

## IMPULSIVE CONTROL OF SYMMETRIC MARKOV PROCESSES AND QUASI-VARIATIONAL INEQUALITIES

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By introducing the notion of impulsive control of a diffusion process A. Bensoussan—J.L. Lions ([1]) showed that if the solution of a quasi-variational inequality has sufficient regularity (twice differentiability and continuity), it turns out to be a pay-off function. Furthermore they constructed the optimal strategy out of the solution. But the regularity problems remained open. On the other hand M. Robin ([7]) has set up an impulsive control problem of a general Markov process with a Feller transition semi-group and has constructed the optimal strategy out of the pay-off function which was characterized however in terms of the semi-group rather than the generator of the basic Markov process. As for the characterization by means of the quasi-variational inequality the regularity of the solution was still assumed in order to identify the solution with the pay-off function like that of Bensoussan-Lions. Regularity problems of elliptic or parabolic quasi-variational inequalities have been studied by L.A. Cafarelli—A. Friedman and others (cf. [2], [5]) under the condition that the diffusion and drift coefficients have sufficient regularity. Cafarelli-Friedman's work, combined with Robin's, establishes completely the relationship between impulsive control problems and quasi-variational inequalities with respect to nice diffusion processes.

Our objective is to extend this relationship to general symmetric Markov processes associated with regular Dirichlet spaces. We prove that the pay-off function is characterized by the weak (maximum) solution of the quasi-variational inequality defined on the Dirichlet space (Theorem 2 in §2). Since we assume only that the Dirichlet space is regular, Theorem 2 establishes the relationship for a wide class of processes. It applies as well to symmetric diffusion process with measurable coefficients and symmetric Markov processes with non local generators (cf. [4]).

Our approach is more potential theoretic than others and accordingly the regularity questions can be dispensed with. Indeed we use the potential theory of Dirichlet spaces and Markov processes developed in [4]. The same method has been used in [6] to establish the relationship between variational inequalities

and optimal stoppings and in [8] to include stopping games.

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### 1. Quasi-variational inequalities on regular Dirichlet spaces

Let  $m(dx)$  be a non-negative everywhere dense Radon measure on a locally compact Hausdorff space  $S$  with countable base. Suppose that  $(\mathcal{F}, \mathcal{E})$  is a regular Dirichlet space relative to  $L^2(dm)$ :

- i)  $\mathcal{F}$  is a dense linear subspace of  $L^2(dm)$ ,
- ii)  $\mathcal{E}$  is a symmetric bilinear form on  $\mathcal{F} \times \mathcal{F}$ ,
- iii)  $\mathcal{F}$  is closed with respect to  $\mathcal{E}_1$ -norm, where  $\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)$ ,  $(u, v)$  denoting inner product of  $L^2(dm)$ ,
- iv) unit contraction operates, that is, if  $v = (0 \vee u) \wedge 1$ ,  $u \in \mathcal{F}$ , then  $v \in \mathcal{F}$  and  $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$ ,
- v)  $\mathcal{F} \cap C_0(S)$  is dense in  $\mathcal{F}$  with  $\mathcal{E}_1$ -norm as well as in  $C_0(S)$  with uniform norm,  $C_0(S)$  denoting the space of all continuous functions on  $S$  with compact support.

DEFINITION 1.1. The capacity of a subset of  $S$  is defined as follows: for open set  $A \subset S$

$$\text{Cap}(A) = \begin{cases} \inf \{ \mathcal{E}_1(u, u); u \in L_A \} & \text{if } L_A \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

where  $L_A = \{u \in \mathcal{F}; u \geq 1 \text{ m-a.e. on } A\}$  and for general set  $B \subset S$

$$\text{Cap}(B) = \inf \{ \text{Cap}(A); B \subset A, A \text{ is open} \}.$$

DEFINITION 1.2. A subset  $B$  of  $S$  with  $\text{Cap}(B) = 0$  is called almost polar and "quasi-everywhere" or "q.e." will mean "except for an almost polar set".

Let  $S_\Delta = S \cup \Delta$  be the one point compactification of  $S$ . When  $S$  is already compact,  $\Delta$  is regarded as an isolated point. Any function on  $S$  is extended to a function on  $S \cup \Delta$  by setting  $f(\Delta) = 0$ .

DEFINITION 1.3. A function  $f$  defined q.e. on  $S$  is said to be quasi-continuous (resp. quasi-continuous in the restricted sense) provided that for each  $\varepsilon > 0$  there exists an open set  $G \subset S$  such that  $\text{Cap}(G) < \varepsilon$  and  $f|_{S-G}$  (resp.  $f|_{S_\Delta-G}$ ) is continuous.

It is known that each  $u \in \mathcal{F}$  admits a quasi-continuous modification  $\tilde{u}$  in the restricted sense in the case that  $(\mathcal{F}, \mathcal{E})$  is a regular Dirichlet space:  $u = \tilde{u}$  m-a.e. and  $\tilde{u}$  is quasi-continuous in the restricted sense (cf. [4]). Hereafter  $\tilde{u}$  denotes a quasi-continuous modification of  $u \in \mathcal{F}$ .  $\tilde{\mathcal{F}}$  denotes the subset of  $\mathcal{F}$  consisting

of all quasi-continuous functions in the restricted sense.

