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STEFAN PROBLEMS WITH THE UNILATERAL BOUNDARY CONDITION ON THE FIXED BOUNDARY IV

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Contents

0. Introduction

We consider the behavior of the solution of the following one-dimensional two phase Stefan problem with the unilateral boundary condition on the fixed boundary: Given the initial data, l and $\phi(x)$, find a critical time T^*, and the two functions $s=s(t)$ and $u=u(x, t)$ defined on $[0, T^*]$ such that

$$(S)\begin{cases}(0.1) & s(0)=l,\ 0<s(t)<1\ (0\leqq t<T^*)\,,\\ (0.2) & u_{xx}-c_0u_t=0\ (0<x<s(t),\ 0<t<T^*)\,,\\ (0.3) & u_{xx}-c_1u_t=0\ (s(t)<x<1,\ 0<t<T^*)\,,\\ (0.4) & \begin{aligned}&\text{(a)}\quad u_x(0, t)\in\gamma_0(u(0, t))\ (0<t<T^*)\,,\\ &\text{(b)}\quad -u_x(1, t)\in\gamma_1(u(1, t))\ (0<t<T^*)\,,\end{aligned}\\ (0.5) & \begin{aligned}&\text{(a)}\quad u(x, 0)=\phi(x)\ (0<x<l)\,,\\ &\text{(b)}\quad u(x, 0)=\phi(x)\ (l<x<1)\,,\end{aligned}\\ (0.6) & u(s(t), t)=0\ (0<t\leqq T^*)\,,\\ (0.7) & b\dot{s}(t)=-u_x^-(s(t), t)+u_x^+(s(t), t)\ (0<t<T^*)\,.\end{cases}$$

The critical time T^*, $0<T^*\leqq\infty$, is defined to be the first time that the free boundary $x=s(t)$ touches the fixed boundary $x=0$ or $x=1$. The quantities c_0, c_1 and b are positive physical parameters of the problem. The assumptions for the boundary condition (0.4) at the fixed boundary are that γ_0 and γ_1 are maximal monotone graphs in $\boldsymbol{R}^2$ such that both $\gamma_0^{-1}(0)\cap[0,\infty[$ and $\gamma_1^{-1}(0)\cap]-\infty, 0]$ are not empty sets. We put these assumptions from the physical reasoning, that is, there are a kind of heater at $x=0$ and a kind of freezer at $x=1$. (0.4) are the unilateral boundary conditions. (0.7) is the so-called Stefan's condition. The author proved the existence and uniqueness of the solution of (S) in [11].

The behvaior of solutions of the problems with the linear boundary conditions on the fixed boundaries are considered by Rubinstein [7, p. 155–181], Friedman [4, 5] and Cannon-Primicerio [2].

The plan of this paper is as follows. In §1 we state main theorems. In §2 we give the proof of Theorem 1, 2 and 3 concerning stationary problems corresponding to the Stefan problem (S). In §3 we state comparison theorems of the solutions of (S). In §4 and §5 we give a priori estimates of the free boundary $s(t)$ and the derivatives of u at the fixed boundary respectively. In §6 we introduce a weak formulation of (S). The results of §4, §5 and §6 are used in the subsequent sections. In §7 and §8 we give the proof of Theorem 4, Theorem 5, Theorem 6 and Corollaries, which give some informations about the solutions of (S). In §9 we give simple examples to clarify the meaning of our results.

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1. Statements of main results

We shall use the notations and definitions introduced in [11]. The assumptions required on the Stefan data $\{l, \phi\}$ are the following (A).

$$(A)\ \begin{cases} 0<l<1. \quad \phi(x)\geqq 0\ (0<x<l),\ \phi(x)\leqq 0\ (l<x<1)\,. \\ \phi(x) \text{ is bounded, and continuous for a.e. } x\in[0, 1]\,. \end{cases}$$

We know the uniqueness and existence of the solution (T^*, s, u) of (S) under the assumption (A) by [11, Theorem 1].

We assume that $\{l, \phi\}$ satisfies (A) throughout this paper.

Now we introduce an elliptic unilateral problem (E) and (E') which are closely related to (S).

$$(E)\ \begin{cases} (1.1)\quad w_{xx}(x)=0\ (0<x<1)\,, \\ (1.2)\quad w_x(0)\in\gamma_0(w(0))\,, \\ (1.3)\quad -w_x(1)\in\gamma_1(w(1))\,, \\ (1.4)\quad w(\mu)=0 \ \text{ for some }\ \mu\in[0, 1]\,. \end{cases}$$

where (1.1)–(1.3) constitute (E').

DEFINITION 1.1. $w(x)$ is a solution of (E) (resp. (E')) if $w(x)$ is a linear function satisfying (1.2), (1.3) and (1.4) (resp. (1.2) and (1.3)).

We state the existence theorem for (E').

Theorem 1. *There exists a solution $w(x)$ of (E'). The constant $w_x(x)$ is uniquely determined and $w_x(x) \leqq 0$.*

REMARK 1.1. (E') has either one solution or an infinite number of solutions.

REMARK 1.2. $w(x)=(B-A)x+A$ is a solution of (E'), if and only if (A, B) is a solution of

$$(\tilde{E}') \begin{cases} (1.5) & B-A \in \gamma_0(A)\,, \\ (1.6) & A-B \in \gamma_1(B)\,. \end{cases}$$

REMARK 1.3. We put $L=A-B$. Thus $(\tilde{E}')$ is equivalent to the following nonlinear problem for (A, B, L),

$$(E^*) \begin{cases} (1.7) & L = A-B\,, \\ (1.8) & A \in (-\gamma_0)^{-1}(L)\,, \\ (1.9) & B \in \gamma_1^{-1}(L)\,. \end{cases}$$

Thus it is easily seen that four points $(B, 0)$, $(A, 0)$, (A, L), $(B, L) \in \boldsymbol{R}^2$ are vertices of a square satisfying (1.8) and (1.9). Consequently our geometrical intuition is available when investigating (E^*), so (E').

We introduce notations to state a uniqueness theorem. Let L be the constant $-w_x(x)$ uniquely defined by Theorem 1. We put

$$L_s = \sup\,\{P-Q;\, P \in (-\gamma_0)^{-1}(L),\, Q \in \gamma_1^{-1}(L),\, P \geqq Q\}\,,$$
$$L_i = \inf\,\{P-Q;\, P \in (-\gamma_0)^{-1}(L),\, Q \in \gamma_1^{-1}(L),\, P \geqq Q\}\,.$$

It is easily seen from Theorem 1 that

$$0 \leqq L_i \leqq L_s \leqq \infty\,.$$

Theorem 2. *The solution of (E') is uniquely determined if and only if any of the following conditions is satisfied,*

(i) $(-\gamma_0)^{-1}(L) \cap \gamma_1^{-1}(L)$ *is a one point set with* $L=0$,
(ii) $(-\gamma_0)^{-1}(L)$ *is a one point set with* $L>0$,
(iii) $\gamma_1^{-1}(L)$ *is a one point set with* $L>0$,
(iv) $L_s=L$ *with* $L>0$,
(v) $L_i=L$ *with* $L>0$.

