

COBORDISM GROUPS OF IMMERSIONS WITH RESTRICTED SELF-INTERSECTION

ANDRAS SZÜCS

(Received August 5, 1982)

0. Introduction. An l -immersion is an immersion without $(l+1)$ -tuple points. An immersion is called *completely regular* if its self-intersections are transversal. In 1971, F. Uchida defined the *cobordism group of l -immersions* as follows, (see [14]).

DEFINITION 0.1. A completely regular l -immersion $f: M^n \rightarrow N^{n+k}$ is *cobordant to zero* if there exists an immersion $F: V \rightarrow W$ where:

- (1) V and W are compact, C^∞ -differentiable manifolds of dimensions $n+1$ and $n+k+1$ respectively, and
- (2) $F: V \rightarrow W$ is a completely regular l -immersion such that $(F|_{\partial V}, \partial V, \partial W) = (f, M, N)$.

Two completely regular l -immersions (f_0, M_0, N_0) and (f_1, M_1, N_1) will be said to be *cobordant* if and only if the disjoint union $(f_0, M_0, N_0) + (-f_1, -M_1, -N_1)$ is cobordant to zero.

Let $C^0(n, k; l)$ denote the set of cobordism classes of completely regular l -immersions of dimensions $\dim M_i = n$, $\dim N_i = n+k$. As usual an abelian group structure is imposed on $C^0(n, k; l)$ by disjoint union.

We shall call these groups the *oriented Uchida groups*. Uchida investigated these groups by geometrical methods. We first reduce the computation of the groups $C^0(n, k; l)$ to algebraic topology and subsequently we compute the ranks of these groups.

Uchida proved that for the non-orientable version of his groups the natural map $C(n, k; l) \rightarrow C(n, k; l+1)$ is a monomorphism. We prove that this holds for the oriented Uchida groups as well. We shall proceed as follows. In Section 1 we describe a space $\Gamma_l(k)$. In Section 2 we show that the bordism groups of this space are isomorphic to the groups $C^0(n, k; l)$. In Section 3 we compute the ranks of the bordism groups of $\Gamma_l(k)$. Before closing the introduction, we give some remarks.

REMARK 0.2. The groups $C^0(n, k; l)$ for $l = \infty$ were considered earlier by Schweitzer [Sc]. He proved that these groups are isomorphic to the bor-

dism groups of the space $\Omega^\infty S^\infty MSO(k)$. $\Omega^\infty S^\infty MSO(k)$ denotes the infinite loop space of infinite suspension of the Thom space $MSO(k)$ of the universal k dimensional vector bundle.

REMARK 0.3. By a result of Barratt and Eccles, for any (connected) space X the space $\Omega^\infty S^\infty X$ is filtrated in some natural way (see [1], [2], [3]). The space $\Gamma_l(k)$, to be constructed in Section 1, actually coincides with the l -th element of this filtration in the case, when $X=MSO(k)$. For details see Section 1.

REMARK 0.4. There is another notion of cobordism groups of immersions in which—using the notation of Definition 0.1.—the manifold N^{n+k} is always the sphere S^{n+k} , and W^{n+k+1} is always the cylinder $S^{n+k} \times I$. This type of cobordism groups was considered for $l=\infty$ by R. Wells [16], and for arbitrary l by P. Vogel [15] and the present author [9]. We remark, that these groups are isomorphic to the homotopy groups $\pi_{n+k}(\Gamma_l(k))$.

1. The space $\Gamma_l(k)$

Below we describe two ways of constructing $\Gamma_l(k)$. The first approach is related to the Γ^+ functor of Barratt and Eccles [1]. The second construction is related to the Pontryagin-Thom construction and its basic idea was suggested to me by M. Gromov.

1.a. Construction of $\Gamma_l(k)$ by Barratt and Eccles (see [1]).

