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ON THE SIMPLICIAL CONE OF SUPERHARMONIC FUNCTIONS IN A RESOLUTIVE COMPACTIFICATION OF A HARMONIC SPACE

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Introduction

In the theory of integrals on convex sets, a lot of materials is supplied from the Dirichlet problem of potential theory and motivates the development of the theory. And a simplicial consideration of cones of continuous or semi-continuous functions, when reflected upon the potential theory, yields remarkable results [2], [3], [6].

In discussing the Dirichlet problem in harmonic spaces, our concern is laid mainly on the problem of coincidence of the Choquet boundary with the set of regular boundary points, and of the unicity of the methods which give reasonable Dirichlet solutions. These problems, rooted on the Keldych's lemma of a classical potential theory that every regular boundary point is a peak point of a harmonic function, were discussed by many authors [1], [3], [4], [11], [12], [13], [14], [15], [18]. Recently, Biledtner-Hansen [2] gave a decisive answer and it turns out that both problems are related deeply to the negligibility of the set of irregular boundary points.

In the present paper, we shall deal with these problems in a resolutive compactification of a harmonic space. The basic property of resolutive compactifications of harmonic spaces in the sense of Constantinescu-Cornea was given by J. Hyvönen [7] and the Wiener compactification, which posesses an extremely heavy boundary, was introduced and discussed [7], [16]. We see that the situation is quite simple in this heavy compactification, or more generally in a saturated (semiregular) compactification. However in an arbitrary resolutive compactification, it is considerably difficult to obtain a neat analogy to the case of relatively compact open subsets.

Let X be a $\mathcal{P}$-harmonic space with countable base in the sense of Constantinescu-Cornea [5], and X^* be a resolutive compactification of X [7], [16]. We shall consider the set $\mathcal{S}$ of functions which are continuous on X^* and superharmonic on X. The purpose of this paper is to derive several results on the Dirichlet problem of X^* from the known results of the cone $\mathcal{S}$ when $\mathcal{S}$ is simplicial. The first two sections are prepared for the introduction of the following sections. Most of the results stated there are known and will be derived from the theory of convex cones. In §3, we define a simplicial compactification. Not all resolutive compactifications are simplicial. The characterization of simplicial compactifications is given in Theorem 3.4. Hinted by [3], we define weakly determining compactifications. Every saturated compactification is weakly determining, and every weakly determining compactification is simplicial if $\mathcal{S}$ contains a negative function. §4 deals with the Choquet boundary defined by the cone $\mathcal{S}$. Every Choquet boundary point is a regular boundary point of the Dirichlet problem. The converse is also valid when X^* is weakly determining. The latter half of this section is devoted to the study of the Choquet boundaries of compactifications which are quotient of the saturated one and the Choquet boundaries of open subsets of X. The results are similar to those obtained for regular boundary points [9], [10]. If we consider the compactification $\bar{G}$ (the closure of G in X^*) for an open subset G of X, the Choquet boundary point lying in X coincides with the Choquet boundary point of Bliedtner-Hansen [2] (Theorem 4.7). In the last section, we define Keldych operators of X^* and give a criterion for compactifications being of type K, i.e., compactifications possessing a unique Keldych operator.

1. The convex cone $\mathcal{S}$

This section and the next serve a preliminary step for the further sections. Whereas the results stated there are almost known or immediately derived from the theory of conevx cones [2], [3], [17], we give the brief proofs for completeness.

Let X be a $\mathcal{P}$-harmonic space with countable base in the sense of Constantinescu-Cornea, and let X^* be a resolutive compactification of X and $\Delta = X^* \backslash X$ [7], [16]. We denote by $\mathcal{S}$ the set of continuous functions on X^* which are superharmonic on X. $\mathcal{S}$ is an inf-stable convex cone. It is assumed that $\mathcal{S}$ contains a bounded function s_0 such that $\inf_{X^*} s_0 > 0$.

Let $\mathcal{M}$ be the set of non-negative Radon measures on X^*. For $\mu, \nu \in \mathcal{M}$ we define $\mu < \nu$ if $\mu(s) \leq \nu(s)$ for every $s \in \mathcal{S}$, and $\mu \sim \nu$ if $\mu < \nu$ and $\nu < \mu$. We also define for $x \in X^*$

$$\mathcal{M}_x = \{\mu \in \mathcal{M};\ \mu(X) = 0,\ \mu < \varepsilon_x\}\ ,$$

where ε_x is the Dirac measure at x.

For an upper bounded function f and $x \in X^*$ we define

$$\hat{f}(x) = \inf\{s(x);\ s \in \mathcal{M},\ s \geq f \text{ on } \Delta\}\ .$$

Proposition 1.1. [17] *If $\mu \in \mathcal{M}$ and f is an upper bounded function on Δ, then $\mu(\hat{f}) = \inf\{\mu(s);\ s \in \mathcal{S},\ s \geq f$ on $\Delta\}$.*

Proof. Let $\alpha=\inf\{\mu(s); s\in\mathcal{S}, s\geq f \text{ on } \Delta\}$. Then $\mu(\hat{f})\leq\alpha<+\infty$. Let $\{s_n\}$ be a decreasing sequence of functions in $\mathcal{S}$ such that $s_n\geq f$ on Δ and $\lim_n\mu(s_n)=\alpha$. It is not difficult to show that

$$\mu(\{x\in X^*; f'(x)>\hat{f}(x)\})=0\,,$$

where $f'=\lim_n s_n$. We have thus $\mu(\hat{f})=\mu(f')=\alpha$, since $f'\geq\hat{f}$.

Proposition 1.2. [3], [17] *If $\mu\in\mathcal{M}$ and f is an upper bounded, upper semi-continuous function on Δ, then there exists $\nu\in\mathcal{M}$ such that $\nu(X)=0$, $\nu<\mu$ and $\nu(f)=\mu(\hat{f})$. Further*

$$\mu(\hat{f})=\sup\{\nu(f); \nu\in\mathcal{M}, \nu(X)=0, \nu<\mu\}\,.$$

Proof. [1°] consider the case where f is continuous. Since

$$P_\mu(\varphi)=\inf\{\mu(s); s\in\mathcal{S}, s\geq\varphi \text{ on } \Delta\}$$

is subadditive and positively homogeneous functional on the space $\boldsymbol{C}(\Delta)$ of functions finite and continuous on Δ, by the Hahn-Banach theorem, there exists a linear functional F on $\boldsymbol{C}(\Delta)$ such that

$$F(\varphi)\leq P_\mu(\varphi) \text{ for every } \varphi\in\boldsymbol{C}(\Delta)\,,$$
$$F(f)=P_\mu(f)=\mu(\hat{f}) \quad \text{(Prop. 1.1)}\,.$$

Since F is positive, F defines a measure $\mu_1\in\mathcal{M}$ such that $\mu_1(X)=0$ and $\mu_1(\varphi)=F(\varphi)$ for every $\varphi\in\boldsymbol{C}(\Delta)$. It is easily seen that $\mu_1<\mu$, thus $\mu(\hat{f})=P_\mu(f)=F(f)=\mu_1(f)\leq\sup\{\nu(f); \nu\in\mathcal{M}, \nu(X)=0, \nu<\mu\}$. The converse inequality is obvious.

[2°] let f be upper bounded and upper semi-continuous.

$$\mathcal{A}_\mu=\{\nu\in\mathcal{M}; \nu(X)=0, \nu<\mu\}$$

is compact in the vague topology of $\mathcal{M}$. Letting

$$\mathcal{G}=\{\varphi\in\boldsymbol{C}(\Delta); f\leq\varphi \text{ on } \Delta\}$$

and considering $\nu(\varphi)$ as a function of ν, we have by [1°]:

$$\begin{aligned}&\sup\{\nu(f); \nu\in\mathcal{A}_\mu\}=\sup[\inf\{\nu(\varphi); \varphi\in\mathcal{G}\}; \nu\in\mathcal{A}_\mu]=\\&\inf[\sup\{\nu(\varphi); \nu\in\mathcal{A}_\mu\}; \varphi\in\mathcal{G}]=\inf\{\mu(\hat{\varphi}); \varphi\in\mathcal{G}\}=\\&\inf[\inf\{\mu(s); s\in\mathcal{S}, s\geq\varphi \text{ on } \Delta\}; \varphi\in\mathcal{G}]\geq\\&\inf\{\mu(s); s\in\mathcal{S}, s\geq f \text{ on } \Delta\}=\mu(\hat{f}) \quad \text{(Prop. 1.1)}.\end{aligned}$$

The converse inequality is obvious.