REMARK 1.4. $(-\gamma_0)^{-1}(L)$ and $\gamma_1^{-1}(L)$ are closed intervals, since γ_0 and

γ_1 are closed graphs in $\boldsymbol{R}^2$.

Consequently we get informations about the existence and uniqueness of solutions of (E) by virtue of Theorem 1 and Theorem 2. The structure of solutions of (E) is as follows, when the set of solution of (E) is not empty.

Theorem 3. *Suppose that (E) has a solution. Then there exist the minimum solution $\underline{w}(x)$ and the maximum solution $\overline{w}(x)$. Furthermore the set W of solutions of (E) is represented by*

$$W=\{\underline{w}(x)+c;\ c\in[0,\ \overline{w}(0)-\underline{w}(0)]\}\ .$$

REMARK 1.5. The condition $\underline{w}(x)\equiv 0$ is equivalent to the condition, $\gamma_0(0)\ni 0$ and $\gamma_1(0)\ni 0$.

Now we state results concerning the behavior of the solutions of (S). We define $\underline{s}$ and $\bar{s}$ by

$$\underline{s}=\begin{cases}0 & \text{if } \underline{w}(x)\equiv 0\,,\\ \text{the unique zero point of } \underline{w}(x) & \text{if } \underline{w}(x)\not\equiv 0\,.\end{cases}$$

$$\bar{s}=\begin{cases}1 & \text{if } \overline{w}(x)\equiv 0\,,\\ \text{the unique zero point of } \overline{w}(x) & \text{if } \overline{w}(x)\not\equiv 0\,.\end{cases}$$

In what follows let $(T^*,\ s,\ u)$ be the solution of (S) corresponding to the data $\{l,\ \phi\}$.

Theorem 4. *Suppose that $T^*=\infty$. Then there exist a real number s^* with $\underline{s}\leqq s^*\leqq\bar{s}$, and a solution $u^*(x)$ of (E) such that*

$$(1.10)\qquad \lim_{t\to\infty} s(t)=s^*\,,$$

$$(1.11)\qquad \lim_{t\to\infty} u(x,\ t)=u^*(x) \text{ in } C([0,\ 1])\,,$$

$$(1.12)\qquad u^*(x)=\underline{w}_x(0)\,(x-s^*)\,.$$

REMARK 1.6. s^* and $u^*(x)$ are determined by the initial data $\{l,\ \phi(x)\}$ in general, when $\underline{w}(x)\not\equiv\overline{w}(x)$.

We can determine s^* and $u^*(x)$ of Theorem 4 in some cases.

Corollary 1. *Let $\gamma_0(0)\ni 0$ and $\gamma_1(0)\ni 0$. Suppose $T^*=\infty$. Then we have $0\leqq s^*\leqq 1$ and $u^*(x)\equiv 0$.*

Corollary 2. *Let $\gamma_0(0)\not\ni 0$ or $\gamma_1(0)\not\ni 0$, and let any one of the conditions in Theorem 2 be satisfied. Suppose $T^*=\infty$. Then we have $s^*=\underline{s}=\bar{s}$ and $u^*(x)=\underline{w}(x)=\overline{w}(x)\not\equiv 0$.*

Corollary 3. *Let $\gamma_0(0)\not\ni 0$ or $\gamma_1(0)\not\ni 0$, and let $\phi(x)\leqq\underline{w}(x)$ (resp. $\phi(x)\geqq$*

$\overline{w}(x)$). *Suppose* $T^*=\infty$. *Then we have* $\underline{s}^*=\underline{s}$ (*resp.* $s^*=\bar{s}$) *and* $u^*(x)=\underline{w}(x)$ (*resp.* $u^*(x)=\overline{w}(x)$).

Corollary 4. *Let* $\gamma_0(0)\not\ni 0$ *or* $\gamma_1(0)\not\ni 0$, *and let* $\underline{w}(x)\leqq\phi(x)\leqq\overline{w}(x)$. *Suppose* $T^*=\infty$. *Then* s^* *is the unique real number such that*

$$\text{(1.13)}\qquad \begin{aligned} &b\, s^*+c_0\int_0^{s^*} u^*(x)dx+c_1\int_{s^*}^1 u^*(x)dx \\ &= b\, l+c_0\int_0^l \phi(x)dx+c_1\int_l^1 \phi(x)dx\,, \end{aligned}$$

and $\underline{w}(x)\leqq u^*(x)=\underline{w}_x(0)\,(x-s^*)\leqq\overline{w}(x)$.

We state some sufficient conditions for $T^*=\infty$.

Theorem 5. *If any one of the following conditions is satisfied, then we have* $T^*=\infty$.

(i) $D(\gamma_0)\not\ni 0$ *and* $D(\gamma_1)\not\ni 0$.

(ii) $D(\gamma_0)\not\ni 0$ (*resp.* $D(\gamma_1)\not\ni 0$). (*E*) *has a solution* $w(x)$ *such that* $w(1)<0$ (*resp.* $w(0)>0$). *The initial function satisfies* $\phi(x)\leqq w(x)$ (*resp.* $\phi(x)\geqq w(x)$).

(iii) (*E*) *has a solution* $w_1(x)$ *and* $w_2(x)$ *such that* $w_1(x)\leqq w_2(x)$, $w_1(0)w_1(1)<0$, $w_2(0)w_2(1)<0$. *The initial function satisfies* $w_1(x)\leqq\phi(x)\leqq w_2(x)$.

(iv) (*E*) *has a solution* $w(x)$ *with a zero point satisfying* $0<m<1$. *The initial data* $\{l, \phi\}$ *satisfies*

$$0<l_0-\varepsilon_0\leqq l_1+\varepsilon_1<1\,,$$

where $l_0=\min(l, m)$, $\phi_0=\min(\phi, w)$, $l_1=\max(l, m)$, $\phi_1=\max(\phi, w)$,

$$\begin{aligned} (-1)^i\varepsilon_i &= -b^{-1}\{c_0\int_0^{l_i}\phi_i(x)\,dx+c_1\int_{l_i}^1\phi_i(x)\,dx\} \\ &\quad +b^{-1}\{c_0\int_0^m w(x)\,dx+c_1\int_m^1 w(x)\,dx\}\,. \end{aligned}$$

We state a sufficient condition for $T^*<\infty$.