Let  $\nu(dx)$  be a given non-negative Radon measure of finite energy integral, that is, there exists for each  $\alpha > 0$  a unique function  $U_\alpha \nu \in \mathcal{F}$  such that

$$(1.1) \quad \mathcal{E}_\alpha(U_\alpha \nu, v) = \int_S v(x) \nu(dx) \quad \text{for each } v \in \mathcal{F} \cap C_0(S).$$

Suppose that  $M$  is an operator defined on  $\tilde{\mathcal{F}}$  such that

$$(M.1) \quad Mu \text{ is a Borel function for any } u \in \tilde{\mathcal{F}},$$

$$(M.2) \quad Mu_1(x) \leq Mu_2(x) \quad \forall x \text{ if } u_1(x) \leq u_2(x) \text{ q.e.},$$

$$(M.3) \quad Mu(x) \geq 0 \quad \forall x \text{ if } u(x) \geq 0 \text{ q.e. and}$$

$$(M.4) \quad \lim_{n \rightarrow \infty} Mu_n(x) = Mu(x) \quad \forall x \text{ if } u_n(x) \downarrow u(x) \text{ q.e.}$$

We consider the following quasi-variational inequality:

$$(1.2) \quad \begin{cases} \mathcal{E}_\alpha(u, v-u) \geq \langle v, \tilde{v}-\tilde{u} \rangle \quad \forall \tilde{v} \leq M\tilde{u} \text{ q.e.} \\ \tilde{u} \leq M\tilde{u} \text{ q.e.} \end{cases}$$

**Theorem 1.** *The above quasi-variational inequality (QVI) (1.2) has the maximum solution.*

Put  $u_0 = U_\alpha \nu$  and  $V_1 = \{v \in \mathcal{F}; \tilde{v} \leq M\tilde{u}_0 \text{ q.e.}\}$ , then we have the unique solution of the following variational inequality (VI) (1.3):

$$(1.3) \quad \begin{cases} \mathcal{E}_\alpha(u, v-u) \geq \langle v, \tilde{v}-\tilde{u} \rangle \quad \forall v \in V_1 \\ u \in V_1, \end{cases}$$

because (1.3) is equivalent to

$$(1.4) \quad \begin{cases} \mathcal{E}_\alpha(u - U_\alpha \nu, u - U_\alpha \nu) \leq \mathcal{E}_\alpha(v - U_\alpha \nu, v - U_\alpha \nu) \quad \forall v \in V_1 \\ u \in V_1 \end{cases}$$

and  $V_1$  is the closed convex subset of Hilbert space  $(\mathcal{F}, \mathcal{E}_\alpha)$ . Let us denote the solution by  $u_1$ . In the same way we can inductively take the solution  $u_n$  of the VI:

$$(1.5) \quad \begin{cases} \mathcal{E}_\alpha(u, v-u) \geq \langle v, \tilde{v}-\tilde{u} \rangle \quad \forall v \in V_n \\ u \in V_n, \end{cases} \quad V_n = \{v \in \mathcal{F}; \tilde{v} \leq M\tilde{u}_{n-1} \text{ q.e.}\} \quad \text{for each } n.$$

At first we note the properties of the solution  $u_n$  of the VI (1.5).

**Lemma.** *The above  $u_n$  has the following properties*

- (i)  $U_\alpha \nu - u_n$  is an  $\alpha$ -almost excessive function and the unique element which minimizes its  $\alpha$ -energy integral in the closed convex subset  $U_\alpha \nu - V_n$  of  $(\mathcal{F}, \mathcal{E}_\alpha)$ :

$$(1.6) \quad \mathcal{E}_\alpha(U_\alpha v - u_n, U_\alpha v - u_n) \leq \mathcal{E}_\alpha(U_\alpha v - v, U_\alpha v - v) \quad \forall v \in V_n$$

for each  $n$ ,

- (ii) (1.7)  $u_n \leq u_{n-1}$   $m$ -a.e. for each  $n$ ,
- (iii) (1.8)  $u_n \geq 0$   $m$ -a.e. for each  $n$  and
- (iv)  $\{u_n\}$  is a  $\mathcal{E}_\alpha$ -Cauchy sequence.

Proof. (i) Since  $u_n$  is the solution of (1.5) it satisfies the following inequality:

$$(1.9) \quad \mathcal{E}_\alpha(u_n - U_\alpha v, v - u_n) \geq 0 \quad \forall v \in V_n.$$

If  $w \geq 0, w \in \mathcal{F}$ , then  $u_n - w \in V_n$ . Therefore it holds that

$$(1.10) \quad \mathcal{E}_\alpha(U_\alpha v - u_n, w) \geq 0 \quad \forall w \geq 0 \text{ } m\text{-a.e., } w \in \mathcal{F}.$$

that is  $U_\alpha v - u_n$  is  $\alpha$ -almost excessive because (1.10) is equivalent to

$$(1.11) \quad u_n \geq 0, e^{-\alpha t} T_t u_n \leq u_n \text{ } m\text{-a.e., } \forall t > 0.$$

Here  $T_t$  is the  $L^2$ -Markov semigroup corresponding to Dirichlet form  $\mathcal{E}$  (cf. [4]). The latter half of (i) follows directly if  $V_1$  in (1.4) is replaced by  $V_n$ .

(ii) Inequality (1.7) with  $n=1$  is obvious because  $U_\alpha v - u_1$  is  $\alpha$ -almost excessive and  $U_\alpha v = u_0$ . Assume that it holds for  $n$ , then  $M\tilde{u}_n \leq M\tilde{u}_{n-1}$   $\forall x$ . Therefore  $\tilde{u}_{n+1} \leq M\tilde{u}_{n-1}$  q.e.. Since  $\tilde{u}_n \leq M\tilde{u}_{n-1}$  q.e. by definition we have  $\tilde{u}_n \vee \tilde{u}_{n+1} \leq M\tilde{u}_{n-1}$  q.e.. On the other hand  $U_\alpha v - u_n \vee u_{n+1} = (U_\alpha v - u_n) \wedge (U_\alpha v - u_{n+1})$  is  $\alpha$ -almost excessive because both  $U_\alpha v - u_{n+1}$  and  $U_\alpha v - u_n$  are  $\alpha$ -almost excessive. So it follows that

$$(1.12) \quad \mathcal{E}_\alpha(U_\alpha v - u_n \vee u_{n+1}, U_\alpha v - u_n \vee u_{n+1}) \leq \mathcal{E}_\alpha(U_\alpha v - u_n, U_\alpha v - u_n)$$

from  $U_\alpha v - u_n \geq U_\alpha v - u_n \vee u_{n+1}$ . By (i) of present Lemma we conclude that  $u_n \vee u_{n+1} = u_n$ , that is,  $u_{n+1} \leq u_n$   $m$ -a.e..