For m and n positive integers, we write C_m^n for the set of strictly monotonically increasing maps $\{1, 2, \dots, m\} \xrightarrow{\mathcal{L}} \{1, 2, \dots, n\}$. Such a map induces a monomorphism $\mathcal{L}_*: S(m) \rightarrow S(n)$ of symmetric groups defined by:

$$\begin{aligned} \mathcal{L}_*(\sigma)(\mathcal{L}(i)) &= \mathcal{L}(\sigma(i)) && \text{for } \sigma \in S(m), i \in \{1, 2, \dots, m\} \\ \text{and } \mathcal{L}_*(\sigma)(j) &= j && \text{for } j \notin \text{Im } \mathcal{L}. \end{aligned}$$

The map \mathcal{L} also induces a map $\mathcal{L}^*: S(n) \rightarrow S(m)$ by the requirement that for any $\sigma \in S(n)$ the following diagram commutes:

$$\begin{array}{ccc} \{1, 2, \dots, m\} & \xrightarrow{\mathcal{L}} & \{1, 2, \dots, n\} \\ \downarrow \mathcal{L}^*(\sigma) & & \downarrow \sigma \\ \{1, 2, \dots, m\} & \xrightarrow{\varphi} & \{1, 2, \dots, n\} \end{array}$$

where $\varphi \in C_m^n$ and $\text{Im } \varphi = \text{Im}(\sigma \circ \mathcal{L})$.

Recall (see [6]) that for any discrete group G the infinite join $G * G * G * \dots$ is a contractible space and G acts on this space freely. Denote this space by WG . Hence the map $\mathcal{L}^*: S(n) \rightarrow S(m)$ induces a map $WS(n) \rightarrow WS(m)$, which we denote by the same symbol \mathcal{L}^* .

Given a space X with base point $*$ consider the disjoint union

$$\bigcup_{n \geq 0} WS(n) \times X^n \quad (1.a.1)$$

where $X^n = X \times \dots \times X$ (n factors).

The following relations generate an equivalence on union (1.a.1)

$$\left. \begin{aligned} (1) \quad & (w, x) \sim (w \cdot \sigma, x \cdot \sigma) \\ (2) \quad & (w, x) \sim (\mathcal{L}^*(w), \mathcal{L}^*(x)) \end{aligned} \right\} \quad (1.a.2)$$

$w \in WS(n)$, $x \in X^n$, $\sigma \in S(n)$, $n \geq m > 0$ and $\mathcal{L} \in C_m^n$ is such that if $i \notin \text{Im } \mathcal{L}$ then $x_i = *$.

On factorizing out by this equivalence we obtain the space which was denoted by Barratt and Eccles by Γ^+X . If in (1.a.1) we take the union only of terms with $n \leq l$ then after factorizing we obtain a space we shall denote by $\Gamma_l(X)$. The space $\Gamma_l(k)$ is $\Gamma_l(X)$ for $X = MSO(k)$.

1.b. Construction of $\Gamma_l(k)$ by M. Gromov.

We define the spaces $\Gamma_l(k)$ by induction on l . For $l=1$ set $\Gamma_1(k)$ to be the Thom space $MSO(k)$.

NOTATION. Let $SO^{(l)}(k)$ denote the wreath product of the oriented orthogonal group $SO(k)$ with the symmetric group $S(l)$. Notice that $SO^{(l)}(k) \subset O(l \cdot k)$. The normal bundle of the manifolds of l -tuple points of a completely regular l -immersion of codimension k admits the group $SO^{(l)}(k)$ as a structure group.

Given a group $G \subset O(N)$ let $\xi_G: E[G] \xrightarrow{R^N} BG$ and MG denote the universal N dimensional vector bundle with structure group G and the Thom space of this bundle respectively. Consider the bundle $\xi = \xi_G$ for $G = SO^{(l)}(k)$. Over any simply connected subset U of the base $BSO^{(l)}(k)$ this bundle decomposes into Whitney sum:

$$\xi|_U = \eta_1 \oplus \dots \oplus \eta_l \quad (1.b.1)$$

where η_1, \dots, η_l are k dimensional oriented vector bundles over U . So we can write $x = x_1 + \dots + x_l$ for $x \in \xi|_U$ and $x_i \in \eta_i$ ($i=1, \dots, l$). Decomposition (1.b.1) will be meaningful even globally if we disregard the order of the terms. After describing a loop in the base space $BSO^{(l)}(k)$ the summands are permuted by the element of the symmetric group $S(l)$, which corresponds to the homotopy class of the loop under the isomorphism $S(l) \approx \pi_1(BSO^{(l)}(k))$.