Theorem 6. *Suppose that* (*E*) *has no solution, then* $T^*<\infty$.

REMARK 1.7. The converse proposition does not hold. In fact, for example, if $\gamma_0=\gamma_1\equiv 0$, then (*E*) has a unique trivial solution $\underline{w}(x)=\overline{w}(x)\equiv 0$, and the behavior of the free boundary $s(t)$ depends on the initial data $\{l, \phi(x)\}$ (see [2, §5]).

In what follows we use the following notations.

$$\begin{aligned} H_0 &= \min\{H\geqq 0;\ H\in\gamma_0^{-1}(0)\}\,, \\ H_1 &= \max\{H\leqq 0;\ H\in\gamma_1^{-1}(0)\}\,. \end{aligned}$$

2. Proof of Theorem 1, 2 and 3

In this section we give the proof of Theorem 1, 2 and 3. We introduce a sequence $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ by

(2.1) $\quad B_1 = H_1\,,$

(2.2) $\quad B_n - A_n \in \gamma_0(A_n) \quad (\text{i.e. } A_n = (I+\gamma_0)^{-1}(B_n))\,,$

(2.3) $\quad A_n - B_{n+1} \in \gamma_1(B_{n+1}) \quad (\text{i.e. } B_{n+1} = (I+\gamma_1)^{-1}(A_n))\,.$

We shall get several estimates for $\{A_n\}$ amd $\{B_n\}$.

Lemma 2.1.
(i) *$H_1 = B_1 \leqq H_0$.*
(ii) *If $H_1 \leqq B_n \leqq H_0$, then $H_1 \leqq A_n \leqq H_0$.*
(iii) *If $H_1 \leqq A_n \leqq H_0$, then $H_1 \leqq B_{n+1} \leqq H_0$.*

Proof. (i) is trivial. We shall show (ii). It follows from (2.2), $0 \in \gamma_0(H_0)$ and the monotonicity of γ_0 that

$$(B_n - A_n)(A_n - H_0) \geqq 0\,.$$

Thus we get $B_n \leqq A_n \leqq H_0$ by $B_n \leqq H_0$. Hence we have $H_1 \leqq A_n \leqq H_0$ by $H_1 \leqq B_n$. Consequently we obtain (ii). We get (iii) in the same way. q.e.d.

We get the following lemma from Lemma 2.1.

Lemma 2.2. *$H_1 \leqq A_n$, $B_n \leqq H_0$ $(n=1, 2, \cdots)$.*

We shall show the monotonicity of $\{A_n\}$ and $\{B_n\}$.

Lemma 2.3.
(i) *$B_1 \leqq B_2$*
(ii) *If $B_n \leqq B_{n+1}$, then $A_n \leqq A_{n+1}$.*
(iii) *If $A_n \leqq A_{n+1}$, then $B_{n+1} \leqq B_{n+2}$.*

Proof. We have (i) by $B_1 = H_1$ and Lemma 2.2. We shall show (ii). We get (2.2) and

$$B_{n+1} - A_{n+1} \in \gamma_0(A_{n+1})\,.$$

Hence we get by the monotonicity of γ_0 that

$$[(B_{n+1} - A_{n+1}) - (B_n - A_n)]\,(A_{n+1} - A_n) \geqq 0$$

Thus we get

$$(B_{n+1} - B_n)(A_{n+1} - A_n) \geqq (A_{n+1} - A_n)^2 \quad (\geqq 0)\,.$$

Thus we get $A_{n+1} \geqq A_n$ easily. Consequently we have (ii). We obtain (iii) in the same way. q.e.d.

Thus we get the next lemma from Lemma 2.2 and 2.3.

Lemma 2.4.

$$H_1 \leqq A_1 \leqq A_1 \leqq \cdots \leqq H_0$$
$$H_1 = B_1 \leqq B_2 \leqq \cdots \leqq H_0 .$$

Proof of Theorem 1. There exist A^* and B^* such that

$$\lim_{n\to\infty} A_n = A^*, \lim_{n\to\infty} B_n = B^* ,$$

by Lemma 2.4. Hence we get (1.5), (1.6) with $A=A^*$, $B=B^*$ in view of (2.2), (2.3) and the closedness of γ_0 and γ_1. Thus we get the existence of a solution of (E') using Remark 1.2.

We shall show that $w_x(x)$ is uniquely determined. Let $w^1(x)$ and $w^2(x)$ be solutions of (E'). It follows from the integration by parts, (1.1), (1.2), (1.3) and the monotonicity of γ_0, γ_1 that

$$\int_0^1 (w_x^1(x) - w_x^2(x))^2 dx$$
$$= (w_x^1(1) - w_x^2(1))(w^1(1) - w^2(1))$$
$$-(w_x^1(0) - w_x^2(0))(w^1(0) - w^2(0)) \leqq 0 .$$

Hence we have $w_x^1(x) \equiv w_x^2(x)$.

We shall show that $w_x(x)$ $(=w_x(0)=w_x(1)) \leqq 0$. We put $A=w(0)$, $B=w(1)$ and $L=-w_x(x)$ $(=A-B)$. Assume that $L=A-B<0$. Then we have from (1.2) and (1.3) that

$$\gamma_0^{-1}(-L) \ni A, \gamma_1^{-1}(L) \ni B, -L>0>L .$$

Hence we get $A \geqq H_0$ (resp. $H_1 \geqq B$) from $\gamma_0^{-1}(0) \ni H_0$ (resp. $\gamma_1^{-1}(0) \ni H_1$) and the monotonicity of γ_0^{-1} (resp. γ_1^{-1}). Thus we have $A \geqq (H_0 \geqq 0 \geqq H_1 \geqq) B$, which is a contradiction. Therefore we obtain $L \geqq 0$. q.e.d.

Proof of Theorem 2. We can easily get the conclusion using Remark 1.3, Theorem 1 and the fact that $(-\gamma_0)^{-1}(L)$ and $\gamma_1^{-1}(L)$ are closed intervals. q.e.d.

Proof of Theorem 3. It is easily seen from Theorem 1 and Remark 1.3 that the set W of solutions of (E) is represented by

$$W = \{-Lx + A;\ A \in I\} ,$$
$$I = (-\gamma_0)^{-1}(L) \cap (I + \gamma_1^{-1})(L) \cap [0, L] ,$$

where L is the non-negative constant $-w_x(x)$ uniquely defined by Theorem 1.