(iii) Since  $\widetilde{U_\alpha v} \geq 0$  q.e. we have  $M\tilde{u}_0 \geq 0$   $\forall x$ . Furthermore  $\tilde{u}_1 \leq M\tilde{u}_0$  q.e. by definition, so we have  $\tilde{u}_1 \vee 0 \leq M\tilde{u}_0$  q.e.. Both  $U_\alpha v - u_1$  and  $U_\alpha v$  being  $\alpha$ -almost excessive,  $U_\alpha v - u_1 \vee 0 = (U_\alpha v - u_1) \wedge U_\alpha v$  is  $\alpha$ -almost excessive. Therefore it follows that

$$(1.13) \quad \mathcal{E}_\alpha(U_\alpha v - u_1 \vee 0, U_\alpha v - u_1 \vee 0) \leq \mathcal{E}_\alpha(U_\alpha v - u_1, U_\alpha v - u_1)$$

from  $U_\alpha v - u_1 \vee 0 \leq U_\alpha v - u_1$   $m$ -a.e.. It implies that  $u_1 \vee 0 = u_1$ , that is  $u_1 \geq 0$   $m$ -a.e.. We can inductively show  $u_n \geq 0$   $m$ -a.e. by similar argument.

(iv) Since  $U_\alpha v - u_n \leq U_\alpha v - u_m$   $m$ -a.e.,  $n \leq m$ , and  $U_\alpha v - u_n \leq U_\alpha v$   $m$ -a.e. for each  $n$  by (ii) and (iii) it holds that

$$(1.14) \quad \mathcal{E}_\alpha(U_\alpha v - u_n, U_\alpha v - u_n) \leq \mathcal{E}_\alpha(U_\alpha v - u_m, U_\alpha v - u_m) \leq \mathcal{E}_\alpha(U_\alpha v, U_\alpha v)$$

for each  $n \leq m$ . Therefore  $\mathcal{E}_\alpha(U_\alpha v - u_n, U_\alpha v - u_n)$  monotonously increases to a

finite number. Since  $w_n = U_\alpha v - u_n$  is  $\alpha$ -almost excessive

$$\begin{aligned} 0 &\leq \mathcal{E}_\alpha(w_n - w_m, w_n - w_m) = \mathcal{E}_\alpha(w_n, w_n) - 2\mathcal{E}_\alpha(w_n, w_m) + \mathcal{E}_\alpha(w_m, w_m) \\ &\leq \mathcal{E}_\alpha(w_m, w_m) - \mathcal{E}_\alpha(w_n, w_n), \quad n \leq m. \end{aligned}$$

Hence  $w_n$  is a  $\mathcal{E}_\alpha$ -Cauchy sequence, so  $u_n$  is also.

Proof of Theorem 1. As the result of (ii) and (iv) of Lemma there exists  $u$  such that  $\mathcal{E}_\alpha(u_n - u, u_n - u) \rightarrow 0$  and  $\tilde{u}_n \downarrow \tilde{u}$  q.e.. We can now prove that this function  $u$  is a solution of the quasi-variational inequality (1.2). We at first note that it follows that

$$\mathcal{E}_\alpha(U_\alpha v - u_n, U_\alpha v - u_n) \leq \mathcal{E}_\alpha(U_\alpha v - v, U_\alpha v - v) \quad \forall \tilde{v} \leq M\tilde{u} = \lim_{n \rightarrow \infty} M\tilde{u}_n \text{ q.e.}$$

from (1.6) because  $\tilde{u}_n \downarrow \tilde{u}$  q.e. implies  $M\tilde{u}_n \downarrow M\tilde{u}$ . Therefore it holds that

$$(1.15) \quad \mathcal{E}_\alpha(U_\alpha v - u, U_\alpha v - u) \leq \mathcal{E}_\alpha(U_\alpha v - v, U_\alpha v - v), \quad \forall \tilde{v} \leq M\tilde{u} \text{ q.e.}$$

since  $\mathcal{E}_\alpha(u_n - u, u_n - u) \rightarrow 0$ . On the other hand, since  $\tilde{u} \leq \tilde{u}_n \leq M\tilde{u}_{n-1}$  q.e. for each  $n$  we have

$$(1.16) \quad \tilde{u} \leq \lim_{n \rightarrow \infty} M\tilde{u}_n = M\tilde{u} \text{ q.e..}$$

(1.15) with (1.16) is equivalent to the QVI (1.2).

Now we are going to prove that the above solution  $u$  of QVI (1.2) is the maximum. Take another solution  $w$  of the QVI

$$\begin{cases} \mathcal{E}_\alpha(w, v - w) \geq \langle v, \tilde{v} - \tilde{w} \rangle \quad \forall \tilde{v} \leq M\tilde{w} \text{ q.e.} \\ \tilde{w} \leq M\tilde{w} \text{ q.e.} \end{cases}$$

In the same way as Lemma we can see  $U_\alpha v - w$  is  $\alpha$ -excessive, so  $U_\alpha v \geq w$ . Therefore  $M\widetilde{U_\alpha v} \geq \tilde{w}$  q.e.. That is  $w \in V_1$ . Since  $U_\alpha v - u_1 \vee w = (U_\alpha v - u_1) \wedge (U_\alpha v - w)$  is  $\alpha$ -almost excessive and  $U_\alpha v - u_1 \vee w \leq U_\alpha v - u_1$  it holds that

$$(1.17) \quad \mathcal{E}_\alpha(U_\alpha v - u_1 \vee w, U_\alpha v - u_1 \vee w) \leq \mathcal{E}_\alpha(U_\alpha v - u_1, U_\alpha v - u_1).$$

Hence we have  $u_1 \geq w$  by similar argument as (iii) of Lemma. In the same way we can inductively see  $u_n \geq w$  for each  $n$ , which implies  $u \geq w$ .