In view of (1.b.1) the following spaces can be defined

$$\begin{aligned} D_l &= \{x \in ESO^{(l)}(k) \mid \|x_j\| \leq 1 \quad \text{for } j = 1, \dots, l\} \\ \partial D_l &= \{x \in D_l \mid \text{there exists } j: 1 \leq j \leq l \text{ such that } \|x_j\| = 1\} \\ D_l^\circ &= D_l \setminus \partial D_l \end{aligned}$$

$$S_l = \{x \in D_l \mid \text{there exist two different } j, j': 1 \leq j \leq l, 1 \leq j' \leq l \text{ such that } \|x_j\| = \|x_{j'}\| = 1\}$$

$$D_l\left(\frac{1}{2}\right) = \left\{x \in D_l \mid \|x\| = \sqrt{\sum_{j=1}^l \|x_j\|^2} \leq \frac{1}{2}\right\}$$

$$\partial D_l\left(\frac{1}{2}\right) = \left\{x \in D_l \mid \|x\| = \frac{1}{2}\right\}$$

$$Y_l = \{x \in D_l^\circ \mid \text{there exists } j, 1 \leq j \leq l \text{ such that } x_j = 0\}$$

$$Z_l = \{x \in \partial D_l \mid x = x_1 + \dots + x_l \text{ and all but one } x_j, 1 \leq j \leq l \text{ are zero}\}$$

EXAMPLE. $l=3, k=1$.

Consider a fibre F of D_l . F is a cube I^3 (see figure 1.) Its surface is $\partial D_l \cap F$. The union of the edges is $S_l \cap F$. The middle-points of the faces form the set $Z_l \cap F$. $Y_l \cap F$ is the union of the planes through the origin parallel to a pair of faces of the cube. $D_l\left(\frac{1}{2}\right)$ and $\partial D_l\left(\frac{1}{2}\right)$ are the ball and sphere of radius $\frac{1}{2}$ with the centre in the centre of the cube. (Figure 1.)

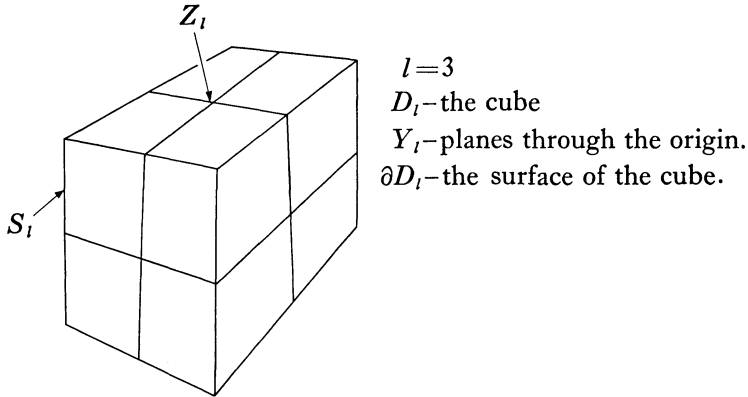


Figure 1.

REMARK 1.b.2. The analogues of these spaces can be defined in the space $E[SO(k) \oplus \dots \oplus SO(k)]$ as well. We will need them in §3.

These spaces will be denoted by the same symbol as in the previous case but with a bar over the symbol.

For example

$$\bar{D}_l = \{x \in E[SO(k) \oplus \dots \oplus SO(k)] \mid \|X_j\| \leq 1 \text{ for } j = 1, \dots, l\}.$$

The construction of $\Gamma_l(k)$. There is a natural fibration $\partial D_l \setminus S_l \rightarrow Z_l$ with structure group $SO^{(l-1)}(k)$ and with fibre $D^k \times D^k \times \dots \times D^k$ ($(l-1)$ factors), where D^k denotes the open unit ball in R^k . This fibration can be induced from the universal bundle $D_{l-1}^0 \rightarrow BSO^{(l-1)}(k)$ by a bundle map (\mathcal{L}, β) :

$$\begin{array}{ccc}
 \partial D_l \setminus S_l & \xrightarrow{\beta} & D_{l-1}^0 \\
 \downarrow \begin{array}{l} D^k \times \dots \times D^k \\ (l-1) \text{ factors} \end{array} & & \downarrow \begin{array}{l} D^k \times \dots \times D^k \\ (l-1) \text{ factors} \end{array} \\
 Z_l & \xrightarrow{\mathcal{L}} & BSO^{(l-1)}(k)
 \end{array}$$

in such a way that $\beta^{-1}(Y_{l-1}) = Y_l \cap \partial D_l$.