A measure $\mu\in\mathcal{M}$ is termed to be *minimal* if

$$\nu\in\mathcal{M},\ \nu(X)=0,\ \nu<\mu\Rightarrow\nu\sim\mu, \text{ i.e., } \nu(s)=\mu(s) \text{ for every } s\in\mathcal{S}\,.$$

Proposition 1.3. [3] *The following assertions are equivalent:*

1) μ *is minimal,*
2) $\mu(\hat{t})=\mu(t)$ *for every* $-t\in\mathcal{S}$,
3) *if* $\nu\in\mathcal{M}$, $\nu(X)=0$, $\nu<\mu$ *and* $\nu\not\sim\mu$, *then there is* $-t\in\mathcal{S}$ *such that* $\mu(\hat{t})=\mu(t)$ *and* $\nu(t)\neq\mu(t)$.

Proof. 1) $\Rightarrow$ 2): let $-t\in\mathcal{S}$, then, by Prop. 1.2, there exists $\nu\in\mathcal{M}$, $\nu(X)=0$, $\nu<\mu$, $\nu(t)=\mu(\hat{t})$. Since μ is minimal we have $\mu(t)=\nu(t)=\mu(\hat{t})$.

2) $\Rightarrow$ 3): this is an immediate consequence of

$$\{t;\ -t\in\mathcal{S},\ \mu(\hat{t})=\mu(t)\}=\{t,\ -t\in\mathcal{S}\}\ .$$

3) $\Rightarrow$ 1): let $\nu\in\mathcal{M}$, $\nu(X)=0$, $\nu<\mu$ and $-t\in\mathcal{S}$, $\mu(\hat{t})=\mu(t)$. Then $\mu(\hat{t})=\inf\{\mu(s);\ s\in\mathcal{S},\ s\geq t \text{ on } \Delta\}\geq\inf\{\nu(s);\ s\in\mathcal{S},\ s\geq t \text{ on } \Delta\}\geq\nu(t)\geq\mu(t)$ implies $\mu\sim\nu$, i.e., μ is minimal.

A function v is called *concave* (resp. *convex*) on X^* if $\mu(v)\leq v(x)$ (resp. $\mu(v)\geq v(x)$) for every $\mu\in\mathcal{M}_x$ and $x\in X^*$. A function is called *affine on* X^* if it is concave and convex on X^*.

Let $\hat{\mathcal{S}}$ be the set of functions lower bounded, lower semi-continuous and concave on X^*. $\hat{\mathcal{S}}$ is a convex cone containing $\mathcal{S}$.

For a convex cone $\mathcal{G}$ such that $\mathcal{S}\subset\mathcal{G}\subset\hat{\mathcal{S}}$ we have

Proposition 1.4. [3] *For every upper bounded, upper semi-continuous function* f *on* Δ *we have* $\hat{f}(x)=\hat{f}^{\mathcal{G}}(x)$ *for every* $x\in X^*$, *where* $\hat{f}^{\mathcal{G}}(x)=\inf\{v(x);\ v\in\mathcal{G},\ v\geq f$ *on* $\Delta\}$.

Proof. We have, by Prop. 1.2, a measure $\nu\in\mathcal{M}_x$ such that $\nu(f)=\hat{f}(x)$. $\nu(f)\leq\inf\{\nu(v);\ v\in\mathcal{G},\ v\geq f \text{ on } \Delta\}\leq\inf\{v(x);\ v\in\mathcal{G},\ v\geq f \text{ on } \Delta\}=\hat{f}^{\mathcal{G}}(x)$, since $\nu(v)\leq v(x)$ for every $v\in\mathcal{G}$. The converse inequality is obvious.

Corollary 1.5. [3] *Let* μ, $\nu\in\mathcal{M}$ *and* $\nu(X)=0$. *Then* $\nu<\mu$ *if and only if* $\nu\underset{\mathcal{G}}{<}\mu$, *where* $\nu\underset{\mathcal{G}}{<}\mu$ *means that* $\nu(v)\leq\mu(v)$ *for every* $v\in\mathcal{G}$.

For supposing $\nu<\mu$, we have for $v\in\mathcal{G}$

$$\begin{aligned}
\nu(v) &= \sup\{\nu(f);\ f\in\boldsymbol{C}(\Delta),\ f\leq v \text{ on } \Delta\}\\
&\leq \sup\{\nu(\hat{f});\ f\in\boldsymbol{C}(\Delta),\ f\leq v \text{ on } \Delta\}\\
&\leq \sup\{\mu(\hat{f});\ f\in\boldsymbol{C}(\Delta),\ f\leq v \text{ on } \Delta\} \quad \text{(Prop. 1.1)}\\
&= \sup\{\mu(\hat{f}^{\hat{\mathcal{S}}});\ f\in\boldsymbol{C}(\Delta),\ f\leq v \text{ on } \Delta\} \quad \text{(Prop. 1.4)}\\
&\leq \sup[\inf\{\mu(v');\ v'\in\hat{\mathcal{S}},\ f\leq v' \text{ on } \Delta\};\ f\in\boldsymbol{C}(\Delta),\ f\leq v \text{ on } \Delta]\\
&\leq \mu(v)\ .
\end{aligned}$$

The converse is trivial.

Corollary 1.6. [3] *$\mathcal{M}_x=\mathcal{M}_x^{\mathcal{G}}$ for every $x\in X^*$, where $\mathcal{M}_x^{\mathcal{G}}=\{\mu\in\mathcal{M};$ $\mu(X)=0,\ \mu\underset{\mathcal{G}}{<}\varepsilon_x\}$.*

Corollary 1.7. [3]

(i) *if μ is $\mathcal{G}$-minimal (i.e., if $\nu\in\mathcal{M}$, $\nu(X)=0$ and $\nu\underset{\mathcal{G}}{<}\mu$, then $\nu(v)=\mu(v)$ for every $v\in\mathcal{G}$), then μ is minimal,*

(ii) *if $\mu\in\mathcal{M}$, $\mu(X)=0$ is minimal, then μ is $\mathcal{G}$-minimal. In particular, for $x\in\Delta$, if ε_x is minimal then ε_x is $\mathcal{G}$-minimal.*

Proposition 1.8. [3]

(i) *let v be upper bounded, upper semi-continuous and concave on Δ, i.e., $\mu(v)\le v(x)$ for every $\mu\in\mathcal{M}_x$ and $x\in\Delta$. If f is lower bounded, lower smei-continuous on Δ and $v<f$, then there exists $s\in\mathcal{S}$ such that $v<s<f$ on Δ.*

(ii) *let v be bounded, lower semi-continuous and concave on Δ. If f is upper bounded, upper semi-continuous and $f\le v$, then there exists $\tilde{s}\in\bar{\mathcal{S}}$ (the closure of $\mathcal{S}$ in $\boldsymbol{C}(\Delta)$ in the topology of sup-norm) such that $f\le\tilde{s}\le v$ on Δ.*

We shall sketch the proof; the detail is refered to [3] Th. 1.5.

(i) is an immediate consequence of $v=\hat{v}$ on Δ. To prove (ii), we construct, by induction, functions $g_n\in\boldsymbol{C}(\Delta)$ and $s_n\in\mathcal{S}$ such that $f\le g_n\le v$ and $g_n\le g_{n+1}\le s_{n+1}\le s_n\le g_n+(1/2^n)s_0$ on Δ. Then $\tilde{s}=\lim_n s_n$ is the required one.