## 2. Impulsive control of symmetric Markov processes

Let  $X = \{\Omega, \mathcal{B}, \mathcal{B}_t, P_x, X_t, \theta_t\}$  be an  $m$ -symmetric standard Markov process of function space type with the state space  $S$ . We assume that its Dirichlet space  $(\mathcal{F}, \mathcal{E})$  is regular. We are now going to repeat Robin's construction of controlled process (cf. [7]) with a little modification and set up an impulsive control problem.

Consider the infinite product space  $\Omega_\infty = \Omega \times \Omega \times \Omega \times \dots$  and define its sub- $\sigma$ fields by

$$(2.1) \quad \mathcal{B}_i^n = \Pi_n^{-1}(\mathcal{B}_i)^{\otimes n}$$

where  $\Pi_n$  is the projection from  $\Omega_\infty$  to the  $n$ -th product  $(\Omega)^n$ .  $\mathcal{B}^n$  is similarly defined. For  $\underline{\omega} = (\omega_1, \omega_2, \dots) \in \Omega_\infty$ , we let

$$(2.2) \quad (\theta_{n,t}\underline{\omega})(s) = (\theta_t\omega_1(s), \dots, \theta_t\omega_n(s)) \\ = (\omega_1(t+s), \dots, \omega_n(t+s)).$$

We note that, if  $\sigma(\underline{\omega})$  is a  $\mathcal{B}^n$ -measurable function on  $\Omega_\infty$ , then  $\sigma(\underline{\omega}) = \bar{\sigma}(\omega_1, \omega_2, \dots, \omega_n)$ ,  $\bar{\sigma}$  being a  $(\mathcal{B})^{\otimes n}$ -measurable function on  $(\Omega)^n$ . Such an identification of  $\sigma$  and  $\bar{\sigma}$  will be made below without mentioning explicitly. It is further noticed that  $P_x$  for each  $x \in S$  can be regarded as a probability measure on  $(\Omega_\infty, \mathcal{B}^1)$ .

A family of subsets  $\{\Gamma_x\}_{x \in S}$  of  $S$  is called *admissible* if the following condition ( $\Gamma$ ) is satisfied:

$$(\Gamma) \quad \text{if } x_n \rightarrow x, x_n, x \in S \text{ and } y_n \in \Gamma_{x_n}, \text{ then there exist } y \in \Gamma_x \text{ and } \{y_{n_k}\} \subset \{y_n\} \\ \text{such that } y_{n_k} \rightarrow y.$$

A sequence  $v = \{(\tau_i, \xi_i)_{i=1}^\infty\}$  of the pairs of random variables  $\tau_i$  and  $\xi_i$  on  $\Omega_\infty$  is called an *admissible control* if the following conditions (v.1)~(v.3) are satisfied for a given admissible  $\{\Gamma_x\}$ :

- (v.1)  $\tau_i$  is a  $\mathcal{B}_i^i$ -stopping time such that  $\tau_i \leq \tau_{i+1}$  for each  $i$  and  $\lim_{i \rightarrow \infty} \tau_i = \infty$
- (v.2)  $\xi_i$  is  $\Gamma_{x_{\tau_i}(\omega_i)}$ -valued  $\mathcal{B}_i^i$ -measurable random variable for each  $i$
- (v.3) for each  $N$  with  $\text{Cap}(N) = 0$  there exists  $\tilde{N} \supset N$  with  $\text{Cap}(\tilde{N}) = 0$  such that  $P_x^i(\xi_i \in \tilde{N}) = 0, x \in S - \tilde{N}$  for each  $i$ , where  $P_x^i$  is a probability measure on  $(\Omega_\infty, \mathcal{B}^i)$  specified below.

The set of all admissible controls are denoted by  $\underline{V}$ . Let us define, for  $y \in S$ , an element  $\delta_y \in \Omega$  by

$$(2.3) \quad \delta_y(t) = y \quad \forall t \geq 0$$

and denote by  $\varepsilon_{\delta_y}$  the probability measure on  $(\Omega, \mathcal{B})$  which is concentrated on  $\delta_y$ .

For a given  $v = \{(\tau_i, \xi_i)_{i=1}^\infty\} \in \underline{V}$ , we are interested in the process  $X_t(\omega_1)$  governed by  $P_x$  up to time  $\tau_1(\omega_1)$ .  $X_t(\omega_1)$  is stopped at time  $\tau_1$  and then our interest is switched to the process  $X_{\tau_1(\omega_1)+t}(\omega_2), t \geq 0$ , governed by  $P_{\xi_1(\omega_1)}$  up to time  $\tau_2(\omega_1, \omega_2)$  and so forth. To formulate such a process, we construct probability measures  $P_x^n$  on  $(\Omega_\infty, \mathcal{B}^n), n = 1, 2, \dots$ , as follows:

First let

$$P_x^1 = P_x \quad \text{on } (\Omega_\infty, \mathcal{B}^1)$$

We can construct a probability measure  $P_x^2$  on  $(\Omega_\infty, \mathcal{B}^2)$  such that

$$(2.4) \quad \begin{cases} P_x^2 = P_x^1 & \text{on } \mathcal{B}_{\tau_1}^1(\subset \mathcal{B}^1) \\ P_x^2(\theta_{2,\tau_1}^{-1}B | \mathcal{B}_{\tau_1}^1) = \varepsilon_{\delta_{X_{\tau_1}}} \otimes P_{\xi_1}(B) & P_x^1\text{-a.s. on } \{\tau_1 < +\infty\} \end{cases}$$

for each  $B \in \mathcal{B}^2$ . Then the process  $X_{\tau_1+t}(\omega_2)$ ,  $t \geq 0$ , is Markovian with respect to  $(\mathcal{B}_{\tau_1+t}^2, P_x^2)$  under the condition  $\mathcal{B}_{\tau_1}^1$ . We define the probability measure  $P_x^{n+1}$  on  $(\Omega_\infty, \mathcal{B}^{n+1})$  inductively by

$$(2.5) \quad \begin{cases} P_x^{n+1} = P_x^n & \text{on } \mathcal{B}_{\tau_n}^n(\subset \mathcal{B}^n) \\ P_x^{n+1}(\theta_{n+1,\tau_n}^{-1}B | \mathcal{B}_{\tau_n}^n) = \varepsilon_{\delta_{X_{\tau_1}(\omega_1)}} \otimes \cdots \otimes \varepsilon_{\delta_{X_{\tau_n}(\omega_n)}} \otimes P_{\xi_n}(B) & P_x^n\text{-a.s. on } \{\tau_n < +\infty\} \end{cases}$$

where  $B \in \mathcal{B}^{n+1}$ .

