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POINT INTERACTION POTENTIAL APPROXIMATION FOR $(-\Delta+U)^{-1}$ AND EIGENVALUES OF THE LAPLACIAN ON WILDLY PERTURBED DOMAIN

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1. Introduction. We remove m balls of radius α/m with centeres $\{w_i^{(m)}\}_{i=1}^m$ from a bounded domain Ω in $\boldsymbol{R}^3$. If m balls are dispersed in a specific configuration as $m\to\infty$, we give a precise asymptotic behaviour of the k-th eigenvalue of the Laplacian in $\overline{\Omega\backslash m \text{ balls}}$ under the Dirichlet condition on its boundary. We use perturbational calculation concerning the Green function of the Laplacian to obtain our theorem. The present work is closely related with Kac [2], Rauch-Taylor [6], Huruslov-Marchenko [3], Papanicolaou-Varadhan [5] and Ozawa [4].

Let Ω be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary γ. Let $w(m)$ be a set of m points $w_1^{(m)}, \cdots, w_m^{(m)}$ in Ω. Let $B(\varepsilon; w)$ be the ball defined by $B(\varepsilon; w)=\{x\in\boldsymbol{R}^3; |x-w|<\varepsilon\}$. Let $0<\mu_1(\varepsilon; w(m))\leq\mu_2(\varepsilon; w(m))\leq\cdots$ be the eigenvalues of $-\Delta(=-div\, grad)$ in $\Omega_{\varepsilon,w(m)}=\Omega\backslash\overline{\bigcup_{i=1}^m B(\varepsilon; w_i^{(m)})}$ under the Dirichlet condition on its boundary $\partial\Omega_{\varepsilon,w(m)}$. We arrange them repeatedly according to their multiplicities.

To state our Theorem we need some assumptions on a distribution of $w(m)$ as $m\to\infty$. A sequence satisfying the following conditions $(C-1)$, $(C-2)$ is said to be of class $\mathcal{O}$ and is written as $\{w(m)\}_{m=1}^\infty\in\mathcal{O}$.

$(C\text{-}1)$ There exists a constant $\tilde{C}>0$ independent of m such that

$$|w_r^{(m)}-w_s^{(m)}|\geq\tilde{C}m^{-1/3} \qquad (r\neq s)$$
$$\mathrm{dist}\,(w_r^{(m)}, \gamma)\geq\tilde{C}m^{-1/3} \qquad (1\leq r\leq m)$$

(C–2) Fix $p\in(0, 1]$. Then, there exists a constant C_p independent of f, m such that

$$\left|\frac{1}{m}\sum_{r=1}^m f(w_r^{(m)})-\int_\Omega f(x)V(x)dx\right|\leq C_p m^{-p/3}\|f\|_{C^p(\bar{\Omega})} \tag{1.1}$$

holds for any $f\in C^p(\bar{\Omega})$ as $m\to\infty$. Moreover

$$\max_j \left| \frac{1}{m} \sum_{\substack{i=1 \\ j \neq i}}^{m} G(w_i^{(m)}, w_j^{(m)}) f(w_i^{(m)}) - \int_\Omega G(y, w_j^{(m)}) V(y) f(y) dy \right|$$
$$\leq C_p m^{-p/3} \|f\|_{C^p(\bar{\Omega})},$$

where $G(x, y)$ denotes the Green function of the Laplacian in Ω under the Dirichlet condition on γ, that is, it satisfies

$$\begin{aligned} \Delta_x G(x, y) &= -\delta(x-y) \qquad & x, y \in \Omega \\ G(x, y) &= 0 & x \in \gamma, \quad y \in \Omega . \end{aligned}$$

EXAMPLE. If $V(x) = (\text{volume of } \Omega)^{-1}$ and

$$\{w_r^{(m)}\}_{r=1}^m = (\overline{\Omega \backslash \{x;\ \text{dist}\,(x, \gamma \leq \bar{C} m^{-1/3}\}})) \cap \text{(the cubic lattice points in } \boldsymbol{R}^3/(\omega m^{-1/3} \boldsymbol{Z})^3), \quad \text{(for suitable } \omega)$$

then it is easily seen that $\{w(m)\}_{m=1}^\infty \in \mathcal{O}$.

We are now in a position to state the following:

Theorem. *Suppose that* $\{w(m)\}_{m=1}^\infty \in \mathcal{O}$ *and* $\alpha > 0$. *Then*

$$\mu_k(\alpha/m;\ w(m)) = \mu_k^V + 0(m^{\varepsilon'-1/6}) \tag{1.2}$$

holds as $m \to \infty$, *where* ε' *is an arbitrary fixed positive number. Here* μ_k^V *denotes the k-th eigenvalue of* $-\Delta + 4\pi\alpha V(x)$ *in* Ω *under the Dirichlet condition on* γ.

We here explain the main idea of our proof of Theorem. Let $G_m(x, y; w(m))$ be the Green function of the Laplacian in $\Omega_{\alpha/m,w}$ under the Dirichlet condition on its boundary. It should be noticed that $\Omega_{\alpha/m,w}$ may not be connected, however we only treat the case where it is connected owing to the conditions (C–1), (C–2). $G_m(x, y; w(m))$ satisfies

$$\begin{aligned} \Delta_x G_m(x, y;\ w(m)) &= -\delta(x-y) \qquad & x, y \in \Omega_{\alpha/m,w} \\ G_m(x, y;\ w(m)) &= 0 & x \in \partial\Omega_{\alpha/m,w} . \end{aligned}$$

Hereafter we abbreviate $w_i^{(m)}$ as w_i for the sake of simplicity. We put

$$\begin{aligned} h_m(x, y;\ w(m)) = \\ G(x, y) + \sum_{s=1}^m (-4\pi\alpha/m)^s \textstyle\sum_{(s)} G(x, w_{i_1}) G(w_{i_1}, w_{i_2}) \\ \cdots G(w_{i_{s-1}}, w_{i_s}) G(w_{i_s}, y) . \end{aligned} \tag{1.3}$$

Here the indicies $(i_1, \cdots, i_s)$ in $\sum_{(s)}$ run over all $1 \leq i_1, \cdots, i_s \leq m$ satisfying $i_1 \neq i_2$, $i_2 \neq i_3, \cdots, i_{s-1} \neq i_s$. This is a good approximation of G_m when α is small enough. For general α, we modify (1.3) to get a nice approximation. It should be remarked that $h_m(x, y; w(m))$ tends to

$$(1.4)\qquad G(x,y)-4\pi\alpha\int_{\Omega}G(x,z)V(z)G(z,y)dz$$
$$+(4\pi\alpha)^2\int_{\Omega}G(x,z_1)V(z_1)G(z_1,z_2)V(z_2)G(z_2,y)dz-\cdots$$

as $m\to\infty$, if α is small enough. We see that (1.4) is equal to the Green function of $\Delta-4\pi\alpha V(x)$, if α is small. Along this line, we can prove that $\mu_k(\alpha/m;w(m))\to\mu_k^V$. Of course, we need rigorous step to prove the remainder estimate in (1.2).