The map β can be extended in a unique way to a continuous map $\bar{\beta}: \partial D_l \rightarrow \Gamma_{l-1}(k)$. Recall that $D_{l-1}^0 \subset \Gamma_{l-1}(k)$. Now the space $\Gamma_l(k)$ by definition is the space $\Gamma_l(k) = D_l \cup_{\beta} \Gamma_{l-1}(k)$, i.e. in the disjoint union $D_l \cup \Gamma_{l-1}(k)$ we identify each point $y \in \partial D_l \subset D_l$ with its image $\bar{\beta}(y) \in \Gamma_{l-1}(k)$.

1.c. Equivalence of the two definition of $\Gamma_l(k)$. Here we show that the two constructions of $\Gamma_l(k)$ are equivalent, i.e. they give the same space (at least up to homotopy).

Notice that

- a) $B SO^{(l)}(k) = \underbrace{B SO(k) \times \dots \times B SO(k)}_{l\text{-factors}} \times_{S(U)} WS(l)$,
- b) $M SO(k) \setminus * = D_1 = \{x \in ESO(k) \mid \|x\| < 1\}$ and
- c) $D_l^0 = D_1 \times \dots \times_{S(U)} D_1 \times WS(l)$.

Hence if we perform identification (1) of (1.a.2) then we obtain $\bigcup_{i=1}^l D_i$. Performing identification (2) we map the fibration $\partial D_i \setminus S_i \rightarrow Z_i$ by a fiberwise linear isomorphism into the fibre bundle $D_{i-1}^0 \rightarrow BSO^{(i-1)}(k)$ and identify the points with their images, i.e. perform the same identification as in the second definition.

2. Connections between the space $\Gamma_l(k)$ and Uchida's groups $C^0(n, k; l)$

Theorem 2.1.

$$C^0(n, k; l) \approx \Omega_{n+k}(\Gamma_l(k)),$$

where $\Omega_{n+k}(\cdot)$ denotes the $(n+k)$ -th bordism group.

More precisely, the following holds.

Theorem 2.1'.

I.1. If $f: M^n \rightarrow N^{n+k}$ is a completely regular l -immersion, then there exists a continuous map $h: N^{n+k} \rightarrow \Gamma_l(k)$ such that $h^{-1}(\tilde{Y}_l) = f(M)$ where $\tilde{Y}_l = \bigcup_{i=1}^l Y_i$.

(Recall that $Y_i \subset D_i^0 \subset \Gamma_l(k)$ for $i \leq l$, hence $\tilde{Y}_l \subset \Gamma_l(k)$.)

The map h is uniquely defined up to homotopy.

I.2. If $h': N^{n+k} \rightarrow \Gamma_l(k)$ is a continuous map, then there exists a continuous map $h: N^{n+k} \rightarrow \Gamma_l(k)$ such that

- a) h' and h are homotopic
- b) $h^{-1}(\tilde{Y}_l)$ is the image of an l -immersion.

II. 1. (relative version of I.1.)

Suppose that M^n and N^{n+k} are manifolds with non empty boundaries, $f: M^n \rightarrow N^{n+k}$ is a completely regular l -immersion such that $f(\partial M) \subset \partial N$ and also $f|_{\partial M}: \partial M \rightarrow \partial N$ is a completely regular l -immersion. Let \tilde{h} be a map $\partial N \rightarrow \Gamma_l(k)$ corresponding to the l -immersion $f|_{\partial M}: \partial M \rightarrow \partial N$. Then there exists a map $h: N \rightarrow \Gamma_l(k)$ such that $h^{-1}(\tilde{Y}_l) = f(M)$ and $h|_{\partial N} = \tilde{h}$.