We are now in a position to formulate our main theorem. Consider the Dirichlet space  $(\mathcal{F}, \mathcal{E})$  associated with the process  $X$ . We suppose that a non-negative Radon measure  $\nu(dx)$  of finite energy integral and non-negative continuous function  $k(x, \xi)$ ,  $x, \xi \in S$ , are given which are to define a pay-off function. It is known that a non-negative continuous additive functional  $A_t(\omega)$  on  $X$  corresponds to  $\nu(dx)$ :

$$(2.6) \quad E_x \left[ \int_0^\infty e^{-\alpha s} dA_s \right] = U_\alpha \nu \quad \text{q.e.}$$

(cf. [4]). Let for  $\underline{\omega} = (\omega_1, \omega_2, \dots) \in \Omega_\infty$

$$(2.7) \quad \underline{A}_t = \begin{cases} A_t(\omega_1), & 0 \leq t \leq \tau_1 \\ \underline{A}_{\tau_1} + A_{t-\tau_1}(\theta_{\tau_1}\omega_2), & \tau_1 < t \leq \tau_2 \\ \underline{A}_{\tau_{n-1}} + A_{t-\tau_{n-1}}(\theta_{\tau_{n-1}}\omega_n), & \tau_{n-1} < t \leq \tau_n \end{cases}$$

and

$$(2.8) \quad y_t(\underline{\omega}) = X_t(\omega_{k+1}) \quad \text{if } t \in [\tau_k, \tau_{k+1}).$$

We can now define the pay-off function  $u^*(x)$  by

$$(2.9) \quad u^*(x) = \inf_{v \in \underline{V}} J_x(v)$$

$$(2.10) \quad J_x(v) = \lim_{n \rightarrow \infty} J_x^n(v)$$

$$(2.11) \quad J_x^n(v) = E_x^n \left[ \int_0^{\tau_n} e^{-\alpha t} d\underline{A}_t + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i) \right].$$

We then introduce the operator  $M$  by

$$(2.12) \quad M\phi(x) = q\text{-essinf}_{y \in \Gamma_x} \{ \phi(y) + k(x, y) \} \\ \equiv \sup \{ c : \text{Cap} \{ y \in \Gamma_x; \phi(y) + k(x, y) < c \} = 0 \}$$

for  $\phi \in \mathcal{F}$ . The fact that this operator  $M$  satisfies (M.1)~(M.4) will be shown later (§3). Recall that Theorem 1 then guarantees the existence of the maximum

solution of the QVI (1.2) associated with the present data  $(\mathcal{F}, \mathcal{E}), \nu$  and  $M$ .

**Theorem 2.** *The pay-off function  $u^*(x)$  defined by (2.9) is a quasi-continuous modification of the maximum solution  $u$  of the QVI (1.2) corresponding to the above  $(\mathcal{F}, \mathcal{E}), \nu$  and  $M$ .*

REMARK. We note that if  $\nu(dx)=f(x)dm$  with a Borel function  $f$  in  $L^2(dm)$ ,  $J_x^n(v)$  is written as

$$(2.13) \quad J_x^n(v) = E_x^n \left[ \int_0^{\tau_n} e^{-\alpha t} f(y_t) dt + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_n}(\omega_n), \xi_n) \right]$$

In the next section we study the operator  $M$  defined by (2.12). All assumptions and notations in section 2 are assumed through the following sections.

### 3. Operator $M$

DEFINITION 3.1. A sequence  $\{F_k\}$  of closed sets such that  $F_k \uparrow$  and  $\text{Cap}(S-F_k) \downarrow 0, k \rightarrow \infty$  is called a nest on  $S$ . A nest  $\{F_k\}$  is said to be  $(m)$ -regular if for each  $k$   $m(U(x) \cap F_k) \neq 0$  for any  $x \in F_k$  and any open neighborhood  $U(x)$  of  $x$ .

Let  $Q$  be a countable family of quasi-continuous function in the restricted sense on  $S$ . Then it is known that there exists a regular nest  $\{F_k\}$  on  $S$  such that  $u|_{F_k \cup \Delta}$  is continuous for each  $k$  for any function  $u \in Q$ .

**Lemma 3.1.** *For any function  $\phi \in \tilde{\mathcal{F}}$   $M\phi$  is a Borel function and has the following representation :*

$$(3.1) \quad M\phi(x) = \lim_{n \rightarrow \infty} \inf_{y \in \Gamma_x^n \cap F_n} \{ \phi(y) + k(x, y) \}$$

where  $\{F_n\}$  is a regular nest and  $\Gamma_x^n$  is a subset of  $S$  which satisfies  $(\Gamma)$ .

Proof. It holds that by definition

$$\text{Cap} \{ y \in \Gamma_x; \phi(y) + k(x, y) < M\phi(x) - \varepsilon \} = 0$$

for any  $\varepsilon > 0$ . Take a regular nest  $\{F_n\}$  such that  $\phi|_{F_n \cup \Delta}$  is continuous for each  $n$ . Put

$$N_x^n = \{ y \in \Gamma_x; \text{there exists a open neighborhood } U_y \text{ such that } \text{Cap}(U_y \cap F_n \cap \Gamma_x) = 0 \}$$

and define

$$\Gamma_x^n = \Gamma_x \cap \left( \bigcup_{y \in N_x^n} U_y \right)^c$$

by above  $U_y$ . Then it is obvious that  $\Gamma_x^n$  satisfies  $(\Gamma)$  because  $\left( \bigcup_{y \in N_x^n} U_y \right)^c$  is closed. Since  $\phi(\cdot)$  and  $k(x, \cdot)$  are continuous on  $\Gamma_x^n \cap F_n$  it follows that



$$\phi(y) + k(x, y) \geq M\phi(x) - \varepsilon \quad \forall y \in \Gamma_x^n \cap F_n$$

from

$$\text{Cap} \{y \in \Gamma_x^n \cap F_n; \phi(y) + k(x, y) < M\phi(x) - \varepsilon\} = 0$$