We make historical remarks. There are some related papers. Under slightly different situation from above Kac [2], Rauch-Taylor [6] treated a probabilistic problem concerning $\mu_k(\alpha/m;w(m))$ when $m\to\infty$. In their notes the convergence

$$\mu_k(\alpha/m;\ \cdot)\to\mu_k^V$$

in probability when we consider $w(m)$ as a random variable in a probability space which is the countable product of a probability space Ω with the probability measure $V(x)dx$, was given. Their studies depend on a probabilistic argument using the theory and a notion of Wiener sausage. Our Theorem is different from theirs in the point that the statement of Theorem is concerning a deterministic result. In [3], various boundary value problem in a region with many small obstacles is treated by potential theoretic approach using the notion of capacity of sets. In a suggestive paper Papanicolaou-Varadhan [5] studied the diffusion problem in a region with small holes by using probabilistic method and got both deterministic and probabilistic results. Here the author emphasize that our method is based on a perturbational calculus using the Green function under singular variation of domains (removing holes) and that this method is new as far as the author concerns. Our method enables us to get an error estimate $0(m^{\varepsilon-1/6})$ in (1.2) which also seems to be new.

We make another remark. When $m=1$, h_m reduces to the integral kernel function $h_\varepsilon(x,y)$ on p. 771 of Ozawa [4]. By using this integral kernel function, we gave an asymptotic formula for the eigenvalues of the Laplacian under singular variation of domains. See [4]. For any $1\le m<\infty$, we can also prove an asymptotic formula for eigenvalues by using h_m. Thus we can say that h_m is a nice asymptotic Green function for all $m=1,2,\cdots,\infty$.

2. Decomposition of the integral kernel function h_m

From now on we abbreviate $\Omega_{\alpha/m,w}$ as Ω_w. Also $B(\alpha/m,w_r)$ is written as B_r, if there is no fear of confusion. We have the following:

Lemma 1. *Suppose that $u\in C^\infty(\Omega_w)$ satisfies*

$$\begin{aligned} &\Delta u(x)=0 && x\in\Omega_w \\ &u(x)=0 && x\in\gamma \\ &\max\{|u(x)|;\, x\in\partial B_r\}=M_r(m) && r=1,\cdots,m. \end{aligned}$$

Then, there exists a constant C independent of $\{w(m)\}\in\mathcal{O}$ such that

$$|u(x)|\leq C(\alpha/m)\sum_{r=1}^{n}|x-w_r|^{-1}M_r(m)$$

holds for any $x\in\Omega_w$.

Proof. See Lemma 6 in [4]. q.e.d.

Now we introduce some integral operators $\boldsymbol{G}_m$, $\boldsymbol{H}_m$, $\boldsymbol{G}$. We put

$$(\boldsymbol{G}_m f)(x)=\int_{\Omega_w} G_m(x,y;w(m))f(y)dy \qquad x\in\Omega_w$$

$$(\boldsymbol{H}_m f)(x)=\int_{\Omega_w} h_m(x,y;w(m))f(y)dy \qquad x\in\Omega_w$$

and

$$(\boldsymbol{G}g)(x)=\int_{\Omega} G(x,y)g(y)dy \qquad x\in\Omega .$$

We put $\kappa=\sup_m \frac{1}{m}\max_i \sum_{\substack{1\leq r\leq m \\ r\neq i}} G(w_i,w_r)$. Then $\kappa<\infty$, by (C–2). The aim of this section is to prove the following:

Proposition 1. *Assume that $\{w(m)\}_{m=1}^{\infty}\in\mathcal{O}$. Fix an arbitrary $\varepsilon'>0$. Then there exists a constant $C_{\varepsilon'}$ independent of m such that*

$$||\boldsymbol{G}_m-\boldsymbol{H}_m||_{L^2(\Omega_w)}\leq C_{\varepsilon'}m^{-1+\varepsilon'}q_{m,\alpha}, \tag{2.1}$$

where $q_{m,\alpha}=1+(\sum_{1\leq s\leq m-1}(4\pi\alpha)^s(4\pi+1)\alpha\kappa^{s-1})+m(4\pi\alpha\kappa)^m$ holds. Here $||T||_{L^2(\Omega_w)}$ denotes the operator norm of a bounded operator T on $L^2(\Omega_w)$.

Firstly we state the following obvious lemma.

Lemma 2. *Put $\boldsymbol{Q}_m=\boldsymbol{G}_m-\boldsymbol{H}_m$. Then*

$$\begin{aligned} &\Delta\boldsymbol{Q}_m f(x)=0 && x\in\Omega_w \\ &\boldsymbol{Q}_m f(x)=0 && x\in\gamma \end{aligned}$$

holds for any $f\in C_0^\infty(\Omega_w)$.

The following Lemma 3 is a crucial step to prove Proposition 1.

Lemma 3. *Assume that $\{w(m)\}\in\mathcal{O}$. Fix an arbitrary $p>3$. Then*

$$\sum_{r=1}^{m} \max_{x\in\partial B_r} |\boldsymbol{Q}_m f(x)| \leq \bar{C}_p q_{m,\alpha} ||f||_{L^p(\Omega_w)}$$

holds for a constant $\bar{C}_p$ *independent of* m *and for any* $f\in C_0^\infty(\Omega_w)$.