2. Simplicial cones

In this section we consider the case where $\mathcal{S}$ is a simplicial cone. Simplicial cones were defined and investigated in a general context [2], [3]. Accoding to it, we define: $\mathcal{S}$ is *simplicial* if $\mathcal{M}_x$ has the unique minimal measure μ_x (with respect to the preorder $<$) for every $x\in X^*$.

Proposition 2.1. [3] *Let $\mathcal{S}$ be simplicial. If u is continuous and convex on Δ (i.e., $\mu(u)\ge u(x)$ for every $\mu\in\mathcal{M}_x$ and $x\in\Delta$), then $\hat{u}$ is affine on X^*.*

Proof. We prove that $\mu_x(u)=\hat{u}(x)$ for every $x\in X^*$, where μ_x is the unique minimal measure of $\mathcal{M}_x$. By Prop. 1.2, we have

$$\mu_x(t)=\hat{t}(x) \text{ for every } t\in-\mathcal{S} \text{ and } x\in X^*.$$

Since $-u$ is concave on Δ, $\widehat{(-u)}=-u$ on Δ; for

$$-u(x)\le\widehat{(-u)}(x)=\sup\{\nu(-u);\ \nu\in\mathcal{M}_x\}\le-u(x) \text{ whenever } x\in\Delta.$$

From this we have

$$\mu_x(-u)=\mu_x(\widehat{(-u)})=\inf(\{\mu_x(s);\ s\in\mathcal{S},\ s\ge(-u) \text{ on } \Delta\}$$

or equivalently,

$$\mu_x(u) = \sup\{\mu_x(t);\ -t\in\mathcal{S},\ t\leq u \text{ on } \Delta\}$$
$$= \sup\{\hat{t}(x);\ -t\in\mathcal{S},\ t\leq u \text{ on } \Delta\}\ .$$

Applying Prop. 1.8 (i) to functions $-u$ and $-u+\varepsilon$ for $\varepsilon>0$. we may find $-t_1 \in\mathcal{S}$ such that $0<u-t_1<\varepsilon$ on Δ. This implies that

$$\hat{u}(x) = \sup\{\hat{t}(x);\ -t\in\mathcal{S},\ t\leq u \text{ on } \Delta\}\ .$$

And, combining this with above, we have $\mu_x(u)=\hat{u}(x)$.

Let $x\in X^*$ and $\mu\in\mathcal{M}_x$.

$$\mu(\hat{u}) = \inf\{\mu(s);\ s\in\mathcal{S},\ s\geq u \text{ on } \Delta\}$$
$$\geq \inf\{\mu_x(s);\ s\in\mathcal{S},\ s\geq u \text{ on } \Delta\}$$
$$\geq \mu_x(u) = \hat{u}(x)$$

and

$$\mu(\hat{u}) = \inf\{\mu(s);\ s\in\mathcal{S},\ s\geq u \text{ on } \Delta\}$$
$$\leq \inf\{s(x);\ s\in\mathcal{S},\ s\geq u \text{ on } \Delta\} = \hat{u}(x)\ .$$

Thus, we have that $\mu(\hat{u})=\hat{u}(x)$ for every $\mu\in\mathcal{M}_x$ and $x\in X^*$, i.e., $\hat{u}$ is affine on X^*.

Proposition 2.2. [3] *Suppose that $\mathcal{S}$ is simplicial or more generally suppose that $\hat{u}$ is affine on X^* for every function u continuous and convex on Δ. If $-t$ and s are continuous and concave on Δ and $t\leq s$ then for every $\varepsilon>0$ there exists, $-\varphi$, ψ which are continuous and concave on Δ satisfying $t\leq\varphi\leq\psi\leq s$ and $\psi-\varphi\leq\varepsilon s_0$ on Δ.*

Proof. Let

$$\mathcal{A} = \{\varepsilon s_0+\varphi-\psi;\ -\varphi,\ \psi\in \boldsymbol{C}(\Delta) \text{ and concave on } \Delta \text{ such that } t\leq\varphi\leq\psi\leq s\}$$

and

$$\mathcal{B} = \{f\in\boldsymbol{C}(\Delta);\ f>0\}\ .$$

The sets $\mathcal{A}$ and $\mathcal{B}$ are non-empty convex subsets of the Banach space $\boldsymbol{C}(\Delta)$ and $\mathcal{B}$ is open. Suppose, for a moment, that $\mathcal{A}\cap\mathcal{B}=\emptyset$, then there exists a continuous linear functional λ on $\boldsymbol{C}(\Delta)$ such that $\lambda<0$ on $\mathcal{A}$ and $\lambda>0$ on $\mathcal{B}$. This functional defines a measure on Δ which will be denoted by the same λ. By Prop. 1.1, there exists $s_1\in\mathcal{S}$ such that $t\leq s_1\leq s$ on Δ and $\lambda(s_1)<\lambda(\hat{t})+\varepsilon\lambda(s_0)$. Since $\hat{t}$ is affine on X^* from our hypothesis, applying Prop. 1.8 (ii) to functions $-s_1$ and $-\hat{t}$ we may find $-\varphi\in\bar{\mathcal{S}}$ such that $-s_1\leq-\varphi\leq-\hat{t}$ on Δ. Clearly, $-\varphi$ is continuous and concave on Δ. Summing up above considerations we have

$$t\leq\hat{t}\leq\varphi\leq s_1\leq s \text{ on } \Delta$$

and

$$\lambda(s_1)-\lambda(\varphi)\leq\lambda(s_1)-\lambda(\hat{t})<\varepsilon\lambda(s_0)\,.$$

Thus, we are led to a contradiction, since $\varepsilon s_0+\varphi-s_1\in\mathcal{A}$ and $\lambda(\varepsilon s_0+\varphi-s_1)>0$. Hence, $\mathcal{A}\cap\mathcal{B}\neq\emptyset$ which assures the required functions.

Theorem 2.3. *Suppose that $\mathcal{S}$ is simplicial or more generally that $\hat{u}$ is affine on X^* if u is continuous and convex on Δ. If $-\varphi$ and ψ are upper bounded, lower semi-continuous and concave on Δ such that $\varphi\leq\psi$, then there exists a function $u^*\in C(X^*)$ which is affine on X^* and $\varphi\leq u^*\leq\psi$ on Δ. Moreover $u^*=H_{u*}$ on X, where H_{u^*} denotes the Dirichlet solution of u^* for X^*.*

Proof. Let $f\in\boldsymbol{C}(\Delta)$ such that $\varphi\leq f\leq\psi$ on Δ. By Prop. 1.8 (ii), there are functions $-v$, $w\in\boldsymbol{C}(\Delta)$, concave on Δ and $\varphi\leq v\leq f\leq w\leq\psi$ on Δ. Using Prop. 2.2 successively, we have the sequences of functions $\{-v_n\}$, $\{w_n\}$ which are continuous and concave on Δ and satisfying on Δ

$$v\leq v_n\leq v_{n+1}\leq w_{n+1}\leq w_n\leq w\,,$$
$$w_n-v_n\leq(1/2^n)s_0\,.$$

The function $u=\lim_n v_n=\lim_n w_n$ is clearly continuous and affine on Δ and $\varphi\leq u\leq\psi$. By assumption, both functions $\hat{u}$ and $(\widehat{-u})$ are upper semi-continuous and affine on X^*. But $\hat{u}=-(\widehat{-u})=u$ on Δ; for

$$\hat{u}(x)=\sup\{\mu(u);\,\mu\in\mathcal{M}_x\}=u(x)\,,$$
$$-(\widehat{-u})(x)=-\sup\{\mu(-u);\,\mu\in\mathcal{M}_x\}=-(-u(x))=u(x)\,.$$

Also, $\hat{u}=H_u=-(\widehat{-u})$ in X; since for $a\in X$ the harmonic measure λ_a (with respect to X^*) is obviously contained in $\mathcal{M}_a$ and $\hat{u}$ is affine on X^* we have

$$H_u(a)=H_{\hat{u}}(a)=\lambda_a(\hat{u})=\hat{u}(a)\,,$$

and in the same way

$$-(\widehat{-u})=H_u\,.$$

Thus, $u^*=\hat{u}$ fulfils the requirements of the theorem.