Therefore

$$\liminf_{n \rightarrow \infty} \inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x, y)\} \geq M\phi(x).$$

In order to get converse inequality put

$$c = \liminf_{n \rightarrow \infty} \inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x, y)\},$$

then

$$\begin{aligned} \text{Cap} \{y \in \Gamma_x; \phi(y) + k(x, y) < c\} &= \text{Cap} \{\Gamma_x \cap (\bigcup_n F_n); \phi(y) + k(x, y) < c\} \\ &\leq \sum_{n=1}^{\infty} \text{Cap} \{y \in \Gamma_x \cap F_n; \phi(y) + k(x, y) < c\} \\ &= \sum_{n=1}^{\infty} \text{Cap} \{y \in \Gamma_x^n \cap F_n; \phi(y) + k(x, y) < c\}. \end{aligned}$$

Hence  $c \leq M\phi(x)$ . Now (3.1) has been proved. On the other hand, since  $\inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x, y)\}$  is a lower semi-continuous function according to the following lemma, we have the conclusion that  $M\phi(x)$  is a Borel function.

**Lemma 3.2.** For any  $\phi \in \tilde{\mathcal{F}}$

$$M_n\phi(x) = \inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + s(x, y)\}$$

is a lower semi-continuous function and has a measurable selection for each  $n$ .

This lemma is a trivial modification of Theorem A in §5, Chap. 2 of [3]. Because  $\Gamma_x^n \cap F_n$  also satisfies  $(\Gamma)$  and  $\phi(\cdot)$  and  $k(x, \cdot)$  are continuous on  $F_n$ .

**Lemma 3.3.** The operator  $M$  defined by (2.12) satisfies  $(M.1) \sim (M.4)$ .

Proof.  $(M.1)$  has been proved in Lemma 3.1.  $(M.2)$  and  $(M.3)$  are obvious. As to  $(M.4)$  it is easily seen that

$$\lim_{n \rightarrow \infty} Mu_n(x) \geq Mu(x).$$

On the other hand

$$Mu_n(x) \leq u_n(y) + k(x, y) \quad \forall y \in \Gamma_x^m \cap F_m,$$

so we have

$$\lim_{n \rightarrow \infty} Mu_n(x) \leq u(y) + k(x, y) \quad \forall y \in \Gamma_x^m \cap F_m$$

for each  $m$ . Then it holds that

$$\begin{aligned} \lim_{n \rightarrow \infty} Mu_n(x) &\leq \lim_{m \rightarrow \infty} \inf_{\gamma \in \Gamma_x^m \cup \mathcal{F}_m} \{u(\gamma) + k(x, \gamma)\} \\ &= Mu(x). \end{aligned}$$

#### 4. Optimal stoppings of Markov processes

We prepare for the proof of Theorem 2 some lemmas on optimal stoppings of Markov processes with which regular Dirichlet spaces are associated.

Let  $\psi_n$  be a given Borel function and  $s_n$  be the unique solution of the following variational inequality:

$$(4.1) \quad \begin{cases} \mathcal{E}_\alpha(s_n, v - s_n) \geq \langle \nu, \bar{v} - \bar{s}_n \rangle \quad \forall v \in \mathcal{F}, \bar{v} \leq \psi_n \text{ q.e.} \\ s_n \in \mathcal{F}, \bar{s}_n \leq \psi_n \text{ q.e.} \end{cases}$$

for each  $n$ .

**Lemma 4.1.** *Suppose that  $\psi_n(x) \downarrow \psi(x) \geq 0$   $\forall x$ , then  $\mathcal{E}_\alpha(s_n - s, s_n - s) \rightarrow 0$  where  $s$  is the unique solution of*

$$(4.2) \quad \begin{cases} \mathcal{E}_\alpha(s, v - s) \geq \langle \nu, \bar{v} - \bar{s} \rangle \quad v \in \mathcal{F}, \bar{v} \leq \psi \text{ q.e.} \\ s \in \mathcal{F}, \bar{s} \leq \psi \text{ q.e.} \end{cases}$$

*Proof.* In a similar way as the proof of Theorem 1 we can easily show that  $s_n \geq s_{n+1}$ ,  $s_n \geq 0$  and  $U_\alpha \nu - s_n$  is an  $\alpha$ -almost excessive function for each  $n$  (cf. Lemma in §1) Therefore we have

$$\mathcal{E}_\alpha(U_\alpha \nu - s_n, U_\alpha \nu - s_n) \leq \mathcal{E}_\alpha(U_\alpha \nu - s_m, U_\alpha \nu - s_m) \leq \mathcal{E}_\alpha(U_\alpha \nu, U_\alpha \nu) \quad n \leq m.$$

So there exists  $s_0 \in \mathcal{F}$  such that  $\mathcal{E}_\alpha(s_n - s_0, s_n - s_0) \rightarrow 0$ . Furthermore  $s_0$  satisfies

$$(4.3) \quad \begin{cases} \mathcal{E}_\alpha(U_\alpha \nu - s_0, U_\alpha \nu - s_0) \leq \mathcal{E}_\alpha(U_\alpha \nu - v, U_\alpha \nu - v) \quad \forall \bar{v} \leq \lim_{n \rightarrow \infty} \psi_n = \psi \text{ q.e.} \\ \bar{s}_0 \leq \psi \text{ q.e.} \end{cases}$$

which is equivalent to (4.2). Hence we conclude  $s_0 = s$  because of uniqueness of the solution of (4.2).