Before we begin our proof of Lemma 3, we give a proof of Proposition 1 assuming Lemma 3. Using Lemmas 1, 2, 3 we get

$$||\boldsymbol{Q}_m||_{L^p(\Omega_w)} \leq C_p m^{-(3/p)} q_{m,\alpha} \qquad (p>3) \tag{2.2}$$

for a constant C_p independent of m. Since

$$\int_{\Omega_w} \boldsymbol{Q}_m u(x)\overline{v(x)}dx = \int_{\Omega_w} u(x)\overline{\boldsymbol{Q}_m v(x)}dx\,,$$

for any $u, v\in C_0^\infty(\Omega_w)$, we have

$$||\boldsymbol{Q}_m||_{L^p(\Omega_w)} \leq C_p m^{-(3/p)} q_{m,\alpha} \qquad (p>3)$$

where $p'=(1-(1/p))^{-1}$. We see that $p>3$ and $p'<(3/2)$. Thus, by the interpolation theorem we get the same bound for $||\boldsymbol{Q}_m||_{L^2(\Omega_w)}$ as in (2.2). We can take $p>3$ sufficiently close enough to 3 and we get Proposition 1.

To show Lemma 3 we use the following decomposition (2.3) of $\boldsymbol{H}_m f$. We put

$$\begin{aligned}
&(I_r^s f)(x) \\
&= \sum_{\substack{i_1\neq r,\, i_2\neq i_1,\\ \cdots,\, i_s\neq i_{s-1}}} G(x, w_{i_1})G(w_{i_1}, w_{i_2})\cdots G(w_{i_{s-1}}, w_{i_s})(\boldsymbol{G}f)(w_{i_s}) \\
&\quad -(4\pi\alpha/m) \sum_{\substack{j_2\neq r,\, j_3\neq i_2,\\ \cdots,\, j_{s+1}\neq j_s}} G(x, w_r)G(w_r, w_{j_2})\cdots G(w_{j_s}, w_{j_{s+1}})(\boldsymbol{G}f)(w_{j_{s+1}})\,.
\end{aligned}$$

Then, it is easy to see that

$$\begin{aligned}
&(\boldsymbol{H}_m f)(x) \\
&= (\boldsymbol{G}f)(x)-(4\pi\alpha/m)G(x, w_r)(\boldsymbol{G}f)(w_r)+\sum_{s=1}^{m-1}(-4\pi\alpha/m)^s(I_r^s f)(x) \\
&\quad +(-4\pi\alpha/m)^m \sum_{\substack{i_1\neq r,\, i_2\neq i_1,\\ \cdots,\, i_m\neq i_{m-1}}} G(x, w_{i_1})G(w_{i_1}, w_{i_2})\cdots G(w_{i_{m-1}}, w_{i_m})(\boldsymbol{G}f)(w_{i_m})
\end{aligned} \tag{2.3}$$

Since $G(x,y)-(4\pi|x-y|)^{-1}=S(x,y)\in C^\infty(\Omega\times\Omega)$, we have

$$G(x, w_r)|_{x\in\partial B_r} = (4\pi(\alpha/m))^{-1}+S(x, w_r)|_{x\in\partial B_r}\,.$$

Therefore,

$$(I_r^s f)(x)|_{x\in\partial B_r} = (L_r^s f)(x)|_{x\in\partial B_r}+(N_r^s f)(x)|_{x\in\partial B_r}\,, \tag{2.4}$$

where

$$(2.5)\quad (L_r^s f)(x)|_{x\in\partial B_r} = \sum_{\substack{i_1\neq r,\, i_2\neq i_1,\\ \cdots,\, i_s\neq i_{s-1}}} (G(x, w_{i_1})-G(w_r, w_{i_1}))G(w_{i_1}, w_{i_2})\cdots G(w_{i_{s-1}}, w_{i_s})(\boldsymbol{G}f)(w_{i_s})$$

and

$$(N_r^s f)(x)|_{x\in\partial B_r} = (-4\pi\alpha/m)\sum_{\substack{i_1\neq r,\, i_2\neq i_1,\\ \cdots,\, i_s\neq i_{s-1}}} S(x, w_r)G(w_r, w_{i_1})\cdots G(w_{i_{s-1}}, w_{i_s})(\boldsymbol{G}f)(w_{i_s}).$$

We have the following:

Lemma 4. *Assume that* $\{w(m)\}\in\mathcal{O}$. *Then, there exists a constant* C *independent of* m *such that*

$$\sum_{r=1}^{m}\sum_{\substack{i=1\\ i\neq r}}^{m} D_{r,i}\leq C\alpha m,$$

where

$$D_{r,i}=\max_{x\in\partial B_r}|G(x, w_i)-G(w_r, w_i)|.$$

Proof. We fix r_0 and we assume that w_{r_0} is contained in $\gamma_\varepsilon=\{x\in\Omega;\ \mathrm{dist}(x, \gamma)<\varepsilon\}$. Let z_0 be the unique point on γ such that $\mathrm{dist}(w_{r_0}, \gamma)=\mathrm{dist}(w_{r_0}, z)$.

Firstly, we examine $D_{r,i}$ in the following case:

$$(2.6)\qquad \max(\mathrm{dist}(w_{r_0}, w_r),\ \mathrm{dist}(w_i, w_r))\leq 2\varepsilon.$$

By (2.6) we see $x, w_{r_0}, w_r, w_i\in B(5\varepsilon, z_0)$. We assume that $\gamma\cap B(5\varepsilon, z_0)$ is flat, that is,

$$(2.7)\qquad \gamma\cap B(5\varepsilon, z_0)\subset\{(x_1, x_2, x_3)\in\boldsymbol{R}^3;\ x_1=0\}.$$

Also we assume that $\Omega\cap B(5\varepsilon, z_0)\subset\{(x_1, x_2, x_3)\in\boldsymbol{R}^3;\ x_1>0\}$. It is easily seen that the assumption (2.7) is by no means restrictive to estimate $D_{r,i}$ in the case (2.6), since we can use local diffeomorphism, local parametrix, ..., etc. Let $x=(x_1, x_2, x_3)$, $y=(y_1, y_2, y_3)$ be points in $B(5\varepsilon, z_0)$. Now we put

$$G_0(x, y)=(4\pi|x-y|)^{-1}$$
$$G_b(x, y)=(4\pi)^{-1}\{(x_1+y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\}^{-1/2}$$