It follows from the maximal monotonicity of γ_0 and γ_1 that $(-\gamma_0)^{-1}(L)$ and $(I+\gamma_1^{-1})(L)$ are closed intervals. Thus I is a bounded closed interval. q.e.d.

3. Comparison theorems

In this section we show the comparison theorems for the Stefan problem (S). The following lemma shows that the free boundary $s(t)$ varies continuously with the continuous change of the initial data $\{l, \phi(x)\}$.

Lemma 3.1. *Let (T_1^*, s_1, u_1) and (T_2^*, s_2, u_2) be the solutions of (S) corresponding to the data $\{l_1, \phi_1\}$ and $\{l_2, \phi_2\}$ respectively. Suppose that $l_1 \leqq l_2$ and $\phi_1 \leqq \phi_2$. Then we have*

$$(3.1) \qquad u_1(x, t) \leqq u_2(x, t) \quad (0 \leqq x \leqq 1,\ 0 \leqq t \leqq T_3^*),$$

$$(3.2) \qquad \begin{aligned} &0 \leqq s_2(t) - s_1(t) \\ &\leqq [1 + K(\|\phi_2\|_{L^\infty(0,1)} + \|\phi_1\|_{L^\infty(0,1)})](l_2 - l_1) \\ &\quad + K\|\phi_2 - \phi_1\|_{L^1(0,1)}, \end{aligned}$$

where $T_3^* = \min(T_1^*, T_2^*)$, $K = b^{-1}\max(c_0, c_1)$.

Proof. We get $u_1 \leqq u_2$ and $s_1 \leqq s_2$ by [11, Proposition 15.1]. It follows from the proof of [11, Proposition 15.1] that

$$(3.3) \qquad \begin{aligned} b(s_2(t) - s_1(t)) &\leqq b(l_2 - l_1) \\ &+ c_0\Big(\int_0^{l_2} \phi_2\, dx - \int_0^{l_1} \phi_1\, dx\Big) + c_1\Big(\int_{l_2}^1 \phi_2\, dx - \int_{l_1}^1 \phi_1\, dx\Big). \end{aligned}$$

Thus we get (3.2) easily. q.e.d.

We prepare simple lemmas which is useful when we investigate the behavior of the solutions of (S).

Lemma 3.2. *Let $w(x)$ be a solution of (E) with a zero point m, $0 < m \leqq 1$ (resp. $0 \leqq m < 1$). Let (T^*, s, u) be the solution of (S) corresponding to the data $\{l, \phi\}$. Suppose that*

$$(3.4) \qquad l \geqq m - \rho \quad (\textit{resp.}\ \leqq m + \rho),$$

$$(3.5) \qquad \phi(x) \geqq w(x) - \chi(\rho) \quad (\textit{resp.}\ \leqq w(x) + \chi(\rho)),$$

where

$$\chi(\xi) = \chi(\xi; w) = \begin{cases} |w_x(0)|\,\xi & \text{if } w \not\equiv 0 \\ \rho & \text{if } w \equiv 0 \end{cases},$$

and ρ is a sufficiently small positive real number, that is,

$$(3.6) \qquad L_\rho < m \quad (\textit{resp.} < 1 - m),$$

(3.7) $\quad \chi(\rho)<w(0)$ (*resp.* $<-w(1)$) *only if* $w\not\equiv 0$.

Here $L_\rho=(1+2KM_\rho)\rho+K\chi(\rho)$, $K=b^{-1}\max(c_0, c_1)$ *and* $M_\rho=\|w\|_{L^\infty(0,1)}+\chi(\rho)$ *Then we have*

(3.8) $\quad s(t)\geqq m-L_\rho$ (*resp.* $\leqq m+L_\rho$) *for* $0\leqq t\leqq T^*$,

(3.9) $\quad u(x, t)\geqq w(x)-\chi(L_\rho)$ (*resp.* $\leqq w(x)+\chi(L_\rho)$) *for* $0\leqq x\leqq 1$, $0\leqq t\leqq T^*$.

Proof. We shall prove the first case of the statement of the lemma. Let (T_1^*, s_1, u_1) be the solution of (S) corresponding to the data $\{l_1, \phi_1\}$, where

$$l_1=m-\rho,$$
$$\phi_1(x)=\begin{cases}\max(w(x)-\chi(\rho), 0) & \text{for } 0\leqq x<m-\rho\\ w(x)-\chi(\rho) & \text{for } m-\rho\leqq x\leqq 1\end{cases}.$$

Thus it follows from (3.4), (3.5), (3.6), (3.7) and Lemma 3.1 applied to $\{l, \phi\}$ and $\{l_1, \phi_1\}$ that $T^*\leqq T_1^*$ and

(3.10) $\quad s(t)\geqq s_1(t)\quad (0\leqq t\leqq T^*)$,

(3.11) $\quad u(x, t)\geqq u_1(x, t)\quad (0\leqq x\leqq 1, 0\leqq t\leqq T^*)$.

Hence we obtain (3.8) using (3.10), Lemma 3.1 and the definition of $\{l_1, \phi_1\}$. We shall show (3.9). We regard $u_1(x, t)$ as the solution of the moving boundary problem with the curve $s_1(t)$. It is easily seen from [11, Lemma 8.1] and the proof of [10, Proposition 10.1] that

(3.12) $\quad u_1(x, t)\geqq w(x)-\chi(L_\rho)\quad (0\leqq x\leqq 1, 0\leqq t\leqq T^*)$.

Consequently we get (3.9) by (3.11) and (3.12). q.e.d.

Lemma 3.3. *Let* (T^*, s, u) *be the solution of* (S) *corresponding to the data satisfying* (A). *Suppose* $T^*=\infty$. *Then for any* $\rho>0$ *there exists* t_ρ *such that*

(3.13) $\quad w^0(x)-\rho\leqq u(x, t)\leqq w^1(x)+\rho\quad (t\geqq t_\rho)$,

where $w^k(x)$ $(k=0, 1)$ *is the solution of the elliptic unilateral problem* (E_k),

$$(E_k)\begin{cases}(3.14)\quad w^k_{xx}(x)=0\quad (0<x<1),\\ (3.15)\quad w^k(k)=0,\\ (3.16)\quad (-1)^{1-k}w^k_x(1-k)\in\gamma_{1-k}(w^k(1-k)).\end{cases}$$

REMARK 3.1 If $w(x)$ is a solution of (E) with a zero point m satisfying $m=0$ (resp. $m=1$), then we see that $w(x)=w^0(x)$ (resp. $w(x)=w^1(x)$).