Corollary 2.4. If $\mathcal{S}$ is simplicial then $\mathcal{S}\cap(-\mathcal{S})\neq\emptyset$.

3. The simplicial compactification

A resolutive compactification X^* is called *simplicial* if $\mathcal{S}$ is simplicial, i.e., $\mathcal{M}_x$ has the unique minimal measure μ_x for every $x\in X^*$.

It is known that if we consider the closure $\bar{G}$ of a relatively compact open subset G as a compactification, then $\bar{G}$ is simplicial [2]. However, as is shown in the following example, not every resolutive compactification is simplicial:

EXAMPLE 3.1. Let

$$X=\{(x_1,x_2)\in \boldsymbol{R}^2; 1<x_1^2+x_2^2<5^2\}\setminus[\{(x_1,x_2)\in \boldsymbol{R}^2; x_1^2+x_2^2=3^2\}\cup\{(2,0)\}\cup\{(4,0)\}]$$

and consider the harmonic structure on X defined by the solutions of the Laplace equation. We compactify X so that the ideal boundary consists of three points $\{a,b,c\}$, where

$$\begin{aligned} a &= \{(x_1, x_2)\in \boldsymbol{R}^2;\ x_1^2+x_2^2 = 1\}\ , \\ b &= \{(x_1, x_2)\in \boldsymbol{R}^2;\ x_1^2+x_2^2 = 3^2\}\cup\{(2, 0)\}\cup\{(4, 0)\}\ , \\ c &= \{(x_1, x_2)\in \boldsymbol{R}^2;\ x_1^2+x_2^2 = 5^2\}\ . \end{aligned}$$

Then X^* is resolutive and $\mathcal{S}$ separates points of Δ. But $\mathcal{S}$ is not simplicial. For, consider the function F continuous and concave on the interval [0, log5], linear on each interval [0, log2], [log2, log4], [log4, log5] and constant on [log2, log4]. $s(x_1, x_2)=F[\log(x_1^2+x_2^2)^{1/2}]\in\mathcal{S}$. If $\mathcal{S}$ is simplicial, then by Prop. 2.1, $-\widehat{(-s)}$ is affine on X^*, moreover it is constant, thus

$$\mu_b(s) = \min[s(a),\ s(c)]\ .$$

Suppose that $\mu_b=l\varepsilon_a+m\varepsilon_b+n\varepsilon_c$, where l,m,n are non-negative; if $s(b)>s(a)>s(c)=0$, then $0=\mu_b(s)=ls(a)+ms(b)$ implies $l=m=0$. In the same way, if $s(b)>s(c)>s(a)=0$, then $m=n=0$, thus $\mu_b=0$. This is however impossible since for the positive constant function u, $\mu_b(u)=u(b)>0$.

Proposition 3.2. *If $\mathcal{S}$ separates points of Δ, i.e., for every distinct points $x,y\in\Delta$ there exist $s_1, s_2\in\mathcal{S}$ such that $s_1(x)s_2(y)\neq s_1(y)s_2(x)$, then a measure μ, $\mu(X)=0$ is minimal if and only if $\mu(\hat{f})=\mu(f)$ for every $f\in\boldsymbol{C}(\Delta)$.*

It is sufficient to prove the "only if" part. By Prop. 1.2,

$$\mu(\hat{f}) = \sup\{\nu(f);\ \nu\in\mathcal{M},\ \nu(X) = 0,\ \nu<\mu\}\ .$$

The minimality means $\mu(s)=\nu(s)$ for every $s\in\mathcal{S}$ and $\mu(s_1-s_2)=\nu(s_1-s_2)$ for every $s_1, s_2\in\mathcal{S}$, and finally $\mu=\nu$, which induces $\mu(\hat{f})=\mu(f)$ since the vector lattice $\mathcal{S}-\mathcal{S}$ is dense in $\boldsymbol{C}(\Delta)$ in the topology of sup norm.

Now we shall give a criterion of a simplicial compactification.

Theorem 3.3. [2],[3] *Suppose that $\mathcal{S}$ separates points of Δ and let $\mathcal{G}$ be a convex cone of lower bounded and lower semi-continuous concave functions on Δ, containing all function continuous and concave on Δ. Then the following assertions are equivalent:*

1) *X^* is simplicial,*

2) *$\hat{u}$ is affine on X^* if $u \in -\mathcal{G}$,*

3) *for every $-u, v \in \mathcal{G}$ which are upper bounded and $u \leq v$ on Δ there exists h continuous and affine on X^* such that $u \leq h \leq v$ on Δ.*

Proof. 1) $\Rightarrow$ 2): if $u \in -\mathcal{G}$ and $\mu \in \mathcal{M}_x$ for $x \in X^*$, by Prop. 1.2, we have $\lambda \in \mathcal{M}$, $\lambda(X)=0$, $\lambda < \mu$ and $\mu(\hat{u})=\lambda(u)$. Since $\hat{u}(x)=\sup\{\nu(u); \nu \in \mathcal{M}_x\}=\mu_x(u)$, where μ_x is the unique minimal measure of $\mathcal{M}_x$, the inequalities

$$\begin{aligned}\mu_x(u) \leq \mu_x(\hat{u}) &= \inf\{\mu_x(s); s \in \mathcal{S}, s \geq u \text{ on } \Delta\} \\ &\leq \inf\{\mu(s); s \in \mathcal{S}, s \geq u \text{ on } \Delta\} = \mu(\hat{u}) = \lambda(u) \leq \mu_x(u)\end{aligned}$$

(the last inequality is the consequence of Cor. 1.5 since $\mu_x < \lambda$) induce

$$\mu(\hat{u}) = \mu_x(u) = \hat{u}(x) .$$

2) $\Rightarrow$ 3): this is proved in Th.2.3.

3) $\Rightarrow$ 1): let $\mu, \mu' \in \mathcal{M}_x$ be minimal for $x \in X^*$. For $t \in -\mathcal{S}$, by Prop.1.2, we have $\lambda \in \mathcal{M}$, $\lambda(X)=0$, $\lambda < \mu$ such that $\mu(\hat{t})=\lambda(t)$. Since μ is minimal $\lambda=\mu$. We prove that $\hat{t}(x)=\mu(t)$; for

$$\begin{aligned}\hat{t}(x) &= \inf\{s(x); s \in \mathcal{S}, s > t \text{ on } \Delta\} \\ &\geq \inf\{\mu(s); s \in \mathcal{S}, s > t \text{ on } \Delta\} \\ &= \mu(\hat{t}) \geq \inf\{\mu(h); h \in \boldsymbol{C}(X^*), \text{ affine on } X^*, h \geq t \text{ on } \Delta\} \\ &= \inf\{h(x); h \in \boldsymbol{C}(X^*), \text{ affine on } X^*, h \geq t \text{ on } \Delta\} \\ &\geq \inf\{v(x), v \in \hat{\mathcal{S}}, v \geq t \text{ on } \Delta\} \\ &= \hat{t}^{\hat{\mathcal{S}}}(x) = \hat{t}(x) \quad \text{(Prop. 1.4)}.\end{aligned}$$

Thus, $\hat{t}(x)=\mu(\hat{t})=\lambda(t)=\mu(t)$. In the same way, $\hat{t}(x)=\mu'(t)$. Hence, $\mu(t)=\mu'(t)$ for every $t \in -\mathcal{S}$, i.e., $\mu=\mu'$.