Then

$$L(x, y)=G(x, y)-G_0(x, y)+G_b(x, y)\in C^\infty((\Omega\cap B(5\varepsilon, z_0))\times(\Omega\cap B(5\varepsilon, z_0))).$$

It is easy to see that

$$\max_{x\in\partial B_r}|G_0(x, w_i)-G_0(w_r, w_i)| \leq(4\pi)^{-1}\max_{x\in\partial B_r}\frac{|w_r-x|}{|x-w_i|\,|w_r-w_i|}\leq C'\alpha m^{-1}|w_r-w_i|^{-2}.$$

Here we used (C–1). Let $\tilde{x}=(-x_1, x_2, x_3)$ be the point of reflection of x with respect to the boundary. Then,

$$\begin{aligned}
&\max_{x\in\partial B_r} |G_b(x, w_i)-G_b(w_r, w_i)| \\
&\quad = \max_{x\in\partial B_r} |G_0(\tilde{x}, w_i)-G_0(\tilde{w}_r, w_i)| \\
&\qquad \leq C'\alpha m^{-1}|\tilde{w}_r-w_i|^{-2}\leq C'\alpha m^{-1}|w_r-w_i|^{-2}.
\end{aligned}$$

It is easy to see that $|L(x, w_i)-L(w_r, w_i)|\leq C'\alpha m^{-1}$. In summing up these facts, we get

$$D_{r,i}\leq C''\alpha m^{-1}(|w_r-w_i|^{-2}+1), \tag{2.8}$$

if (2.6) holds. We also have (2.8) when $\operatorname{dist}(w_{r_0}, w_r)\leq 2\varepsilon$ and $\operatorname{dist}(w_i, w_r)>2\varepsilon$. Thus,

$$\sum_{\substack{i=1\\ i\neq r}}^{m} D_{r,i}\leq \bar{C}\alpha$$

holds for a constant $\bar{C}$ independent of m, w_r satisfying $\operatorname{dist}(w_r, w_{r_0})\leq 2\varepsilon$. Notice that γ_ε is covered by a finite number of open balls which are written as $B(2\varepsilon, w_{r_j})$ for some $w_{r_j}\in\gamma_\varepsilon$. Therefore we get

$$\sum_{w_r\in\gamma_\varepsilon}\sum_{\substack{i=1\\ i\neq r}}^{m} D_{r,i}\leq C_0\alpha m.$$

By a similar argument as above we see that

$$\sum_{w_r\notin\gamma_\varepsilon}\sum_{\substack{i=1\\ i\neq r}}^{m} D_{r,i}$$

does not exceed $C_1\alpha m$. We complete our proof of Lemma 4. q.e.d.

As a consequence of Lemma 4 we get the following;

Lemma 5. *Assume that* $\{w(m)\}\in\mathcal{O}$. *Then*

$$\sum_{s=1}^{m-1}(4\pi\alpha/m)^s\sum_{r=1}^{m}\max_{x\in\partial B_r}|L_r^s f(x)|\leq C_1\Big(\sum_{s=1}^{n-1}(4\pi\alpha)^s\kappa^{s-1}\alpha\Big)\|f\|_{L^p(\Omega_w)} \tag{2.9}$$
$$(f\in C_0^\infty(\Omega_w))$$

holds for any fixed $p>3/2$. *Here* C_1 *is a constant independent of* m.

Proof. By the Sobolev embedding theorem we see that

$$\sup_{x\in\Omega}|(\boldsymbol{G}f)(x)|\leq C_0\|f\|_{L^p(\Omega_w)}\qquad (f\in C_0^\infty(\Omega_w)).$$

Therefore, we have (2.9) using Lemma 4. q.e.d.

We put

$$P_m = \sum_{r=1}^{m} \sum_{\substack{i=1 \\ i \neq r}}^{m} \max_{x \in \partial B_r} |S(x, w_r)| G(w_r, w_i) .$$

Then we have the following:

Lemma 6. *Assume that* $\{w(m)\} \in \mathcal{O}$. *Then* $P_m \leq Cm^2$ *holds for a constant* C *independent of* m.

Proof. From (C–2) we get

$$\max_r \left| \frac{1}{m} \sum_{\substack{1 \leqq i_1 \leqq m \\ i_1 \neq r}} G(w_r, w_{i_1}) - (GV)(w_r) \right| \leq Cm^{-1/3} .$$

Therefore

$$(2.10) \qquad P_m \leq m \sum_{r=1}^{m} \max_{x \in \partial B_r} |S(x, w_r)| (GV)(w_r) + Cm^{2/3} \sum_{r=1}^{m} \max_{x \in \partial B_r} |S(x, w_r)| .$$

We again use γ_ε to estimate (2.10). It is easy to see that

$$\sum_{w_r \notin \gamma_\varepsilon} \max_{x \in \partial B_r} |S(x, w_r)| \leq C_2$$

holds for some constant C_2 independent of $\{w(m)\} \in \mathcal{O}$. Let w_{r_0}, z_0, $G_0(x, y)$, $\cdots$ be as before. We also assume, and it suffices to assume that $\gamma \cap B(5\varepsilon, z_0)$ is flat. For the case $\bar{B}_r \cap \gamma_\varepsilon \neq \phi$, we see that

$$(2.11) \qquad \max_{x \in \partial B_r} |S(x, w_r)| \leq \max_{x \in \partial B_r} |G_b(x, w_r)| + \max_{x \in \partial B_r} |L(x, w_r)|$$
$$\leq C_3(d(w_r, \gamma))^{-1} + C(\alpha/m) .$$

Here we used (C–1) to estimate $\max_{x \in \partial B_r} |G_b|$. By (2.11) and (C–1) we get

$$\sum_{w_r \notin \gamma_\varepsilon} \max_{x \in \partial B_r} |S(x, w_r)| \leq C_4 m^{4/3} .$$

Therefore the second term in the right hand side of (2.10) does not exceed Cm^2.