Proof of I.1. and II.1. First of all observe that the analogues of the spaces $D_l, \partial D_l, S_l, Y_l, Z_l$ can be defined for an arbitrary bundle ν with structure group $SO^{(l)}(k)$. These spaces will be denoted by $D_l(\nu), \partial D_l(\nu)$ etc. respectively. Now the proof of the theorem goes by induction on l . For $l=1$ the statement follows from the Pontryagin-Thom construction. For a completely regular l -immersion $f: M^n \rightarrow N^{n+k}$ denote by $K_l(f)$ and $T_l(f)$ the set $\{y \in N^{n+k} \mid \text{card } f^{-1}(y) = l\}$ and its tubular neighbourhood, respectively. ($\dim K_l(f) = n - (l-1)k$, $\dim T_l(f) = n + k$).

Let ν_f denote the normal bundle of $K_l(f)$ in N . We can identify $T_l(f)$ with $D_l(\nu_f)$ in such a way that $f(M) \cap T_l(f)$ corresponds to $Y_l(\nu_f)$.

Let h_1 be a bundle map from the bundle $\nu_f: T_l(f) \rightarrow K_l(f)$ into the universal bundle $\xi: ESO^{(l)}(k) \xrightarrow{R^{lk}} BSO^{(l)}$ such that $Y_l(\nu_f) = (h_1)^{-1}(Y_l(\xi_l))$.

In case II.1. i.e. when ∂M is not empty then $\partial K_l(f) \subset \partial N$, $T_l(f) \cap \partial N = \partial T_l(f)$ and we also demand h_1 to coincide with \tilde{h} on $\partial T_l(f)$.

Consider the manifold $N \setminus T_l(f)$. Then

1. Its boundary is $\partial(N \setminus T_l(f)) = \partial T_l(f)$.
2. $f(M) \cap (N \setminus T_l(f))$ is the image of an $(l-1)$ immersion into $N \setminus T_l(f)$.
3. $f(M) \cap \partial(N \setminus T_l(f))$ is the image of an $(l-1)$ immersion into $\partial(N \setminus T_l(f))$ to be denoted by f_1 .
4. $h_1|_{\partial(N \setminus T_l(f))}$ is the map corresponding to f_1 i.e. $h_1^{-1}(Y_l) \cap \partial(N \setminus T_l(f)) =$ image of f_1 .

By the induction assumption there exists a map $h_2: N \setminus T_l(f) \rightarrow \Gamma_{l-1}(k)$ such that $h_2^{-1}(Y_{l-1}) = f(M) \cap \partial(N \setminus T_l(f))$ and h_2 coincides with h_1 on the boundary (i.e. $h_2|_{\partial(N \setminus T_l(f))} = h_1$). Then the map $h: N \rightarrow \Gamma_l(k)$ defined by the formulae $h|_{N \setminus T_l(f)} = h_2$, $h|_{T_l(f)} = i_l \circ h_1$, (where i_l is the inclusion $D_l^i \subset \Gamma_l(k)$), is correctly defined and this is the required map. Q.E.D.

Proof of I.2. The map h' can be approximated by a map h which is trans-

versal to the subspace $\partial D_l\left(\frac{1}{2}\right)$ of $\Gamma_l(k)$. Then $h^{-1}\left(\partial D_l\left(\frac{1}{2}\right)\right)$ will be a submanifold in N^{n+k} (of codimension l), and it divides the manifold N^{n+k} into two parts: N_1 and N_2 , where $N_1 = h^{-1}\left(D_l\left(\frac{1}{2}\right)\right)$. $N_1 \cup N_2 = N^{n+k}$ and $N_1 \cap N_2 = h^{-1}\left(\partial D_l\left(\frac{1}{2}\right)\right)$. Denote by N'_1 and N'_2 the manifolds $N_1 \cap \partial N$ and $N_2 \cap \partial N$. Let U_i be the set $Y_i \cap \partial D_l\left(\frac{1}{2}\right)$. We may suppose that the map $h|_{N_1 \cap N_2}: N_1 \cap N_2 \rightarrow D_l\left(\frac{1}{2}\right)$ is transversal to U_i . Then the restriction of the map $h: N \rightarrow \Gamma_l(k)$ to the set $N_1 \cap N_2$ is transversal to Y_i . Hence the map $h|_{N_1}: N_1 \rightarrow D_l\left(\frac{1}{2}\right)$ can be made transversal to $Y_i \cap D_l\left(\frac{1}{2}\right)$ without changing it on $N_1 \cap N_2$. Let $r_l: \Gamma_l(k) \setminus D_l\left(\frac{1}{2}\right) \rightarrow \Gamma_{l-1}(k)$ be a retraction. Such a retraction does exist and can be defined uniquely because