Proof. We shall show the right inequality of (3.13). It follows from [10, Proposition 10.1] that

$$u(x, t) \leqq v^1(x, t) \quad (0 \leqq x \leqq 1,\ t \geqq 0),$$

where $v^1(x, t)$ is the solution of the parabolic unilateral problem (P_1) with the initial data $\psi(x)=\phi(x)$, which is introduced in [12, §5]. Hence we get the conclusion using [12, Proposition 5.1]. We can get the left inequality of (3.13) in the same way. q.e.d.

4. A priori estimates of the free boundary $s(t)$

In this section we prove the following proposition which give some a priori estimates of the free boundary $s(t)$.

Proposition 4.1. *Let (T^*, s, u) be the solution of (S) corresponding to the data $\{l, \phi\}$.*

(i) *If $D(\gamma_0) \not\ni 0$, then there exist positive constants δ and δ_σ depending on $\sigma \in]0, T^*[$ such that*

$$s(t) > \delta \quad (0 \leqq t \leqq T^*),$$
$$u(0, t) > \delta_\sigma \quad (\sigma \leqq t \leqq T^*).$$

(ii) *If $D(\gamma_1) \not\ni 0$, then there exist positive constants δ and δ_σ depending on $\sigma \in]0, T^*[$ such that*

$$s(t) < 1-\delta \quad (0 \leqq t \leqq T^*),$$
$$u(1, t) < -\delta_\sigma \quad (\sigma \leqq t \leqq T^*).$$

(iii) *If $D(\gamma_0) \not\ni 0$ and $D(\gamma_1) \not\ni 0$, then $T^* = \infty$ and there exist positive constants δ and δ_σ depending on $\sigma \in]0, \infty[$ such that*

$$\delta < s(t) < 1-\delta \quad (0 \leqq t < \infty),$$
$$u(0, t) \geqq \delta_\sigma \quad (\sigma \leqq t < \infty),$$
$$u(1, t) \leqq -\delta_\sigma \quad (\sigma \leqq t < \infty).$$

We may assume that $\{l, \phi\}$ satisfies the following condition,

$$(H) \quad \begin{array}{l} 0<l<1,\ \phi(0) \in D(\gamma_0),\ \phi(l)=0,\ \phi(1) \in D(\gamma_1), \\ \phi \in C^{0,1}([0, 1]):\ |\phi(x')-\phi(x)| \leqq K|x'-x| \quad (0 \leqq x \leqq x' \leq 1), \end{array}$$

since u satisfies (0.4) and [11, (1.14)]. In what follows we assume (H). We note that the conclusions of [11, Theorem 2] hold by virtue of (H).

We shall show (i) of Proposition 4.1. We investiagte the behavior of the free boundary according to the shape of γ_0. Since γ_0 is a maximal monotone graph in $\boldsymbol{R}^2$ with $D(\gamma_0) \not\ni 0$ and $\gamma_0(H_0) \ni 0$ $(H_0 \geqq 0)$, we see that

$$D(\gamma_0)\cap[0, H_0] = \begin{cases} (\mathrm{i}) & [B, H_0], \text{ where } 0<B\leqq H_0\,, \\ (\mathrm{ii}) &]B, H_0], \text{ where } 0<B<H_0\,, \\ (\mathrm{iii}) &]0, H_0]\,. \end{cases}$$

Lemma 4.1. *Suppose that γ_0 satisfies* (i) *or* (ii). *Then we have*

(4.1) $\quad s(t)\geqq d \quad (0\leqq t\leqq T^*)\,,$

(4.2) $\quad u(0, t)\geqq B \quad (0\leqq t\leqq T^*)\,,$

where $d=2^{-1}\min(B/A, l)$, $A=\max[-\min(-\|\phi\|_{L^\infty(l,1)}, H_1)/(1-l), K]$.

Proof. Suppose that $s(t)=d$ for some $t\in[0, T^*]$. Let $t^*=\inf\{t\in[0, T^*[\,;\, s(t)=d\}$. Clearly $\dot{s}(t^*)\leqq 0$. Consider the function

$$w(x, t) = A(d-x) \ (s(t)\leqq x\leqq 1,\ 0\leqq t\leqq t^*)\,,$$

which satisfies

$$\begin{aligned} &w_{xx}-c_1 w_t = 0 \ (s(t)<x<1,\ 0<t<t^*)\,, \\ &w(s(t), t) = A(d-s(t))\leqq 0 = u(s(t), t) \ (0\leqq t\leqq t^*)\,, \\ &w(1, t) = A(d-1)\leqq A(l-1) \\ &\qquad\leqq \min(-\|\phi\|_{L^\infty(l,1)}, H_1)\leqq u(1, t) \ (0\leqq t\leqq t^*)\,, \\ &w(x, 0) = A(d-x)\leqq K(d-x)\leqq K(l-x)\leqq \phi(x) \ (l\leqq x\leqq 1)\,, \end{aligned}$$

by [11, (1.13)], (H) and the definition of t^*, d, A. Thus we have

$$w(x, t)\leqq u(x, t) \ (s(t)\leqq x\leqq 1,\ 0\leqq t\leqq t^*)$$

by the elementary maximum principle. Hence we have

(4.3) $\quad u_x^+(s(t^*), t^*)\geqq -A\,.$

Next we introduce the function

$$z(x, t) = B(1-x/d) \ (0\leqq x\leqq s(t),\ 0\leqq t\leqq t^*)\,.$$

It follows that

$$\begin{aligned} &z_{xx}-c_0 z_t = 0 \ (0<x<s(t),\ 0<t<t^*)\,, \\ &z(s(t), t) = B(1-s(t)/d)\leqq 0 = u(s(t), t) \ (0\leqq t\leqq t^*)\,, \\ &z(0, t) = B\leqq u(0, t) \ (0\leqq t\leqq t^*)\,, \\ &z(x, 0)\leqq B-Ax\leqq \phi(0)-Kx\leqq \phi(x) \ (0\leqq x\leqq l)\,, \end{aligned}$$

by (0.4), $\phi(0)\in D(\gamma_0)$, (H) and the definition of t^*, d, A. Hence we have

$$z(x, t)\leqq u(x, t) \ (0\leqq x\leqq s(t),\ 0\leqq t\leqq t^*)$$

by the maximum principle. Thus we have

(4.4) $\quad u_x^-(s(t^*), t^*) \leqq -B/d$.

Consequently we see from (0.7), (4.3), (4.4) and the definition of d that

$$b\dot{s}(t^*) \geqq B/d - A > 0,$$

which is in contradiction to $\dot{s}(t^*) \leqq 0$. q.e.d.