We here want to estimate the first term of (2.10). We see that

$$(2.12) \qquad |Gf(w_r)| = |Gf(w_r) - Gf(z_0)| \leq C \|Gf\|_{C^1(\Omega)} \operatorname{dist}(w_r \gamma)$$
$$\leq \tilde{C} \|f\|_{L^p(\Omega_w)} \operatorname{dist}(w_r, \gamma) \qquad (f \in C_0^\infty(\Omega_w)) .$$

for a fixed $p > 3$. Combining with (2.11) and (2.12) we see that the first term in the right hand side of (2.10) does not exceed Cm^2. We have the desired estimate. q.e.d.

As an easy consequence of Lemma 6 we get the following:

Lemma 7. *Assume that* $\{w(m)\}\in\mathcal{O}$. *Then*

$$(2.13)\qquad (4\pi\alpha/m)^s\sum_{r=1}^{m}\max_{x\in\partial B_r}|N_r^s f(x)|\leq C_2\Big(\sum_{s=1}^{m-1}(4\pi\alpha)^{s+1}\kappa^{s-1}\Big)\|f\|_{L^p(\Omega_w)}$$
$$(f\in C_0^\infty(\Omega_w))$$

holds for any fixed $p>3$. *Here* C_2 *is a constant independent of* m.

Proof of Lemma 3. Since we have Lemmas 5, 7, we have only to estimate

$$(2.14)\qquad \sum_{r=1}^{m}\max_{x\in\partial B_r}|(\boldsymbol{G}f)(x)-(4\pi\alpha/m)G(x, w_r)(\boldsymbol{G}f)(w_r)|$$

and

$$(2.15)\qquad \textstyle\sum_r \text{(the forth term in (2.3))}\,.$$

It is easy to see that (2.14) does not exceed

$$(2.16)\qquad \max_{x\in\partial B_r}|\boldsymbol{G}f(x)-\boldsymbol{G}f(w_r)|+C'(\alpha/m)\sum_{r=1}^{m}\max_{x\in\partial B_r}|S(x, w_r)|\,|\boldsymbol{G}f(w_r)|\,.$$

By (2.11), (2.12) we see that the second in (2.16) does not exceed

$$(2.16)\qquad C'\alpha\|f\|_{L^p(\Omega_w)}\qquad (p>3, f\in C_0^\infty(\Omega_w))\,.$$

The first term in (2.16) also does not exceed (2.17). We see that

$$|(2.15)|\leq Cm(4\pi\alpha\kappa)^m\|f\|_{L^p(\Omega_w)}\qquad (p>3/2, f\in C_0^\infty(\Omega_w))\,.$$

In summing up these facts we get Lemma 3. q.e.d.

3. Proof of Theorem

Let $\{\varphi_k(x)\}_{k=1}^\infty$ be an orthonomal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta+4\pi\alpha V(x)$ under the Dirichlet condition on γ. Let $\tilde{\boldsymbol{H}}_m$ be the integral operator given by

$$(\tilde{\boldsymbol{H}}_m f)(x)=\int_\Omega h_m(x, y, w(m))f(y)dy\qquad x\in\Omega\,.$$

Then it is easy to see that $g_m(x)=\chi_{\Omega_w}(\tilde{\boldsymbol{H}}_m\varphi_k)(x)-\boldsymbol{H}_m(\chi_{\Omega_w}\varphi_k)(x))=\chi_{\Omega_w}(\tilde{\boldsymbol{H}}_m(\tilde{\chi}_{\Omega_w}\varphi_k))(x)$ satisfies $\Delta g_m(x)=0$ for $x\in\Omega_w$ and $g_m(x)=0$ for $x\in\gamma$, where χ_{Ω_w} (resp. $\tilde{\chi}_{\Omega_w}$) denotes the characteristic function on $\bar{\Omega}_w$ (resp. $\Omega\backslash\Omega_w$. By a simple consideration we see that $\beta_{m,w(m)}=\sum_{r=1}^{m}\{\max|g_m(x)|\,; x\in\partial B_r\}$ does not exceed the term which is given by replacing f in Lemma 3 by $\tilde{\chi}_{\Omega_w}\varphi_k$. We put $p=4$. By Lemma 1 we get the following:

Lemma 8. *Assume that* $\{w(m)\}\in\mathcal{O}$. *Fix* k. *Then*

$$(3.1)\qquad \|\chi_{\Omega_w}\tilde{\boldsymbol{H}}_m\varphi_k-\boldsymbol{H}_m(\chi_{\Omega_w}\varphi_k)\|_{L^2(\Omega_w)}\leq C(\alpha/m)^{7/4}q_{m,\alpha}$$

holds for a constant C *independent of* m.

We have the following:

Lemma 9. *Assume that* $\{w(m)\}\in\mathcal{O}$. *Fix* k. *Then, there exist constants* C *and* $C_{\varepsilon'}$ *independent of* m *such that*

$$(3.2)\qquad \|(\tilde{\boldsymbol{H}}_m-(\mu_k^V)^{-1})\varphi_k\|_{L^2(\Omega_w)}\leq C'\alpha m^{-1}p_{m,\alpha}+C'(4\pi\alpha)^3m^{-1/3}+C_{\varepsilon'}m^{\varepsilon'-(1/6)}$$

holds, where

$$p_{m,\alpha}=4\pi\alpha m^{2/3}q_{m,\alpha}+(4\pi\alpha\kappa)^m\kappa^{-1}m^{5/3}+\alpha^{-1}(4\pi\alpha)^{m+1}\kappa^{m-1}.$$

Proof.

$$L_m(w_{i_p};\psi)=\frac{1}{m}\sum_{\substack{1\leq i_{p+1}\leq m\\ i_{p+1}\neq i_p}}G(w_{i_p},w_{i_{p+1}})\psi(w_{i_{p+1}})-(\boldsymbol{G}V\psi)(w_{i_p})$$