1. $D_l \setminus D_l\left(\frac{1}{2}\right)$ can be retracted onto ∂D_l by projection along the radii.
2. $\Gamma_l(k) \setminus D_l\left(\frac{1}{2}\right) = D_l \setminus D_l\left(\frac{1}{2}\right) \cup_{\partial D_l} \Gamma_{l-1}(k)$.

By the induction assumption the map $r_l \circ h|_{N_2}: N_2 \rightarrow \Gamma_{l-1}(k)$ can be perturbed to achieve the preimage of $\tilde{Y}_{l-1}(k)$ to be the image of an $(l-1)$ -immersion f into N_2 . This perturbation can be performed in such a way that it leaves $r_l \circ h$ unchanged on $N_1 \cap N_2$, since $r_l \circ h|_{N_1 \cap N_2}$ is already transversal to U_i . Now take the union of the image of the above $(l-1)$ immersion with the set $h^{-1}\left(Y_i \cap D_l\left(\frac{1}{2}\right)\right)$. This union forms the image of an l -immersion into N^{n+k} . This is the required l -immersion.

REMARK 2.2. Barratt and Eccles proved that the space $\Gamma_l X$ is stably homotopically equivalent to the wedge space $(\Gamma_{l-1} X) \vee (\Gamma_l X / \Gamma_{l-1} X)$. Now from the isomorphism $\tilde{\Omega}_{n+1}(SY) \approx \tilde{\Omega}_n(Y)$ and theorem 2.1 it follows that the natural map $C^0(n, k; l-1) \rightarrow C^0(n, k; l)$ is a monomorphism.

3. Computation of the group $\Omega_{n+k}(\Gamma_l k) \otimes Q$

It is well known (see [5]) that the rank of the bordism groups of an arbitrary space X can be computed by the formula

$$\Omega_*(X) \otimes Q \approx H_*(X; Q) \otimes \Omega_* \tag{3.1}$$

where Ω_* denotes the sum $\bigoplus_{i=0}^{\infty} \Omega_i$ of Thom's cobordism groups. By Theorem

2.1. of [B-E 3] we know that

$$\tilde{H}_*(\Gamma_i(k); Q) = \bigoplus_{i=0}^i \tilde{H}_*(\Gamma_i(k)/\Gamma_{i-1}(k); Q) \quad (3.2)$$

The factor space $\Gamma_i(k)/\Gamma_{i-1}(k)$ coincides with the Thom space $M\mathcal{S}O^{(i)}(k)$. So the computation of groups of Uchida is reduced to the computation of groups $H^*(M\mathcal{S}O^{(i)}(k); Q)$.

Lemma 3.3. *The group $\tilde{H}^*(M\mathcal{S}O^{(i)}(k); Q)$ is isomorphic to the subring of the tensor product $\tilde{H}^*(M\mathcal{S}O(k); Q) \otimes \cdots \otimes \tilde{H}^*(M\mathcal{S}O(k); Q)$ (i factors) consisting of those elements, which are invariant under the natural $S(i)$ action of this product.*

The natural $S(i)$ action on $\tilde{H}^(M\mathcal{S}O(k); Q) \otimes \cdots \otimes \tilde{H}^*(M\mathcal{S}O(k); Q)$ can be described as follows. Given*

$\sigma \in S(i)$ and $a_1 \otimes a_2 \otimes \cdots \otimes a_i \in \tilde{H}^(M\mathcal{S}O(k); Q) \otimes \cdots \otimes \tilde{H}^*(M\mathcal{S}O(k); Q)$ we define $\sigma(a_1 \otimes \cdots \otimes a_i)$ to be equal to $a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)}$.*