Lemma 4.2. *Suppose that* γ_0 *satisfies* (iii). *Then we have*

(4.5) $\quad s(t) \geqq d_1 \ (0 \leqq t \leqq T^*)$,

(4.6) $\quad u(0, t) \geqq A_1 d_1 \ (0 \leqq t \leqq T^*)$,

where $\quad d_1 = 2^{-1} \min(B_1/A_1, l),\ A_1 = A+1$,

$\quad B_1 = \min(\max\{\xi; \gamma_0(\xi) \ni -A_1\}, \phi(0))$,

$\quad A = \max(-\min(-\|\phi\|_{L^\infty(l,1)}, H_1)/(1-l), K)$.

Proof. Suppose that $s(t)=d_1$ for some $t \in [0, T^*]$. Let $t^* = \inf\{t \in [0, T^*[\ ; s(t)=d_1\}$. Clearly $\dot{s}(t^*) \leqq 0$. We can get

(4.7) $\quad u_x^+(s(t^*), t^*) \geqq -A$

by the argument used in the proof of previous lemma. We introduce the function

$$z(x, t) = A_1(d_1 - x) \ (0 \leqq x \leqq s(t),\ 0 \leqq t \leqq t^*).$$

It follows that

(4.8) $\quad z_{xx} - c_0 z_t = 0 \ (0 < x < s(t),\ 0 < t < t^*)$,

(4.9) $\quad z(s(t), t) = A_1(d_1 - s(t)) \leqq 0 = u(s(t), t) \ (0 \leqq t \leqq t^*)$,

(4.10) $\quad z(x, 0) \leqq A_1 d_1 - Kx \leqq B_1 - Kx \leqq \phi(0) - Kx \leqq \phi(x) \ (0 \leqq x \leqq l)$,

(4.11) $\quad (z_x(0, t) - u_x(0, t))\,(z(0, t) - u(0, t))^+$

$$= (-A_1 - u_x(0, t))\,(A_1 d_1 - u(0, t))^+$$

$$\begin{cases} = 0 & \text{for } u(0, t) \geqq A_1 d_1, \\ \geqq (-A_1 - \min\{\eta; \eta \in \gamma_0(B_1)\})\,(A_1 d_1 - u(0, t)) & \\ & \text{for } u(0, t) < A_1 d_1 \ (< B_1), \end{cases}$$

$$\geqq 0$$

by (H), (0.4) (a) and the definition t^*, d_1, A_1. Hence we have

$$z(x, t) \leqq u(x, t) \ (0 \leqq x \leqq s(t),\ 0 \leqq t \leqq t^*),$$

by [10, Lemma 10.1]. Thus we have

(4.12) $\quad u_x^-(s(t^*), t^*) \leqq -A_1$.

Consequently we see from (0.7), (4.7), (4.12) that

$$b\dot{s}(t^*) \geqq A_1 - A = 1 > 0 ,$$

which is in contradiction to $\dot{s}(t^*) \leqq 0$. Thus we get (4.5). Therefore we obtain $z(x, t) \leqq u(x, t)$ on $\bar{D}^0_{T^*}$ using (4.5), (4.8), (4.9), (4.10), (4.11) and [10, Lemma 10.1]. Hence we get (4.6) by $z(0, t) \leqq u(0, t)$ $(0 \leqq t \leqq T^*)$. q.e.d.

Proof of Proposition 4.1. We get (i) by Lemma 4.1 and 4.2. We get (ii) by the argument similar to the proof of (i). We obtain (iii) by (i) and (ii). q.e.d.

5. A priori estimates of $u_x(0, t)$ and $u_x(1, t)$

We show the following a priori estimates of the derivatives of the solution of (S) at the fixed boundary.

Proposition 5.1. *Let (T^*, s, u) be the solution of (S) corresponding to the data $\{l, \phi\}$. Then there exists a positive constant C_σ depending on $\sigma \in]0, T^*[$ such that*

(5.1) $\quad |u_x(0, t)| \leqq C_\sigma$ *a.e.* $t \in [\sigma, T^*[$,

(5.2) $\quad |u_x(1, t)| \leqq C_\sigma$ *a.e.* $t \in [\sigma, T^*[$.

Proof. We may assume the condition (H) which is introduced in §4, since u satisfies (0.4) and [11, (1.14)]. We shall show (5.1). It is easily seen from [10, Lemma 10.1] and the proof of [10, Lemma 6.1] that

(5.3) $\quad u_x(0, t) \leqq C_{1,\sigma}$ a.e. $t \in [\sigma, T^*[$,

where $C_{1,\sigma}$ is a positive constant depending on σ. We shall obtain the estimate from below. We consider two cases.

Case (i). $D(\gamma_0) \not\ni 0$.

It is easily seen from [10, Lemma 10.1], Proposition 4.1 (i) and the proof of [10, Lemma 6.1] that

(5.4) $\quad -C_{2,\sigma} \leqq u_x(0, t)$ a.e. $t \in [\sigma, T^*[$,

where $C_{2,\sigma}$ is a positive constant depending on σ.

Case (ii). $D(\gamma_0) \ni 0$.

It follows from [11, (1.12)] that $u_x(0, t) \geqq 0$ if $u_x(0, t)$ exists and $u(0, t) = 0$. Thus we have

(5.5) $\quad -C_{3,\sigma} \leqq u_x(0, t)$ a.e. $t \in [\sigma, T^*[$, where $C_{3,\sigma}=-\max\{\eta;\ \eta\in\gamma_0(0)\}\geqq 0$.

Hence we obtain (5.1) by (5.3), (5.4), (5.5). We can get (5.2) in the same way. q.e.d.

6. Weak formulation of the Stefan problem (S)

We give a weak formulation of the Stefan problem (S), which will be useful for us to investigate the asymptotic behavior of the solutions.

Proposition 6.1. *Let (T^*, s, u) be the solution of (S) corresponding to the data $\{l, \phi\}$, then we have*

$$\tag{6.1} \begin{aligned} &\int_0^{s(t)} c_0 u_t v\,dx+\int_{s(t)}^1 c_1 u_t v\,dx+b\,\dot{s}(t)\,v(s(t))\\ &+\int_0^1 u_x v_x\,dx+u_x(0,t)\,v(0)-u_x(1,t)\,v(1)=0 \end{aligned}$$

a.e. $t\in[0, T^*]$, where $v(x)\in H^1(0,1)$ is arbitrary.