By (1.3) we get

$$(3.3)\qquad \begin{aligned}\mu_k^V(\tilde{\boldsymbol{H}}_m\varphi_k)(x)&=\tilde{\boldsymbol{H}}_m(4\pi\alpha V\varphi_k)(x)-\tilde{\boldsymbol{H}}_m(\Delta\varphi_k)(x)\\ &=\varphi_k(x)-\sum_{s=1}^m R_m^s(x),\end{aligned}$$

where

$$R_m^1(x)=m^2(4\pi\alpha/m)^2\Big(\frac{1}{m}\sum_{i=1}^m G(x,w_i)\varphi_k(w_i)-(\boldsymbol{G}V\varphi_k)(x))\Big)$$

$$R_m^2(x)=m^2(-4\pi\alpha/m)\sum_{i_1=1}^m G(x,w_{i_1})L_m(w_{i_1};\varphi_k)),\ \cdots$$

$$\begin{aligned}&R_m^{m-1}(x)\\ &\quad=m^2(-4\pi\alpha/m)^m\textstyle\sum_{(m-1)}G(x,w_{i_1})G(w_{i_1},w_{i_2})\cdots G(w_{i_{m-2}},w_{i_{m-1}})L(w_{i_{m-1}};\varphi_k)\\ &R_m^m(x)\\ &\quad=m(-4\pi\alpha/m)^{m+1}\textstyle\sum_{(m)}G(x,w_{i_1})G(w_{i_1},w_{i_2})\cdots G(w_{i_{m-1}},w_{i_m})(\boldsymbol{Q}V\varphi_k)(w_{i_m})\end{aligned}$$

for $x\in\partial B_r$. We rearrange them as follows: We put

$$T_{m,r}^1(x)=R_m^1(x)+m^2(-4\pi\alpha/m)^3G(x,w_r)L_m(w_r;\varphi_k)$$

$$\begin{aligned}T_{m,r}^s(x)=m^2(-4\pi\alpha/m)^{s+1}&\sum_{\substack{i_1\neq r,\,i_2\neq i_1,\\ \cdots,\,i_s\neq i_{s-1}}}(G(x,w_{i_1})-(4\pi\alpha/m)G(x,w_r)G(w_r,w_{i_1}))\\ &\times G(w_{i_1},w_{i_2})\cdots G(w_{i_{s-1}},w_{i_s})L_m(w_{i_s},\varphi_k)\end{aligned}$$

for $2 \leq s \leq m-2$,

$$T_{m,r}^{m-1}(x) = m^2(-4\pi\alpha/m)^m \sum_{\substack{i_1 \neq r, i_2 \neq i_1, \\ \cdots, i_{m-1} \neq i_{m-2}}} G(x, w_{i_1})G(w_{i_1}, w_{i_2}) \cdots G(w_{i_{m-2}}, w_{i_{m-1}})L(w_{i_{m-1}}, \varphi_k)$$

$$T_{m,r}^m(x) = R_m^m(x) .$$

Then $\sum_{s=1}^m R_m^s(x) = \sum_{s=1}^m T_{m,r}^s(x)$ for any $r=1, \cdots, m$.

By (C–2) we see that $|L(w_{i_p}, \varphi_k)| \leq Cm^{-1/3}||\varphi_k||_{C^1(\Omega)}$. Then, by the same argument as before we have

$$\sum_{1=r}^{m} \sum_{s=2}^{m-2} \max_{x \in \partial B_r} |T_{m,r}^s(x)| \leq C4\pi\alpha m^{2/3} q_{m,\alpha} \tag{3.4}$$

$$\sum_{r=1}^m \max_{x \in \partial B_r} |T_{m,r}^{m-1}(x)| \leq C(4\pi\alpha\kappa)^m \kappa^{-1} m^{5/3} .$$

It is easy to see that

$$\max_{x \in \partial B_r} |T_{m,r}^m(x)| \leq C\alpha^{-1}(4\pi\alpha)^{m+1}\kappa^{m-1} . \tag{3.5}$$

Here we used the following fact to get (3.5).

$$\max_{\substack{x \in \partial B_r \\ i \neq r}} |G(x, w_i)| \leq C(m/\alpha) .$$

We see that $\sum_{s=2}^m R_m^s(x)$ is harmonic in Ω_w and is zero on γ. Thus, by Lemma 1 we get

$$\begin{aligned}
(3.6) \qquad & ||\sum_{s=2}^m R_m^s||_{L^2(\Omega_w)} . \\
& \leq C(\alpha/m) \sum_{r=1}^m \max_{x \in \partial B_r} |\sum_{s=2}^m R_m^s(x)| \\
& \leq C(\alpha/m) \sum_{r=1}^m \max_{x \in \partial B_r} |\sum_{s=1}^m T_{m,r}^s(x)) - R_m^1(x)| \\
& \leq C(\alpha/m) \sum_{r=1}^m \max_{x \in \partial B_r} (|\sum_{s=2}^m T_{m,r}^s(x)| + m^2(4\pi\alpha/m)^3| |G(x, w_r)| |L_m(w_r, \varphi_k)|) \\
& \leq C(\alpha/m) (\sum_{r=1}^m \sum_{s=2}^m \max_{x \in \partial B_r} |T_{m,r}^s(x)|) + C(4\pi\alpha)^3 m^{-1/3} .
\end{aligned}$$

By (3.4), (3.5) and (3.6), we have

$$||\sum_{s=2}^m R_m^s||_{L^2(\Omega_w)} \leq C'\alpha m^{-1} p_{m,\alpha} + C'(4\pi\alpha)^3 m^{-1/3} . \tag{3.7}$$

We now want to estimate $||R_m^1||_{L^2(\Omega_w)}$. We have

$$||R_m^1||_{L^2(\Omega)}^2 \leq (4\pi\alpha)^4(X_1(m)+X_2(m))\,,$$

where

$$X_1(m) = \left|\frac{1}{m}\sum_{i=1}^{m}(G^2V\varphi_k)(w_i)\varphi_k(w_i) - \int_\Omega ((GV\varphi_k)(x))^2dx\right|$$

$$X_2(m) = \frac{1}{m}\left|\frac{1}{m}\sum_{i,j=1}^{m} G^{(2)}(w_i, w_j)\varphi_k(w_i)\varphi_k(w_j) - \sum_{j=1}^{m}(G^2V\varphi_k)(w_j)\varphi_k(w_j)\right|.$$