Proof. There exists a fibre bundle of pairs $(D_i, \partial D_i) \rightarrow BS(i)$, with fibres $(\bar{D}_i, \partial \bar{D}_i)$, where $BS(i)$ is the classifying space for the group $S(i)$

$$\text{i.e. } \pi_j(BS(i)) = \begin{cases} 0 & \text{if } j > 1 \\ S(i) & \text{if } j = 1. \end{cases}$$

The second term of the spectral sequence of this fibration is

$$E_2^{**} = H^*(BS(i); \tilde{H}^*(M\mathcal{S}O(k) \wedge \cdots \wedge M\mathcal{S}O(k); Q)) \quad (\text{local coefficients})$$

$$(\text{The coefficients are } \tilde{H}^*(M\mathcal{S}O(k) \wedge \cdots \wedge M\mathcal{S}O(k); Q))$$

because $\bar{D}_i | \partial \bar{D}_i = M\mathcal{S}O(k) \wedge \cdots \wedge M\mathcal{S}O(k)$. The final term E_∞ is associated to the reduced cohomology ring $\tilde{H}^*(M\mathcal{S}O^{(i)}(k); Q)$.

It is known, that $H^j(BS(i); A) = 0$ for any $j > 0$ if in the coefficient group A (which is an $S(i)$ module) each element can be divided by $i!$ (see [4] Corollary 2.7, chapt. XII). The group $H^0(BS(i); A)$ can be identified with the submodule $A^{S(i)}$ of the $S(i)$ module A , consisting of all $S(i)$ invariant elements. Hence the spectral sequence described above is trivial i.e. $E_2 = E_\infty$ and $\tilde{H}^*(M\mathcal{S}O^{(i)}(k); Q)$ is isomorphic to the ring of $S(i)$ invariant elements of $\tilde{H}^*(M\mathcal{S}O(k) \wedge \cdots \wedge M\mathcal{S}O(k); Q) = \tilde{H}^*(M\mathcal{S}O(k); Q) \otimes \cdots \otimes \tilde{H}^*(M\mathcal{S}O(k); Q)$.

Q.E.D.

Finally we derive an explicit formula for the ranks of Uchida's groups. In view of the above this is merely a combinatorial problem. We introduce some notation. Let $\pi(m)$ denote the number of partitions of the positive integer m into sum of integers $m = \sum_{i=1}^{\infty} x_i$, $x_1 \geq x_2 \geq \cdots > 0$. Let $\pi_k(m)$ denote the number of those partitions where $k \geq x_1$. For a graded Q -module $J = J_0 \oplus J_1 \oplus \cdots$, let $p_J(t) = \sum_{i=1}^{\infty} (\dim J_i) \cdot t^i$ denote the *dimension generating function* of J . The di-

mension generating function of the Thom module $\Omega_* \otimes Q$ is $P_\Omega(t) = \sum_{i=0}^{\infty} \pi(i) \cdot t^{4i} = \prod_{i=1}^{\infty} 1/(1-t^{4i})$ (3.3)

Given $P_J(t)$ for some graded Q -module J , we wish to determine the dimension generating function of the symmetrized tensor product $S(J, m) = J \circ J \circ \dots \circ J$ (m times).

Setting $d_i = \dim J_i$ and $P_{S(J,m)}(t) = \sum_{i=1}^{\infty} D_i t^i$ (3.4)

we clearly have $D_i = \sum \prod_{j=1}^m d_{\beta_j}$. (3.5)

where the summation extends over all integral solutions $(\beta_1, \dots, \beta_m)$ of the equation $\beta_1 + \dots + \beta_m = i$, such that $0 \leq \beta_1 \leq \dots \leq \beta_m$.

In our case, the role of J will be played by the cohomology ring $H^*(MSO(k); Q) \approx H^{*-k}(BSO(k); Q)$. It is well known, that

- a) $H^*(BSO(2s+1); Q) = Q[p_1, \dots, p_s]$ and
- b) $H^*(BSO(2s); Q) = Q[p_1, \dots, p_{s-1}, X_{2s}]$ where $\deg p_i = 4i$, $\deg X_{2s} = 2s$.