**Lemma 4.2.** *Put*

$$t_n(x) = \inf_{\tau} E_x \left[ \int_0^\tau e^{-\alpha s} dA_s + e^{-\alpha \tau} M_n \phi(X_\tau) \right] \text{ q.e.}$$

where  $\phi \in \tilde{\mathcal{F}}$ , then  $t_n$  is a quasi-continuous modification of the solution  $s_n$  of the variational inequality (4.1) for each  $n$  in which  $\psi_n$  is considered  $M_n \phi$ . Furthermore there exists an optimal stopping time.

*Proof.* Since  $U_\alpha \nu - s_n$  is  $\alpha$ -almost excessive by similar argument as Lemma 1.1 there corresponds a non-negative Radon measure  $\mu_n$  of finite energy

integral such that

$$\mathcal{E}_\alpha(U_\alpha v - s_n, v) = \int \mu_n(dx) \bar{v}(x) \quad \forall v \in \mathcal{F}.$$

Therefore it follows that

$$(4.4) \quad \begin{cases} \int \mu_n(dx) (\bar{s}_n(x) - \bar{v}(x)) \geq 0 \quad \forall \bar{v} \leq M_n \phi \text{ q.e., } v \in \mathcal{F} \\ \bar{s}_n \leq M_n \phi \text{ q.e., } s_n \in \mathcal{F} \end{cases}$$

from (4.1) with  $\psi_n = M_n \phi$ . Put

$$(4.5) \quad L_n = \{x \in \bigcup_{k=1}^\infty F_k; \bar{s}_n(x) < M_n \phi(x)\},$$

where  $\{F_k\}$  is a regular nest corresponding to the family of quasi-continuous functions  $\{\bar{s}_n\}$ . Take an arbitrary point  $x_0 \in L_n$ , then  $x_0 \in F_{k_0}$  for some  $k_0$ . On the other hand, Since  $M_n \phi(x)$  is a lower semi-continuous function there exists a sequence of continuous functions  $c_j^n(x)$  such that  $c_j^n(x) \uparrow M_n \phi(x), j \rightarrow \infty, \forall x$ . Therefore  $c_{j_0}^n(x_0) > s_n(x_0)$  for sufficiently large  $j_0$ , which implies that there exists a neighborhood  $U(x_0)$  of  $x_0$  such that

$$s_n(x) < c_{j_0}^n(x) \quad \forall x \in F_{k_0} \cap U(x_0).$$

Accordingly there exists a neighborhood  $V(x_0)$  and  $v_n \in \mathcal{F} \cap C_0(S)$  such that

$$\bar{V}(x_0) \subset U(x_0),$$

$$\text{Supp } v_n \subset U(x_0), v_n(x) > 0 \text{ on } V(x_0)$$

and

$$\bar{s}_n(x) + v_n(x) \leq M_n \phi(x)$$

because the Dirichlet space  $(\mathcal{F}, \mathcal{E})$  is regular. Therefore

$$-\int \mu_n(dx) v_n(x) \geq 0$$

which implies  $\mu_n(V(x_0)) = 0$ . Since  $x_0 \in L_n$  is arbitrary we conclude that

$$(4.6) \quad \mu_n(L_n) = 0.$$

Next, we have

$$(4.7) \quad \bar{s}_n(x) \leq M_n \phi(x) \text{ q.e.}$$

On the other hand let  $S-N$  be a defining set of the additive functional  $A_t$  (cf. [1]) and put  $\tau_n = \inf \{t; X_t \in L_n^c \cap \{S-N\}\}$ , then

$$(4.8) \quad P_x(X_{\tau_n} \in L_n^c \cap (S-N)) = 1 \quad x \in (\bigcup_{n=1}^\infty F_n) \cap (S-N)$$

for the benefit of lower semi-continuity of  $M_n\phi$  and quasi-continuity of  $s_n$ .

From (4.6), (4.7) and (4.8) in addition to the fact that there corresponds a non-negative additive functional  $A_t^n$  to the  $\alpha$ -almost excessive function  $U_\alpha v - s_n$  such that

$$\widetilde{U_\alpha v - s_n} = E_x\left[\int_0^\infty e^{-\alpha s} dA_s^n\right] \text{ q.e.}$$

our present lemma follows in the same way as Theorem in [6].

**Lemma 4.3.** *Put*

$$(4.9) \quad t(x) = \inf_\tau E_x\left[\int_0^\tau e^{-\alpha s} dA_s + e^{-\alpha\tau} M\phi(X_\tau)\right],$$

then  $t(x)$  is a quasi-continuous modification of the solution of the variational inequality (4.2) in which  $\psi$  is considered as  $M\phi(x)$ .

Proof. Let  $s(x)$  be the solution of (4.2) with  $\psi = M\phi$ , then  $U_\alpha v - s$  is  $\alpha$ -almost excessive and there corresponds a non-negative continuous additive functional  $A_t^0$  such that

$$(4.10) \quad U_\alpha v(x) - s(x) = E_x\left[\int_0^\infty e^{-\alpha t} dA_t^0\right] \text{ q.e..}$$

On the other hand we have

$$(4.11) \quad \mathfrak{s}(x) \leq M\phi(x) \text{ q.e..}$$

From (4.10) and (4.11) it follows that

$$\begin{aligned} \mathfrak{s}(x) &= E_x\left[\int_0^\infty e^{-\alpha t} dA_t\right] - E_x\left[\int_0^\infty e^{-\alpha t} dA_t^0\right] \\ &= E_x\left[\int_0^\tau e^{-\alpha t} dA_t\right] - E_x\left[\int_0^\tau e^{-\alpha t} dA_t^0\right] + E_x\left[e^{-\alpha\tau} \mathfrak{s}(X_\tau)\right] \\ &\leq E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} \mathfrak{s}(X_\tau)\right] \\ &\leq E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} M\phi(X_\tau)\right] \text{ q.e.,} \end{aligned}$$

for any stopping time  $\tau$ . Therefore it holds that

$$(4.12) \quad \mathfrak{s}(x) \leq t(x) \text{ q.e..}$$

Now it is clear that

$$t(x) \leq \inf_\tau E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} M_n\phi(X_\tau)\right] = \mathfrak{s}_n(x) \text{ q.e..}$$

Since  $\mathcal{E}_\alpha(s_n - s, s_k - s) \rightarrow 0$  by lemma 4.1 we obtain  $\mathfrak{s}_n(x) \downarrow \mathfrak{s}(x)$  q.e.. Hence

$$(4.13) \quad t(x) \leq \mathfrak{s}(x) \text{ q.e.}$$

(4.12) and (4.13) give our conclusion.