Here $G^{(2)}(x, y)$ is given by

$$G^{(2)}(x, y) = \int_\Omega G(x, z)G(z, y)dz\,.$$

It is well known that $G^{(2)}(x, y)$ has the diagonal singularity of the type $|x-y|$, thus,

$$\max_{x\in\bar{\Omega}} ||G^{(2)}(\cdot, x)||_{C^p(\Omega)} \leq C'' < \infty$$

for any fixed $0<p<1$. Therefore, by (C–2) we have

$$X_2(m) \leq \bar{C}\max_{x\in\bar{\Omega}}\left|\frac{1}{m}\sum_{i=1}^{m} G^{(2)}(w_i, x)\varphi_k(w_i) - (G^2V\varphi_k)(x)\right| \leq C_* m^{-p/3} \qquad (0<p<1)\,.$$

By (C–2) we have $X_1(m) \leq Cm^{-1/3}$. In summing up these facts we see that there exists a constant $C_{\varepsilon'}>0$ independent of m such that

$$||R_m^1||_{L^2(\Omega)} \leq C_{\varepsilon'} m^{\varepsilon'-(1/6)} \tag{3.8}$$

holds for any fixed $\varepsilon'>0$. We sum up (3.3), (3.7), (3.8) we get the desired result. q.e.d.

By Proposition 1, Lemmas 8, 9 we have the following:

Lemma 10. *Assume that* $\{w(m)\}\in\mathcal{O}$ *and* α *satisfies* $4\pi\alpha\kappa<1/2$. *Fix* k. *Then*

$$||(G_m - (\mu_k^V)^{-1})\varphi_k||_{L^2(\Omega_w)} \leq \bar{C}_{\varepsilon'} m^{\varepsilon'-(1/6)} \qquad (\varepsilon'>0) \tag{3.9}$$

holds for a constant $\bar{C}_{\varepsilon'}$ *independent of* m.

We have the following:

Lemma 11. *Assume that* $\{w(m)\}\in\mathcal{O}$ *and* α *satisfies* $4\pi\alpha\kappa<1/2$. *Then*

$$||\tilde{\chi}_{\Omega_w} H_m||_{L^2(\Omega)} \mapsto 0$$

as $m\mapsto\infty$

Proof. Notice that

$$\left(\int_{\Omega\setminus\Omega_w} dx \int_\Omega |\textstyle\sum_{(s)} G(x, w_{i_1}) \cdots G(w_{i_s}, y)|^2 dy\right)^{1/2}$$
$$\leq C(\kappa m)^{s-1}\left(\int \sum_{i=1}^m G(x, w_i)^2 dx\right)^{1/2} \leq C''(\kappa m)^{s-1} \qquad (s \geq 1)\,.$$

Since $4\pi\alpha\kappa < 1/2$, we get

$$\lim_{m\to\infty}\left(\int_{\Omega\setminus\Omega_w} dx \int_\Omega |h_m(x, y; w(m))|^2 dy\right)^{1/2} \to 0\,. \qquad \text{q.e.d.}$$

Lemma 12. *Suppose that α satisfies $4\pi\alpha \max(||GV||_{L^2(\Omega)}, \kappa) < 1/2$. Assume that $\{w(m)\} \in \mathcal{O}$. Then*

$$||\tilde{\boldsymbol{H}}_m - H_\infty||_{L^2(\Omega)} \to 0$$

as $m \to \infty$, where $\boldsymbol{H}_\infty$ denotes the operator given by

$$\boldsymbol{G} + \sum_{s=1}^{\infty} (-4\pi\alpha)^s \boldsymbol{G}(V\boldsymbol{G})^s\,.$$

Proof. We fix $u, v \in L^2(\Omega)$. We estimate

$$\frac{1}{m^3}\textstyle\sum_{(3)} (\boldsymbol{G}u)(w_{i_1}) G(w_{i_1}, w_{i_2}) G(w_{i_2}, w_{i_3}) (\boldsymbol{G}v)(w_{i_3}) - \int_\Omega u(x)(\boldsymbol{G}(V\boldsymbol{G})^3 v)(x) dx$$
$$= J_1(m) + J_2(m) + J_3(m)\,,$$

where

$$J_1(m) = \frac{1}{m}\sum_{1\leq i_1\leq m} (\boldsymbol{G}u)(w_{i_1})\left[\frac{1}{m}\sum_{\substack{1\leq i_1\leq m\\ i_2\neq i_1}} G(w_{i_1}, w_{i_2})\right.$$
$$\left.\times\left\{\frac{1}{m}\sum_{\substack{1\leq i_3\leq m\\ i_3\neq i_2}} G(w_{i_2}, w_{i_3})(\boldsymbol{G}v)(w_{i_3}) - (\boldsymbol{G}V\boldsymbol{G}v)(w_{i_2})\right\}\right]$$

$$J_2(m) = \frac{1}{m}\sum_{1\leq i_1\leq m} (\boldsymbol{G}u)(w_{i_1})\left\{\frac{1}{m}\sum_{\substack{1\leq i_2\leq m\\ i_2\neq i_1}} G(w_{i_1}, w_{i_2})(\boldsymbol{G}V\boldsymbol{G}v)(w_{i_2}) - (\boldsymbol{G}(V\boldsymbol{G})^2 v)(w_{i_1})\right\}$$

$$J_3(m) = \frac{1}{m}\sum_{1\leq i_1\leq m} (\boldsymbol{G}u)(w_{i_1})(\boldsymbol{G}(V\boldsymbol{G})^2)(w_{i_1}) - \int_\Omega u(x)(\boldsymbol{G}(V\boldsymbol{G})^3 v)(x) dx\,.$$

We have

$$J_1(m) \leq C||u||_{L^2(\Omega)}\kappa m^{-1/6}||\boldsymbol{G}v||_{C^{1/2}(\Omega)} \leq C^2\kappa m^{-1/6}||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}\,,$$
$$J_2(m) \leq C||u||_{L^2(\Omega)} m^{-1/6}||\boldsymbol{G}V\boldsymbol{G}v||_{C^{1/2}(\Omega)} \leq C^2 m^{-1/6}(8\pi\alpha)^{-1}||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}$$
$$J_3(m) \leq C^2 m^{-1/6}(8\pi\alpha)^{-2}||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}\,.$$

Thus

$$||(m^{-3}\textstyle\sum_{(3)} \cdots - \boldsymbol{G}(V\boldsymbol{G})^3||_{L^2(\Omega)} \leq C^2 m^{-1/6}(\kappa + (8\pi\alpha)^{-1} + (8\pi\alpha)^{-1})$$