Hence

a) if $k = 2 \cdot s + 1$ then $p_J(t) = t^k \sum_{i=0}^{\infty} \pi_s(i) \cdot t^{4i} = t^k \prod_{i=1}^s (1-t^{4i})^{-1}$
 b) if $k = 2 \cdot s$ then $p_J(t) = t^k \cdot (1-t^{2s})^{-1} \cdot \prod_{i=1}^{s-1} (1-t^{4i})^{-1}$ (3.6)

Now our final result is

Theorem 3.7. Let $U_{k,i}(t) = \sum_{n=0}^{\infty} \text{rank } C^0(n, k; I) t^{n+k}$ denote the rank generating function of Uchida's groups. Then

$$U_{k,i}(t) = \sum_{n=0}^i P_{S(J,m)}(t) \cdot P_\Omega(t)$$

where $P_\Omega(t)$ is defined by (3.3), and $P_{S(J,m)}(t)$ by (3.4), (3.5) and (3.6).

Proof. Immediate by formula (3.1), (3.2), Lemma 3.1. and Theorem 2.1. Q.E.D.

REMARK 3.8. In a very similar way one can compute the nonorientable Uchida's groups too (see [9]).

REMARK 3.9. One can generalize Uchida's groups admitting maps with simplest singularities. The computation of the obtained groups must be quite available using the appropriate analogue of the space $\Gamma_i(k)$ for the maps with Σ^1 -type singularities (see [10] and [11]).

PROBLEM. To construct the analogue of the spaces $\Gamma_i(k)$ for the more complicated singularities and investigate their homotopy types. This must lead to some geometrical consequences about the realization of homology (or

bordism) classes of manifolds by immersions or maps with simple singularities (see [10], Theorem B).

References

- [1] M.G. Barratt and P.J. Eccles: Γ^+ -structures I, *Topology* **13** (1974), 25–45.
- [2] M.G. Barratt and P.J. Eccles: Γ^+ -structures II, *Topology* **13** (1974), 113–126.
- [3] M.G. Barratt and P.J. Eccles: Γ^+ -structures III, *Topology* **13** (1974), 199–207.
- [4] H. Cartan and S. Eilenberg: *Homological algebra*, Princeton, 1956.
- [5] P.E. Conner and E.E. Floyd: *Differentiable periodic maps*, Springer, 1964.
- [6] J. Milnor: *Construction of universal fibre bundles*, *Ann. of Math.* **63** (1956), 272–284.
- [7] P. Schweitzer: *Joint cobordism of immersions*, *Lecture Notes in Math.* 168, Springer, 1970, 267–282.
- [8] A. Szűcs: *Cobordism groups of l-immersions I* (in Russian), *Acta Math. Acad. Sci. Hungar.* **27** (1976), 343–358.
- [9] A. Szűcs: *Cobordism groups of l-immersions II* (in Russian) *Acta Math. Acad. Sci. Hungar.* **28** (1976), 93–102.
- [10] A. Szűcs: *Analog of the Thom space for maps with Σ^1 -type singularities* (in Russian), *Mat. Sb.* **108** (1979), 433–456.
- [11] A. Szűcs: *Cobordism of maps with simplest singularities*, *Lecture Notes in Math.* 788, Springer, 1980.
- [12] F. Uchida: *Exact sequences involving cobordism groups of immersions*, *Osaka J. Math.* **6** (1969), 397–408.
- [13] F. Uchida: *The structures of the cobordism groups $B(n, k)$ of bundles over manifolds with involution*, *Osaka J. Math.* **7** (1970), 193–202.
- [14] F. Uchida: *Cobordism groups of immersions*, *Osaka J. Math.* **8** (1971), 207–218.
- [15] P. Vogel: *Cobordisme d'immersions*, *Ann. Sci. Ecole Norm. Sup. (4)* **7** (1974), 317–357.
- [16] R. Wells: *Cobordism of immersions*, *Topology* **5** (1966), 281–294.

Department of Analysis I
 Eötvös University
 Múzeum krt. 6–8
 Budapest, Hungary H-1088