REMARK. We can prove Theorem 3.3 for any convex cone $\mathcal{G}$ of lower bounded, lower semi-continuous concave functions on Δ containing the restrictions on Δ of all functions of $\mathcal{S}$. Let

2') *$\hat{u}$ is affine on X^* if u is upper semi-continuous and convex on Δ,*

2'') *$\hat{u}$ is affine on X^* if $u \in -\bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ denotes the uniform closure of $\mathcal{S}$ in $\boldsymbol{C}(\Delta)$,*

2''') *$\hat{u}$ is affine on X^* if u is the restriction on Δ of a function of $-\mathcal{S}$,*

3') *for every $-t, s \in \mathcal{S}$ and $t < s$ on Δ, there exists h continuous and affine on X^* such that $t \leq h \leq s$.*

By Th.3.3, 1)$\Rightarrow$2'); obviously 2')$\Rightarrow$2) and 2')$\Rightarrow$2'')$\Rightarrow$2'''), but it is easily proved that 2''')$\Rightarrow$2''). From the proof of Th.2.3, we get also 2'')$\Rightarrow$3) and 3')$\Rightarrow$1). Thus, we have conclusively:

$$1) \Rightarrow 2') \Rightarrow 2) \Rightarrow 2''') \Leftrightarrow 2'') \Rightarrow 3) \Rightarrow 3') \Rightarrow 1) .$$

Theorem 3.4. *The propositions* 1), 2), 2′), 2″), 2‴), 3), *and* 3′) *are equivalent.*

In [7], J. Hyvönen defined the oder relation in the resolutive compactifications of X. Let X^* and X^{**} be resolutive compactifications of X. We call X^* is a *quotient* of X^{**} and denote it by '$X^* \leq X^{**}$ if there is a continuous mapping π of X^{**} onto X^* such that $\pi(a)=a$ for every $a \in X$. The mapping π is called *canonical.*

A resolutive compactification X^* is termed to be *saturated* (or *semi-regular*) if all Dirichlet solutions $H_{F|\Delta}$ of $F \in \boldsymbol{C}(X^*)$ can be extended continuously on X^*. X^* is saturated if and only if X^* is homeomorphic to $X^{Q(X^*)}$, where $X^{Q(X^*)}$ is the Q-compactification of X [7] and

$$Q(X^*) = \{F \mid X;\, F \in \boldsymbol{C}(X^*)\} \cup \{H_{F|\Delta};\, F \in \boldsymbol{C}(X^*)\} .$$

As in [8], [9], we can prove that $X^{Q(X^*)}$ is the smallest saturated compactification possessing X^* as a quotient, i.e., if X^{**} is saturated and $X^* \leq X^{**}$ then $X^{Q(X^*)} \leq X^{**}$. Furthermore, if $X^* \leq X^{**} \leq X^{Q(X^*)}$ then the smallest saturated compactification of X^{**} is $X^{Q(X^*)}$, i.e., $X^{Q(X^*)}$ is homeomorphic to $X^{Q(X^{**})}$.

We introduce here the *harmonic boundary* Γ of X^*:

$$\Gamma = \cap \{\Gamma_p;\, p \text{ is a potential on } X\} ,$$

where $\Gamma_p = \{x \in \Delta;\, \underline{\lim}_x\, p = 0\}$. The harmonic boundary Γ plays an important role in the theory of resolutive compactifications [7], [16]. It is known that if $X^* \leq X^{**}$ then $\pi(\Gamma^{**}) = \Gamma^*$, where Γ^* (resp. Γ^{**}) denotes the harmonic boundary of X^* (resp. X^{**}) and π is the canonical mapping of X^{**} onto X^*. Further, all points of $\Gamma^{Q(X^*)}$ (the harmonic boundary of $X^{Q(X^*)}$) are regular with respect to the Dirichlet problem for $X^{Q(X^*)}$. The Dirichlet solutions H_f are solved for data functions f defined only on Γ [7].

Proposition 3.5. *If* $X^* \leq X^{**} \leq X^{Q(X^*)}$ *then* Γ^* *is homeomorphic to* Γ^{**}.

Proof. It is sufficient to show that the canonical mapping π of $X^{Q(X^*)}$ onto X^* is one-to-one on $\Gamma^{Q(X^*)}$. For $\varphi \in \boldsymbol{C}(\Gamma^{Q(X^*)})$ and $\varepsilon > 0$ we may find, by the same argument as in [8], a function $f \in \boldsymbol{C}(\Gamma^*)$ such that $\sup_X |H_f - H^Q_\varphi| < \varepsilon$, where H_f (resp. H^Q_φ) is the Dirichlet solution of f (resp. φ) for X^* (resp. $X^{Q(X^*)}$). From this it is easy to construct $f_0 \in \boldsymbol{C}(\Gamma^*)$ such that $H_{f_0} = H^Q_\varphi$. Suppose that $\tilde{x}_1, \tilde{x}_2 \in \Gamma^{Q(X^*)}$ and $\pi(\tilde{x}_1) = \pi(\tilde{x}_2) = x$. Then $\tilde{x}_1 = \tilde{x}_2$, for if $\tilde{x}_1 \neq \tilde{x}_2$, then there exists $\varphi \in \boldsymbol{C}(\Gamma^{Q(X^*)})$ and $f_0 \in \boldsymbol{C}(\Gamma)$ such that $\varphi(\tilde{x}_1) \neq \varphi(\tilde{x}_2)$ and $H^Q_\varphi = H_{f_0}$. We have then $H^Q_{f_0 \circ \pi} = H_{f_0} = H^Q_\varphi$. Since all points of $\Gamma^{Q(X^*)}$ are regular, $\varphi = f_0 \circ \pi$ on $\Gamma^{Q(X^*)}$. Thus we have led to a contradictoin: $\varphi(\tilde{x}_1) = f_0[\pi(\tilde{x}_1)] = f_0[\pi(\tilde{x}_2)] = \varphi(\tilde{x}_2)$.

Theorem 3.6. *A saturated compactification* X^* *is simplicial and the minimal measure* μ_x *is*

$$\mu_x = \begin{cases} \lambda_x & \text{if } x \in X \\ \varepsilon_x & \text{if } x \in \Gamma \\ \nu & \text{if } x \in \Delta \backslash \Gamma\,, \end{cases}$$

where λ_x *is the harmonic measure of* x *and* ν *is the measure such that* $\nu(f)=H_f(x)$ *for every* $f\in \boldsymbol{C}(\Delta)$.

Proof. We note that $\{H_f; f\in \boldsymbol{C}(\Delta)\}\subset \mathcal{S}$. For $t\in -\mathcal{S}$, $H_t=-H_{(-t)}\in \mathcal{S}\cap(-\mathcal{S})$, i.e., H_t is affine on X^*; and $\hat{t}=H_t$ on X^*, for $s\in\mathcal{S}$, $t\leq s$ on Δ implies $t\leq H_t\leq H_s\leq s$, thus, by Th.3.4, X^* is simplicial.

If $x\in X$, then $\lambda_x\in\mathcal{M}_x$ and $\lambda_x(X^*\backslash\Gamma)=0$ [7], which means $\lambda_x(\hat{t})=H_{\hat{t}}(x)=H_t(x)=\lambda_x(t)$, since $\hat{t}=t$ on Γ. For $x\in\Gamma$, $\varepsilon_x(\hat{t})=\varepsilon_x(t)$. And finally, for $x\in\Delta\backslash\Gamma$, $\nu\in\mathcal{M}_x$, $\nu(X^*\backslash\Gamma)=0$ and $\nu(\hat{t})=\nu(t)$. By Prop.1.3, λ_x, ε_x and ν are minimal.

REMARKS. 1. Later (§4) we shall see that in a saturated compactification X^* the Choquet boundary of X^* for $\mathcal{S}$ is Γ.

2. It is obvious that $\lambda_x\neq\varepsilon_x$ for $x\in X$ and $\nu\neq\varepsilon_x$ for $x\in\Delta\backslash\Gamma$.

Hinted by the notion of weakly determining sets [3], we shall define that a resolutive compactification X^* is *of type* (*WD*) if for every $s\in\mathcal{S}$, there exists an upper directed family $\{h_\iota\}$ of functions continuous and affine on X^* such that

$$\sup_\iota h_\iota(x) = \underline{\lim}_x H_s \qquad \text{for every } x\in\Gamma\,.$$

Porposition 3.7. *Every saturated compactification is of type* (*WD*) *and a compactification of type* (*WD*) *is simplicial if* $\mathcal{S}$ *contains strictly negative functions.*

Proof. The first half is an immediate consequence of $H_s=-H_{(-s)}\in\mathcal{S}\cap(-\mathcal{S})$.