Similarly,

$$||(m^{-s}\sum_{(s)}\cdots)-G(VG)^s||_{L^2(\Omega)}\le C^2m^{-1/6}((8\pi\alpha)^{1-s}+\sum_{h=2}^{s}\kappa^{s-h}(8\pi\alpha)^{2-h})\qquad (s\ge 4)$$

and

$$||(m^{-2}\sum_{(2)}\cdots)-G(VG)^2||_{L^2(\Omega)}\le C^2m^{-1/6}(1+(8\pi\alpha)^{-1}).$$

In summing up these facts, we get

$$||\tilde{H}_m-(C+\sum_{s=1}^{m}(-4\pi\alpha)^sG(VG)^s)||_{L^2(\Omega)}\le C^2m^{-1/6}\{(1+(8\pi\alpha)^{-1})$$
$$+(4\pi\alpha)^2\sum_{s=3}^{m}((s-1)+(8\pi\alpha)^{-1})2^{-s+2}\}.$$

We complete the proof of Lemma 12. q.e.d.

Now we are in a final step to prove our Theorem. We use the following:

Lemma 13 (Lemma 2 in [4]). *Let X be a Hilbert space over $\boldsymbol{C}$ with the inner product $(\ ,\ )$ and the norm $||\ ||$. Let E be a compact self-adjoint operator in X. We fix $\tau\in\boldsymbol{R}\setminus\{0\}$. Assume that there are $\psi_j\in X$, $j=1,\cdots,N$ satisfying $||(E-\tau)\psi_j|<\varepsilon$, $||\psi_j||=1$, and $|(\psi_j,\psi_k)|\le(2N)^{-2}$ for $j\neq k$. Then there are at least N eigenvalues τ_j^*, $j=1,\cdots,N$ of E, counted by multiplicity, satisfying*

$$|\tau_j^*-\tau|\le 2N\varepsilon,\qquad i=1,\cdots,N.$$

Proof of Theorem. Firstly we treat the case where α satisfies $4\pi\alpha\max(||GV||_{L^2(\Omega)},\kappa)<1/2$. Let $Z=\mathrm{mult}(\mu_k^V)$ denote the multiplicity of μ_k^V. Let $\varphi_k^V,\cdots,\varphi_{k+Z-1}^V$ be an orthonomal basis of the eigenspace of $-\Delta+4\pi\alpha V$ associated with φ_k^V. We know that

$$\lim_{m\to\infty}(\varphi_k^V,\varphi_h^V)_{L^2(\Omega_w)}=\delta_k^h. \tag{3.10}$$

Using Lemma 10 we see that there exists at least $\mathrm{mult}(\mu_k^V)$ eigenvalues $\mu_k^*(m)^{-1}\le\cdots\le\mu_{k+Z-1}^*(m)^{-1}$ of G_m, counted by multiplicities, satisfying

$$|\mu_{k+j}^*(m)-\mu_k^V|\le C_{\varepsilon'}m^{\varepsilon'-(1/6)}\qquad (j=0,\cdots,\tilde{Z}-1). \tag{3.10}$$

Here $\tilde{Z}\ge Z$.

It should be remarked that we restrict ourselves to the case where Ω_w is connected, since we suppose that $\{w(m)\}\in\mathcal{O}$. Let $\Phi_{k+j}^m(x)$ be the normalized eigenfunction of G_m satisfying

$$(G_m-\mu_{k+j}^*(m)^{-1})\Phi_{k+j}^m=0.$$

And let $\tilde{\Phi}_{k+j}^m(x)$ be an extension of $\Phi_k^m(x)$ putting zero on $\Omega\setminus\bar{\Omega}_w$. By Proposition 1, Lemmas 11, 12 we have

$$\lim_{m\to\infty} ||(\boldsymbol{H}_\infty - \mu^*_{k+j}(m)^{-1})\Phi^m_{k+j}||_{L^2(\Omega)} = 0 .$$

Since we have (3.10) and $(\Phi^m_{k+j}, \Phi^m_{k+h})_{L^2(\Omega)}=\delta^h_j$, $(j, h=0, \cdots, \tilde{Z}-1)$, we see that there exists at least $\tilde{Z}$-dimensional eigenspace of $\boldsymbol{H}_\infty$ whose eigenvalues is μ^V_k. Thus, $\tilde{Z}\leq Z$. As a consequence of these facts we get Theorem under the assumption on α.

We now give a proof of Theorem for general α.Fix α. Let $\lambda>0$ be a sufficiently large number such that the Green function $G_\lambda(x, y)$ and the Green operator $\boldsymbol{G}_\lambda=(-\Delta+\lambda)^{-1}$ of $\Delta-\lambda$ under the Dirichlet condition satisfies

$$(4\pi\alpha) \max_j \sum_{\substack{1\leq i\leq m \\ i\neq j}} G_\lambda(w_i, w_j)<m/2$$

and

$$(4\pi\alpha)||\boldsymbol{G}_\lambda V||_{L^2(\Omega)}<1/2 ,$$

respectively. It is well known that such α exists. Using such λ we discuss everything by changing Δ to $\Delta-\lambda$, and we get Theoıem q.e.d.

References

[1] A. Bensoussan, J.L. Lions and G.C. Papanicolaou: Asymptotic methods in periodic structures, North-Holland, Amsterdam, 1978.

[2] M. Kac: *Probabilistic method in some problems of scattering theory,* Rocky Mountain J. Math. **4** (1974), 511–538.

[3] E.I. Huruslov and V.A. Marchenko: Boundary value problems in regions with fine-grained boundaries, Kiev, 1974. (in Russian).

[4] S. Ozawa: *Singular variation of domains and eigenvalues of the Laplacian,* Duke Math. J. **48** (1981), 767–778.

[5] G.C. Papanicolaou and S.R.S. Varadhan: *Diffusion in region with many small holes,* Lecture Notes in Control and Information, Springer, 75 (1980), 190–206.

[6] J. Rauch and M. Taylor: *Potential and scattering theory on wildly perturbed domains,* J. Funct. Anal. **18** (1975), 27–59.

[7] B. Simon: Functional integration and quantum physics, Academic Press, 1979.

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