REMARK 6.1. We put

$$U(x,t)=\beta(u(x,t))=\begin{cases} c_0\,u(x,t)+b & \text{for } u(x,t)\geqq 0\,,\\ c_1\,u(x,t) & \text{for } u(x,t)<0\,. \end{cases}$$

Then (6.1) is written formally as follows.

$$\tag{6.2} \begin{aligned} &\frac{d}{dt}\int_0^1 U\,v\,dx+\int_0^1 \beta^{-1}(U)_x v_x\,dx\\ &+\gamma_0(\beta^{-1}(U(0,t)))v(0)+\gamma_1(\beta^{-1}(U(1,t)))\,v(1)\ni 0\,. \end{aligned}$$

It may be natural that (6.2) is a weak formulation of (S). However we do not investigate (6.2) here.

REMARK 6.2. When γ_0 and γ_1 are single valued functions, Cannon & DiBenedetto [3], Visintin [8], Niezgodka-Pawlow-Visintin [6] investigated the problem similar to (6.2) with some additional conditions in n-dimensional case.

Proof of Proposition 6.1. We have

$$\begin{aligned} &\int_0^{s(t)} c_0 u_t v\,dx+\int_{s(t)}^1 c_1 u_t v\,dx\\ &=\int_0^{s(t)} u_{xx} v\,dx+\int_{s(t)}^1 u_{xx} v\,dx\\ &=[u_x v]_0^{s(t)-}-\int_0^{s(t)} u_x v_x\,dx+[u_x v]_{s(t)+}^1-\int_{s(t)}^1 u_x v_x\,dx\\ &=(u_x^-(s(t),t)-u_x^+(s(t),t))\,v(s(t))-u_x(0,t)\,v(0)+u_x(1,t)\,v(1)\\ &\quad-\int_0^1 u_x v_x\,dx \end{aligned}$$

using [11, Theorem 1], (0.2), (0.3). Thus we get (6.1) by (0.7). q.e.d.

7. Proof of Theorem 4, 5 and 6

In this section we give the proof of Theorem 4, 5 and 6. Let $T^*=\infty$. It follows from [11, Theorem 1] and Proposition 5.1 that

$$|u(x, t)| \leqq \text{const.} \quad \text{on } [0, 1]\times[1, \infty[\,, \tag{7.1}$$

$$|u(x', t)-u(x, t)| \leqq \text{const.}\ |x'-x|^{1/2} \quad \text{on } [0, 1]\times[1, \infty[\,, \tag{7.2}$$

$$\sup_{t\geqq 1}\{\int_0^1 u_x(x, t)^2\, dx\} + \int_1^\infty |\dot{s}(t)|^3\, dt + \int_1^\infty\int_0^1 u_t(x, t)^2\, dxdt < \infty\,, \tag{7.3}$$

$$u_x(0, \cdot)\in L^\infty(1, \infty),\ u_x(1, \cdot)\in L^\infty(1, \infty)\,. \tag{7.4}$$

Thus it is easily seen from (7.1), (7.2) and (7.3) that there exist a subsequence $\{t_n\}$, a real number $s^*\in[0, 1]$ and a function $u^*(x)\in C([0, 1])$ such that

$$s(t_n) \to s^*\,, \tag{7.5}$$

$$\dot{s}(t_n) \to 0\,, \tag{7.6}$$

$$u(x, t_n) \to u^*(x) \quad \text{in } C([0, 1])\,, \tag{7.7}$$

$$u_x(x, t_n) \to u_x^*(x) \quad \text{weakly in } L^2(0, 1)\,, \tag{7.8}$$

$$\int_0^1 u_t(x, t_n)^2\, dx \to 0 \tag{7.9}$$

as $t_n \to \infty$. Moreover we may assume from (0.4), (7.4), (7.7) and the closedness of γ_0 and γ_1 that there exist real numbers g_0 and g_1 such that

$$u_x(0, t_n) \to g_0\in\gamma_0(u^*(0))\,, \tag{7.10}$$

$$-u_x(1, t_n) \to g_1\in\gamma_1(u^*(1))\,. \tag{7.11}$$

Consequently we see from (6.1), (7.6), (7.8), (7.9), (7.10) and (7.11) that

$$\int_0^1 u_x^* v_x\, dx + g_0\, v(0) + g_1\, v(1) = 0$$

for all $v\in H^1(0, 1)$. Hence

$$\begin{cases} u_{xx}^*(x) = 0 \qquad (0<x<1)\,, \\ u_x^*(0) = g_0\in\gamma_0(u^*(0))\,, \\ -u_x^*(1) = g_1\in\gamma_1(u^*(1))\,, \\ u^*(s^*) = 0\,, \quad 0\leqq s^*\leqq 1\,. \end{cases}$$

Therefore $u^*(x)$ is a solution of (E).

We shall show that

(7.12) $s(t) \to s^*$,

(7.13) $u(x, t) \to u^*(x)$ in $C([0, 1])$,

as $t \to \infty$ without taking a subsequence. For any given $\varepsilon > 0$, there exists t_N such that

(7.14) $|s(t_N) - s^*| < \varepsilon$,

(7.15) $|u(x, t_N) - u^*(x)| < \varepsilon \quad (0 \leqq x \leqq 1)$

by (7.5) and (7.7). Consequently for any $\varepsilon > 0$ there exists $t_\varepsilon\,(\geqq t_N)$ such that

$$|s(t) - s^*| < \varepsilon \qquad (t \geqq t_\varepsilon),$$
$$|u(x, t) - u^*(x)| < \varepsilon \qquad (0 \leqq x \leqq 1,\ t \geqq t_\varepsilon),$$

from (7.14), (7.15), Lemma 3.2, Lemma 3.3 and Remark 3.1.

Thus we complete the proof of Theorem 4.

Theorem 6 is obvious from Theorem 4.

We obtain (i), (ii), (iii) of Theorem 5 using Lemma 3.1 and Proposition 4.1. To prove Theorem 5 (iv), we can apply Lemma 3.1 and the inequality (3.3) to the solutions of (S) corresponding to the data $\{l_0, \phi_0\}$, $\{m, w\}$ and the data $\{m, w\}$, $\{l_1, \phi_1\}$.

8. Proof of Corollary 1, 2, 3 and 4

We give the proof of Corollaries.

Proof of Corollary 1. It is evident from Theorem 4, Theorem 3 and Remark 1.5. q.e.d.

Proof of Corollary 2. It is obvious from Theorem 4 and Theorem 2. q.e.d.

Proof of Corollary 3. It is obvious from Theorem 4, Lemma 3.1 and the fact that $\underline{w}(x)$ (resp. $\overline{w}(x)$) is the minimum (resp. maximum) solution of (E).

Proof of Corollary 4. We may assume $\underline{w}(x) < \overline{w}(x)$ in view of Corollary 2. We see from Lemma 3.1 and [10, Proposition 10.1] that

(8.1) $\underline{w}(x) \leqq u(x, t) \leqq \overline{w}(x) \quad (t \geqq 0)$.