To prove the second half, in view of Th.3.4, it is sufficient to prove that for every $-t, s\in\mathcal{S}$ with $t<s$ on Δ, there exists h continuous and affine on X^* such that $t\leq h\leq s$ on Δ. Then, for some $\eta>0$, $t<t-\eta s_1<s$ on Δ, where s_1 is a strictly negative function in $\mathcal{S}$. Since $t<t-\eta s_1\leq H_t-\eta H_{s_1}\leq H_s\leq s$ on X, $t(x)<\underline{\lim}_x H_s$ for every $x\in\Delta$, and therefore we have a function h continuous and affine on X^* such that $t<h\leq s$ on Γ and $t\leq h\leq s$ on Δ.

4. The Choquet boundary

In the following, we suppose that $\mathcal{S}$ separates points of Γ. We define the *Choquet boundary*

$$Ch_{\mathcal{S}}X^* = \{x\in X^*;\ \mathcal{M}_x = \{\varepsilon_x\}\}\,.$$

First we shall show

Proposition 4.1.

$$Ch_{\mathcal{S}} X^* \subset \Delta_{\text{reg}} \subset \Gamma ,$$

where Δ_{reg} denotes the set of regular boundary points.

To prove Prop. 4.1, we prepare

Lemma 4.2. *Given $s \in \mathcal{S}$ and $x \in \Delta$, there exists $\lambda \in \mathcal{M}_x$ with $\lambda(\Delta \backslash \Gamma)=0$ such that $\lambda(s)=\underline{\lim}_x H_s$.*

Proof. We consider the saturated compactification $X^{Q(X^*)}$ defined in the preceding section. Since H_s has the continuous extension on $X^{Q(X^*)}$ (we denote this extension by the same H_s), we may find a point $\tilde{x} \in \pi^{-1}(x) \cap X^{Q(X^*)}$ such that $\underline{\lim}_x H_s = H_s(\tilde{x})$, where π is the canonical mapping of $X^{Q(X^*)}$ onto X^*. The mapping of $C(\Delta)$, $f \to H_f(\tilde{x})$ define sa positive Radon measure λ on Γ. Since $\lambda(s') = H_{s'}(\tilde{x}) \leq \overline{\lim}_x H_{s'} \leq s'(x)$ for every $s' \in \mathcal{S}$, $\lambda \in \mathcal{M}_x$.

Proof of Prop. 4.1. By Lemma 4.2, for every $s \in \mathcal{S}$ we obtain $\lambda \in \mathcal{M}_x$ satisfying $\lambda(\Delta \backslash \Gamma)=0$ and $\lambda(s)=\underline{\lim}_x H_s$. If $x \in Ch_{\mathcal{S}} X^*$ then, since $\lambda=\varepsilon_x$, $s(x)= \lambda(s)=\underline{\lim}_x H_s \leq \overline{\lim}_x H_s \leq s(x)$, i.e., $\lim_x H_s = s(x)$ for every $s \in \mathcal{S}$ and, since $\mathcal{S}-\mathcal{S}$ is uniformly dense in $C(\Gamma)$, $\lim_x H_f = f(x)$ for every $f \in C(\Delta)$. This implies that $x \in \Delta_{\text{reg}} \subset \Gamma$, since Γ is the carrier of harmonic measures [7].

The following proposition shows a role of the harmonic boundary in the theory. We define

$$\mathcal{M}_x^* = \{\mu \in \mathcal{M}_x;\ \mu(X^* \backslash \Gamma) = 0\} .$$

Proposition 4.3. *For every $\mu \in \mathcal{M}_x$ there exists $\nu \in \mathcal{M}_x^*$ such that $\nu < \mu$. In particular, we have*

$$Ch_{\mathcal{S}} X^* = \{x \in X^*;\ \mathcal{M}_x^* = \{\varepsilon_x\}\} .$$

Proof. Since $P_\mu(f)=\inf\{\mu(s);\ s \in \mathcal{S},\ s \geq f \text{ on } \Gamma\}$ defines a subadditive and positively homogeneous functional on $C(\Gamma)$, there exists a positive Radon measure ν on Γ satisfying $\nu(f) \leq P_\mu(f)$ for every $f \in C(\Gamma)$. We can derive readily that $\nu \in \mathcal{M}_x^*$, $\nu < \mu$. If $\mathcal{M}_x^* = \{\varepsilon_x\}$ and $\mu \in \mathcal{M}_x$ then $\varepsilon_x < \mu < \varepsilon_x$, this implies that $\mu(f) = f(x)$ for every $f \in C(\Gamma)$, and finally, $x \in \Gamma$, $\mu = \varepsilon_x$.

Proposition 4.4. [3] *If X^* is of type (WD) then, $Ch_{\mathcal{S}} X^* = \Delta_{\text{reg}}$.*

Proof. Suppose that $x \in \Delta_{\text{reg}}$ and $\nu \in \mathcal{M}_x^*$. By definition, for every $s \in \mathcal{S}$ there exists an upper directed family $\{h_\iota\}$ of functions continuous and affine on X^* such that $\underline{\lim}_x H_s = \sup_\iota h_\iota(x)$ for every $x \in \Gamma$. On the other hand, we may find a minimal measure $\mu \in \mathcal{M}_x^*$ so that $\mu < \nu$. Since $h_\iota \leq s$ on Γ we have $h_\iota \leq s$ on Δ. Therefore $s(x) = \underline{\lim}_x H_s = \sup_\iota h_\iota(x) \leq \sup\{t(x);\ t \in -\mathcal{S},\ t \leq s \text{ on } \Delta\} \leq$

$\sup\{\mu(t);\ t\in-\mathcal{S},\ t\leq s \text{ on } \Delta\}=-\inf\{\mu(s');\ s'\in\mathcal{S},\ s'\geq -s \text{ on } \Delta\}=-\mu(\widehat{-s})$. Since μ is minimal, by Prop.1.3 $\mu(-s)=\mu(\widehat{-s})$, therefore $s(x)\leq\mu(s)\leq\nu(s)\leq s(x)$, i.e., $\nu(s)=s(x)$ for every $s\in\mathcal{S}$. Finally, since $\mathcal{S}$ is total in $\boldsymbol{C}(\Gamma)$, we have $\nu=\varepsilon_x$ and $x\in Ch_{\mathcal{S}}\,X^*$.

Theorem 4.5. *Suppose that $X^*\leq X^{**}\leq X^{Q(X^*)}$, and let π be the canonical mapping of X^{**} onto X^* and $\mathcal{S}_1$ be the set of functions continuous in X^{**} and superharmonic in X. Then $Ch_{\mathcal{S}}\,X^*\subset\pi\,(Ch_{\mathcal{S}_1}X^{**})$.*

Proof. Since, by Prop.3.5, Γ^* is homeomorphic to Γ^{**}, we identify these harmonic boundaries and denote it by Γ. Making a proper identification we have $\mathcal{S}\subset\mathcal{S}_1$, thus $\mathcal{M}_x^{*\mathcal{S}_1}\subset\mathcal{M}_x^{*\mathcal{S}}$ and $\mathcal{M}_x^{*\mathcal{S}}=\{\varepsilon_x\}$ implies $\mathcal{M}_x^{*\mathcal{S}_1}=\{\varepsilon_x\}$, which means $Ch_{\mathcal{S}}\,X^*\subset\pi(Ch_{\mathcal{S}_1}X^{**})$.

Next, for an open subset $G(\neq X)$ of X^* we consider the harmonic space $X_0=G\cap X$. The closure $\bar{X}_0$ of X_0 in X^* is a resolutive compactification [10]. We write $\Delta(X_0)=\bar{X}_0\backslash X_0=(\Delta\cap\bar{G})\cup\partial G$, where $\partial G=(\bar{G}\backslash G)\cap X$, and $\mathcal{S}_0=\{s;$ continuous on $\bar{X}_0$, superharmonic in $X_0\}$.

