, Theorem 4.1]). *Let G be a domain in R^n and let F be a relatively closed subset of G, which is of capacity zero. Suppose that f: $G \setminus F \to R^n$ is a quasiregular mapping for which $f(G \setminus F)^c$ is of positive capacity. Then f is uniquely extended to a continuous mapping $\tilde{f}$: $G \to \bar{R}^n$ such that the restriction f^* of $\tilde{f}$ to $G \setminus \tilde{f}^{-1}(\infty)$ is quasiregular. Furthermore $K_O(f^*) = K_O(f)$ and $K_I(f^*) = K_I(f)$.*

Proof. If G^c is of capacity zero, then $(G \setminus F)^c = G^c \cup F$ is also of capacity zero. Hence f is constant or else a contradiction arises (Theorem 4), from which the theorem is obvious. Hereafter we suppose that G^c is of positive capacity. Further we may assume that f is not constant. Then f is a uniformly continuous mapping of $(G\backslash F, c_G)$ into $(\bar{R}^n, q)$ (Remark 2) as $c_{G\backslash F}=c_G$ on $G\backslash F$. Since $G\backslash F$ is dense everywhere in G and $(\bar{R}^n, q)$ is a complete metric space, f is uniquely extended to a continuous mapping $\tilde{f}: G\to\bar{R}^n$. $\tilde{f}(F)$ contains no non-empty open set, because owing to the way of path lifting ([12]) and a modulus inequality under quasiregular mappings ([8]), we can show that $\tilde{f}(F)$ is of capacity zero. Therefore since F is 0-dimensional, it follows from [15, Theorem 9 and Corollary to Theorem 4] that $\tilde{f}$ is locally sense-preserving discrete, open, and hence f^* is sense-preserving, locally BVB (Remark 3) as the local sense-preservingness implies obviously the sense-preservingness.

To see that f^* is quasiregular, it remains to be proved that the condition (*) holds for a constant $K>0$. Let $D\subset G\backslash \tilde{f}^{-1}(\infty)$, $D'\supset f^*(D)$ be any domains. Then we have

$$\begin{aligned} c_{D'}(f(x), f(y)) &\leq c_{f(D\backslash F)}(f(x), f(y)) \\ &\leq K_I(f)c_{D\backslash F}(x, y) \\ &= K_I(f)c_D(x, y) \end{aligned}$$

for all $x, y\in D\backslash F$, and hence

$$c_{D'}(f^*(x), f^*(y))\leq K_I(f)c_D(x, y)$$

for all $x, y\in D$ since $F\cap D$ is nowhere dense in D. It is obvious that $K_O(f^*)=K_O(f)$, $K_I(f^*)=K_I(f)$, since $F\cap\tilde{f}^{-1}(\infty)$ is of Lebesgue measure zero. q.e.d.

Remark 4. The $\tilde{f}$ in Theorem 6 is, in fact, quasimeromorphic in the sense stated in [7].

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