It is easily shown that $\gamma_0(u) = \{\underline{w}_x(0)\}$ (resp. $\gamma_1(u) = \{-\underline{w}_x(0)\}$) for u satisfying $\underline{w}(0) < u < \overline{w}(0)$ (resp. $\underline{w}(1) < u < \overline{w}(1)$). Hence it is easily seen from (8.1) that

$$u_x(0, t) = \underline{w}_x(0) \qquad (\text{a.e. } t > 0),$$
$$u_x(1, t) = \underline{w}_x(0) \qquad (\text{a.e. } t > 0).$$

Thus it follows from [11, Proposition 9.1] that

$$\begin{aligned} & b\, s(t) + c_0 \int_0^{s(t)} u(x, t)\, dx + c_1 \int_{s(t)}^1 u(x, t)\, dx \\ = & b\, s(\sigma) + c_0 \int_0^{s(\sigma)} u(x, \sigma) dx + c_1 \int_{s(\sigma)}^1 u(x, \sigma)\, dx , \end{aligned}$$

for any $\sigma \in]0, t[$. Letting $t \to \infty$ and $\sigma \to 0$, we get (1.13) by Theorem 4 and [11, Theorem 1]. We see the uniqueness of s^* satisfying (1.13). In fact, the function,

$$E(p) = b\, p + c_0 \int_0^p \underline{w}_x(0)\, (x-p)\, dx + c_1 \int_p^1 \underline{w}_x(0)\, (x-p)\, dx ,$$

is continuous and strictly increasing in p, since $\underline{w}_x(0) < 0$. q.e.d.

9. Some examples

To make clear the meaning of our results, we give some simple examples. Let (T^*, s, u) be the solution of (S) corresponding to the initial data $\{l, \phi\}$. The unilateral condition (0.4) will be taken concretely.

EXAMPLE 1. Let $u_x(0, t) \equiv -\alpha$ and $u(1, t) \equiv -\beta$, where $0 \leqq \alpha < \beta$. Then $T^* < \infty$ and $s(T^*) = 0$.

In fact, we may apply Theorem 6 and Proposition 4.1 (ii) by taking

$$\gamma_0(u) = \begin{cases} -\alpha & (u \leqq M) \\ [-\alpha, \infty[& (u = M), \\ \emptyset & (u > M) \end{cases} \qquad \gamma_1(u) = \begin{cases}]-\infty, \infty] & (u = -\beta) \\ \emptyset & (u \neq -\beta) \end{cases},$$

where M is a sufficiently large real number (for example, $M = \|\phi\|_{L^\infty(0,l)} + \alpha + 1$).

EXAMPLE 2. Let $u_x(0, t) \equiv -\alpha$ and $u(1, t) \equiv -\beta$, where $0 < \beta \leqq \alpha$. Then either of the following situations occurs corresponding to the initial data.

(i) $T^* < \infty$ and $s(T^*) = 0$,

(ii) $T^* = \infty$, $\lim_{t \to \infty} s(t) = (\alpha - \beta)/\alpha$ and

$$\lim_{t \to \infty} u(x, t) = \alpha(1-x) - \beta .$$

Moreover a sufficient condition for (ii) is that $\alpha > \beta$ and $\phi(x) \geqq \alpha(1-x) - \beta$.

In fact, we may apply Theorem 4, Proposition 4.1 (ii) and Theorem 5 (ii) by taking γ_0 and γ_1 as above.

REMARK 9.1. From the physical intuition and numerical experiments by a computer it may be thought that (i) occurs when

$$-\{b\, l + c_0 \int_0^l \phi(x)\, dx + c_1 \int_l^1 \phi(x)\, dx\}$$

is sufficiently large, though we do not have a mathematical proof at present.

EXAMPLE 3. Let $u_x(0, t)=1-1/u(0, t)$, and $u(1, t)\leqq -1$, $u_x(1, t)\leqq 0$, $u_x(1, t)(u(1, t)+1)=0$ (Signorini type boundary condition). Then we have $T^*=\infty$,

$$\lim_{t\to\infty} s(t) = 1-1/\sqrt{2},$$

$$\lim_{t\to\infty} u(x, t) = -\sqrt{2}x+\sqrt{2}-1 \qquad \text{in } C([0, 1]).$$

In fact, we may apply Theorem 4, Corollary 2 by taking $\gamma_0(u)=1-1/u$ and

$$\gamma_1(u) = \begin{cases} 0 & (u<-1) \\ [0, \infty[& (u=-1) \\ \emptyset & (u>-1) \end{cases}.$$

EXAMPLE 4. Let

$u\geqq 1$, $u_x\leqq -3$ $(u=1)$, $u_x=-3$ $(1<u<2)$, $u_x=u-5$ $(u\geqq 2)$,

on the fixed boundary $x=0$, and

$u\leqq -1$, $u_x\leqq -3$ $(u=-1)$, $u_x=-3$ $(-1>u>-2)$, $u_x=-u-5$ $(u\leqq -2)$,

on the other fixed boundary $x=1$. Then we have $T^*=\infty$,

$$\lim_{t\to\infty} s(t) = s^*, \qquad 1/3\leqq s^*\leqq 2/3,$$

$$\lim_{t\to\infty} u(x, t) = -3\,(x-s^*) \qquad \text{in } C([0, 1]).$$

Moreover we have the following properties.

(i) If $\phi(x)\leqq -3x+1$ (resp. $\phi(x)\geqq -3x+2$), then $s^*=1/3$ (resp. $s^*=2/3$).

(ii) If $-3x+1\leqq\phi(x)\leqq -3x+2$ (with $1/3\leqq l\leqq 2/3$), then s^* is the unique real number such that $1/3\leqq s^*\leqq 2/3$ and

$$b\,s^*+3\{c_0\,(s^*)^2-c_1\,(1-s^*)^2\}/2$$
$$= b\,l+c_0\int_0^l \phi(x)\,dx+c_1\int_l^1 \phi(x)\,dx.$$

In fact, we may apply Theorem 4, Corollary 3, Corollary 4 and Theorem 5 (i) by taking

$$\gamma_0(u) = \begin{cases} \emptyset \quad (u<1) \\ ]-\infty, -3] \quad (u=1) \\ -3 \quad (1<u<2) \\ u-5 \quad (u\geqq 2) \end{cases}, \qquad \gamma_1(u) = \begin{cases} \emptyset \quad (u>-1), \\ [3, \infty[\quad (u=-1) \\ 3 \quad (-2<u<-1) \\ u+5 \quad (u\leqq -2) \end{cases}.$$

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