Theorem 4.6.

$$(Ch_{\mathcal{S}_0}\bar{X}_0)\cap G\cap\Delta\subset(Ch_{\mathcal{S}}\,X^*)\cap G\,.$$

Proof. Assume that $x\in(Ch_{\mathcal{S}_0}\bar{X}_0)\cap G\cap\Delta$. For $f\in\boldsymbol{C}(\Delta)$ and $s\in\mathcal{S}$, such that $s\geq f$ on Δ, we define

$$\varphi=\begin{cases} f & \text{on } G\cap\Delta \\ \sup s & \text{on } \overline{\partial G}\,. \end{cases}$$

The function φ is upper semi-continuous on $\Delta(X_0)$. From Prop. 1.2 and our assumption,

$$\begin{aligned}\hat{\varphi}^{\mathcal{S}_0}(x)&=\inf\{s'(x);\ s'\in\mathcal{S}_0,\ s'\geq\varphi \text{ on } \Delta(X_0)\}\\ &=\sup\{\nu(\varphi);\ \nu(\bar{X}_0\backslash\Delta(X_0))=0,\ \nu\underset{\mathcal{S}_0}{<}\varepsilon_x\}=\varphi(x)\,.\end{aligned}$$

Hence, for evrey $\varepsilon>0$ we may find $s'\in\mathcal{S}_0$ such that $s'\geq\varphi$ on $\Delta(X_0)$ and $s'(x)<\varphi(x)+\varepsilon=f(x)+\varepsilon$. The function

$$s_1=\begin{cases} s & \text{on } X^*\backslash\bar{X}_0 \\ \inf(s,s') & \text{on } \bar{X}_0 \end{cases}$$

is in $\mathcal{S}$ and $s_1\geq f$ on Δ. Thus we have

$$f(x)\leq\hat{f}(x)\leq s_1(x)\leq s'(x)<f(x)+\varepsilon\,.$$

This implies that $f(x)=\hat{f}(x)$ for every $f\in\boldsymbol{C}(\Delta)$, and finally $x\in Ch_{\mathcal{S}}\,X^*$.

Two parts of Choquet boundaries $(Ch_{\mathcal{S}_0}\bar{X}_0)\cap G\cap\Delta$ and $(Ch_{\mathcal{S}}X^*)\cap G$ do not coincid in general. This is seen in the example of [10] (p. 182), which is the example showing the same situation for regularity.

In [2], Bliedtner and Hansen considered the Choquet boundary of an open subset G of X and showed that every point of the Choquet boundary has a local property. More precisely, let

$$S(X_0)=\{v\in \boldsymbol{C}(\bar{G}\cap X);\ \text{superharmonic in } X_0 \text{ and } |v|\leq p \text{ for a potential } p \text{ on } X\},$$
$$\hat{f}^{S(X_0)}(x)=\inf\{v(x);\ v\in S(X_0),\ v\geq f \text{ on } \bar{G}\cap X\},$$
$$\mathcal{M}_x(S(X_0))=\{\mu;\ \text{positive Radon measure on } \bar{G}\cap X, \mu(v)\leq v(x) \text{ for all } v\in S(X_0)\},$$
$$Ch_{S(X_0)}(\bar{G}\cap X)=\{x\in\bar{G}\cap X;\ \mathcal{M}_x(S(X_0))=\{\varepsilon_x\}\}.$$

Then, if $x\in Ch_{S(X_0)}(\bar{G}\cap X)$ and U is a relatively compact open neighborhood of x, then $x\in Ch_{S(G\cup U)}(\overline{G\cap U})$ and *vice-versa* ([12], Prop.3.11)

We remark that if $\hat{t}^{S(X_0)}(x)=t(x)$ for every $t\in -P$, where P is the set of finite continuous potentials on X, then $x\in Ch_{S(X_0)}(\bar{G}\cap X)$. For, letting $\mu\in\mathcal{M}_x(S(X_0))$, we have

$$\begin{aligned} t(x)=\hat{t}^{S(X_0)}(x) &= \inf\{v(x);\ v\in S(X_0),\ v\geq t \text{ on } \bar{G}\cap X\}\\ &\geq \inf\{\mu(v);\ v\in S(X_0),\ v\geq t \text{ on } \bar{G}\cap X\}\\ &\geq \mu(t)\geq t(x).\end{aligned}$$

By the approximation theorem, $\mu(f)=f(x)$ for every $f\in \boldsymbol{C}_0(\bar{G}\cap X)$.

Theorem 4.7.

$$(Ch_{\mathcal{S}_0}\bar{X}_0)\cap\partial G=Ch_{S(X_0)}(\bar{G}\cap X).$$

Proof. First, assume that $x\in(Ch_{\mathcal{S}_0}\bar{X}_0)\cap\partial G$, $t\in -P$. The function

$$\varphi=\begin{cases} t & \text{on } \partial G\\ 0 & \text{on } \bar{G}\cap\Delta\end{cases}$$

is upper semi-continuous and $\varphi\leq 0$ on $\Delta(X_0)$. By Prop. 1.2, we have

$$\begin{aligned} t(x)=\varphi(x)=\hat{\varphi}^{\mathcal{S}_0}(x) &= \inf\{s(x);\ s\in\mathcal{S}_0,\ s\geq\varphi \text{ on } \Delta(X_0)\}\\ &\geq \inf\{\inf(s,\ -t)(x);\ s\in\mathcal{S}_0,\ s\geq\varphi \text{ on } \Delta(X_0)\}\\ &\geq \inf\{v(x);\ v\in S(X_0),\ v\geq t \text{ on } \partial G\}\\ &= \inf\{v(x);\ v\in S(X_0),\ v\geq t \text{ on } \bar{G}\cap X\}\\ &= \hat{t}^{S(X_0)}(x).\end{aligned}$$

Thus we have $\hat{t}^{S(X_0)}(x)=t(x)$ for every $t\in -P$, and, by the preceding remark, $x\in Ch_{S(X_0)}(\bar{G}\cap X)$.

Next, suppose that $x\in Ch_{S(X_0)}(\bar{G}\cap X)$. Let $D=G\cap U$, where U is a relatively compact open neighborhood of x. By a local property of the Choquet boundary point, $x \in Ch_{S(D)}(\bar{D})$. For $t \in -\mathcal{S}_0$ and $s \in \mathcal{S}_0$ such that $s \geq t$, the function

$$\varphi = \begin{cases} t & \text{on } \Delta(G) \cap D \\ \sup s & \text{on } \bar{G} \cap \partial U \end{cases}$$

is upper semi-continuous and upper bounded on ∂D. Then $\hat{\varphi}^{S(D)}(x) = \varphi(x)$, thus for every $\varepsilon > 0$ we may find $v \in S(D)$ such that $v \geq \varphi$ on ∂D and

$$v(x) < \hat{\varphi}^{S(D)}(x) + \varepsilon = \varphi(x) + \varepsilon = t(x) + \varepsilon .$$

The function

$$s_1 = \begin{cases} s & \text{on } \bar{G} \setminus D \\ \inf(v, s) & \text{on } U \cap \bar{G} \end{cases}$$

is in $\mathcal{S}_0$ and $s_1 \geq t$ on $\Delta(X_0)$. Hence,

$$\hat{t}^{\mathcal{S}_0}(x) \leq s_1(x) \leq v(x) < t(x) + \varepsilon ,$$

which means that $\hat{t}^{\mathcal{S}_0}(x) = t(x)$ for every $t \in -\mathcal{S}_0$, and, by Prop. 1.3, we can conclude that $x \in (Ch_{\mathcal{S}_0} \bar{X}_0) \cap \partial G$.

REMARK. The regular boundary points enjoy the same properties stated in Theorem 4.5, Theorem 4.6 and Theorem 4.7 [9], [10]. It is remarkable and interesting to point out this similarity of the Choquet boundary and the set of regular boundary points.

5. Keldych operators

In a relatively compact open set U, the unicity of Keldych operators, that is operators forming reasonable Dirichlet solutions, depends on the set of irregular boundary points. This was established by J. Lukeš [13], in virtue of a theorem of Bliedtner-Hansen [2]. In the case of arbitrary open set the author proved that modifying the definition of Keldych operators, the same result holds for normalized Dirichlet solutions [11]. In a resolutive compactification, while we can not expect too much we obtain several results which are of some interest.

Throughout this section, when we consider a simplicial cone $\mathcal{S}$, μ_x always denotes the unique minimal measure of $\mathcal{M}_x$.

A *Keldych operator* $\mathcal{L}$ is defined to be a mapping of $\boldsymbol{C}(\Delta)$ into the space of harmonic functions on X such that

1) for every $a \in X$, $\mathcal{L}_f(a)$ defines a positive Radon measure ν_a on Δ,
2) $\mathcal{L}_s \leq s$ on X for every $s \in \mathcal{S}$.

It is clear that the Dirichlet solutions H_f form a Keldych operator, and

$H_f(a)=\lambda_a(f)$, where λ_a is the harmonic measure. And if X^* is simplicial then $\mathcal{L}_f(a)=\mu_a(f)$ is a Keldych operator, for, by Prop. 2.1, $\hat{t}(x)=\mu_x(t)$ is affine for every $t\in-\mathcal{S}$, hence $\mu_a(t)=\hat{t}(a)=\lambda_a(\hat{t})$ is harmonic on X.

Lemma 5.1. [13] *Let ν_a be a measure associated with a Keldych operator $\mathcal{L}$. Then for every $a\in X$, $\nu_a<\lambda_a$. And, if $\mathcal{S}$ is simplicial, $\mu_a<\nu_a$.*

Proof. $\mathcal{L}_s\leq s$ for every $s\in\mathcal{S}$. This implies $\nu_a\in\mathcal{M}_a$. Since $\mathcal{L}_s$ is harmonic, bounded above and $\overline{\lim}\,\mathcal{L}_s\leq s$ on Δ, we have $\mathcal{L}_s\leq H_s$. Thus $\nu_a<\lambda_a$.

We denote by $\mathcal{L}_{\text{reg}}$ the set of $x\in\Delta$ such that

$$\lim_{a\to x}\mathcal{L}_f(a)=f(x) \qquad \text{for every } f\in \boldsymbol{C}(\Delta)\,.$$

Propositton 5.2. [13] *If $\mathcal{S}$ separates points of Δ,*

$$Ch_{\mathcal{S}}\,X^*\subset\mathcal{L}_{\text{reg}}\subset\Delta_{\text{reg}}\,.$$

Proof. As in the proof of Lemma 4.2, for every $s\in\mathcal{S}$ we may construct a measure $\nu\in\mathcal{M}_x$ such that $\nu(s)=\underline{\lim}_x\mathcal{L}_s$. Thus, if $x\in Ch_{\mathcal{S}}X^*$ then $s(x)=\lim_x\mathcal{L}_s$ for every $s\in\mathcal{S}$ and, since $\overline{\mathcal{S}-\mathcal{S}}=\boldsymbol{C}(\Delta)$, $f(x)=\lim_x\mathcal{L}_f$ for every $f\in\boldsymbol{C}(\Delta)$; that is, $x\in\mathcal{L}_{\text{reg}}$.

Next, we suppose that $x\in\mathcal{L}_{\text{reg}}$. Then, by Lemma 5.1,

$$s(x)=\lim_{a\to x}\nu_a(s)\leq\underline{\lim}_{a\to x}\lambda_a(s)=\underline{\lim}_{a\to x}H_s(a)\leq s(x)\,,$$

which means $s(x)=\underline{\lim}_{a\to x}H_s(a)$ and similarly, $s(x)=\overline{\lim}_{a\to x}H_s(a)$. Therefore we conclude that $x\in\Delta_{\text{reg}}$.

A set $E\subset\Delta$ is called *polar* if there exists a non-negative superharmonic function v on X such that $\lim_{a\to x}v(a)=+\infty$ for every $x\in E$. E is polar if and only if $\bar{H}_{\chi_E}=0$, where χ_E is the characteristic function of E.

A resolutive compactification is called to be *of type K* if it has a unique Keldych operator, i.e., $\mathcal{L}_f=H_f$.

In the sequel, we assume that $\mathcal{S}$ separates points of Δ.

Proposition 5.3. *If $\Delta\backslash Ch_{\mathcal{S}}X^*$ is polar then X^* is of type K.*

Proof. Suppose that $\mathcal{L}$ is a Keldych operator. Let v be a non-negative superharmonic function on X such that $\lim_x v=+\infty$ for every $x\in\Delta\backslash Ch_{\mathcal{S}}X^*$. For $f\in\boldsymbol{C}(\Delta)$ and $\varepsilon>0$, we consider the superharmonic function $w=H_f-\mathcal{L}_f+\varepsilon v$. It is readily derived that $\underline{\lim}\,w\geq0$ on Δ; in fact, for $x\in Ch_{\mathcal{S}}X^*$ $\lim_x H_f=\lim_x\mathcal{L}_f=f(x)$, and for $x\in\Delta\backslash Ch_{\mathcal{S}}X^*$ $\lim_x w=+\infty$, since $|H_f|$ and $|\mathcal{L}_f|$ are dominated by some function in $\mathcal{S}$. Hence, $H_f-\mathcal{L}_f+\varepsilon v\geq0$ and $H_f\geq\mathcal{L}_f$. Similarly we have $\mathcal{L}_f\geq H_f$.

Theorem 5.4. *Let*

i) $\Delta\backslash Ch_{\mathcal{S}}X^*$ is negligible, i.e., $\lambda_a(\Delta\backslash Ch_{\mathcal{S}}X^*)=0$ for every $a\in X$,

ii) $\Delta\backslash Ch_{\mathcal{S}}X^*$ is polar,

iii) X^* is of type K,

iv) $\lambda_a(\hat{t})=\lambda_a(t)$ for every $a\in X$ and $t\in-\mathcal{S}$.

Then we have i)$\Rightarrow$ii)$\Rightarrow$iii). *Further, if X^* is metrizable and simplicial then* iii)$\Rightarrow$iv)$\Rightarrow$i).

Proof. From the definition of polar sets and Prop. 5.3, i)$\Leftrightarrow$ii)$\Rightarrow$iii) are derived immediately. Suppose that X^* is metrizable and simplicial. Then iii) $\Rightarrow$iv); for in this case we have $\lambda_a=\mu_a$ and $\lambda_a(\hat{t})=H_{\hat{t}}(a)=\mu_a(t)=\lambda_a(t)$. To prove the last part of the theorem, we consider a countable family $\{s_n\}$ of $\mathcal{S}$ which is total in $\boldsymbol{C}(\Delta)$. Then,

$$Ch^{\mathcal{S}}X^* = \bigcap_{n=1}^{\infty}\{x\in\Delta;\, s_n(x) = -(\widehat{-s_n})\,(x)\}\ .$$

Hence, if $\Delta\backslash Ch_{\mathcal{S}}X^*$ is not negligible, then there exists $a\in X$ and $s_n\in\mathcal{S}$ such that $\lambda_a(\{x\in\Delta;\, s_n(x)>-(\widehat{-s_n})\,(x)\})>0$. Therefore, $\lambda_a(-s_n)<\lambda_a(\widehat{-s_n})$.

Corollary 5.5. *Let X^* be metrizable and simplicial compactification of type K, and $X^*\leq X^{**}\leq X^{Q(X^*)}$. Then X^{**} is of type K.*

This is an immediate consequence of Th.4.5 and Th.5.4.

REMARK. The set of Keldych operators forms a convex set. $\lambda_a(f)=H_f(a)$ is an extreme point of this set and, if X^* is simplicial, $\mu_a(f)$ is also extreme.

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