**5. Proof of Theorem 2**

Now we are going to prove Theorem 2. Let us introduce the set  $\underline{V}_n$  of admissible controls which have  $n$  jump times at most:

$$(5.1) \quad \underline{V}_n = \{v \in \underline{V}; \tau_{n+1}(\omega) = \infty\}$$

for each  $n$ . Put

$$(5.2) \quad u_n^*(x) = \inf_{v \in \underline{V}_n} E_x^{n+1} \left[ \int_0^{\tau_{n+1}} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i) \right]$$

and

$$(5.3) \quad w_n(x) = \inf_{\tau} E_x \left[ \int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha \tau} M w_{n-1}(X_{\tau}) \right]$$

for each  $n$  where  $w_0(x) = \widetilde{U_{\alpha} v}(x)$ .

Theorem 2 is a consequence of the following two propositions.

**Proposition 5.1.**  $w_n(x)$  is a quasi-continuous modification of the solution  $u_n$  of the variational inequality (1.5).

**Proposition 5.2.** It holds that

$$(5.4) \quad w_n(x) = u_n^*(x) \text{ q.e..}$$

Proposition 5.1 is a direct consequence of Lemma 4.3. For the proof of Proposition 5.2 we prepare the following two lemmas.

**Lemma 5.3.** It holds that

$$(5.5) \quad w_n(x) = \lim_{k_1 \uparrow \infty} \cdots \lim_{k_n \uparrow \infty} w_{k_n \cdots k_1}^n(x) \text{ q.e.}$$

where

$$(5.6) \quad w_{k_n \cdots k_1}^n(x) = \inf_{\tau} E_x \left[ \int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha \tau} M_{k_n} w_{k_{n-1} \cdots k_1}^{n-1}(X_{\tau}) \right] \quad n = 2, 3, \dots$$

and

$$(5.7) \quad w_{k_1}^1(x) = \inf_{\tau} E_x \left[ \int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha \tau} M_{k_1} \widetilde{U_{\alpha} v}(X_{\tau}) \right].$$

Proof. Because of Lemma 4.1 it follows that

$$(5.8) \quad w_{k_1}^1(x) \downarrow w_1(x) \text{ q.e., } k_1 \uparrow \infty$$

from  $M_{k_1} \widetilde{U_{\alpha} v}(x) \downarrow M \widetilde{U_{\alpha} v}(x) \forall x, k_1 \uparrow \infty$ , in the same way as the the proof of Lemma 4.3. Let us assume that

$$(5.9) \quad w_{n-1}(x) = \lim_{k_1 \uparrow \infty} \cdots \lim_{k_n \uparrow \infty} w_{k_{n-1} \dots k_1}^{n-1}(x) \text{ q.e.}$$

Then it follows that

$$(5.10) \quad \lim_{k_n \uparrow \infty} w_{k_n \dots k_1}^{n-1}(x) = \inf_{\tau} E_x \left[ \int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha \tau} M w_{k_{n-1} \dots k_1}^{n-1}(X_{\tau}) \right]$$

from  $M_{k_n} w_{k_{n-1} \dots k_1}^{n-1}(x) \downarrow M w_{k_{n-1} \dots k_1}^{n-1}(x) \forall x, k_n \uparrow \infty$ , in the same way as above. On the other hand it holds that

$$(5.11) \quad M w_{n-1}(x) = \lim_{k_1 \uparrow \infty} \cdots \lim_{k_{n-1} \uparrow \infty} M w_{k_{n-1} \dots k_1}^{n-1}(x) \forall x$$

by our assumption and the property of  $M$ . Making use of Lemma 4.1 we obtain our present lemma from (5.10) and (5.11).

**Lemma 5.4.** *Let  $\bar{X}=(\bar{\Omega}, \bar{\mathcal{B}}, \bar{\mathcal{P}}_x, \bar{X}_t)$  be a  $m$ -symmetric Markov process associated with a regular Dirichlet space  $(\mathcal{F}, \mathcal{E})$  and*

$$H(M\phi; x) = \inf_{\tau} \bar{E}_x \left[ \int_0^{\tau} e^{-\alpha s} d\bar{A}_s + e^{-\alpha \tau} M\phi(\bar{X}_{\tau}) \right] \quad \phi \in \tilde{\mathcal{F}},$$

then it holds that

$$\bar{E}_x \left[ \int_{\sigma}^{\tau} e^{-\alpha s} d\bar{A}_s + e^{-\alpha \tau} M\phi(\bar{X}_{\tau}) \mid \bar{\mathcal{B}}_{\sigma} \right] \geq e^{-\alpha \sigma} H(M\phi; \bar{X}_{\sigma})$$

for any stopping time  $\sigma, \tau$  such that  $\sigma \leq \tau$ . Here  $\bar{A}_t$  is an additive functional of  $\bar{X}$  corresponding to the Radon measure  $\nu(dx)$  of finite energy integral.

Proof. At first we note that  $H(M\phi; x) \in \tilde{\mathcal{F}}, U_{\alpha} \nu(x) - H(M\phi; x)$  is  $\alpha$ -almost excessive and  $H(M\phi; x) \leq M\phi(x)$  q.e. by Lemma 4.3. Therefore  $e^{-\alpha t} \{U_{\alpha} \nu(\bar{X}_t) - H(M\phi; \bar{X}_t)\}$  is a  $(\bar{P}_x, \bar{\mathcal{B}}_t)$  supermartingale for q.e.  $x$ . Hence

$$\begin{aligned} & \bar{E}_x [e^{-\alpha \tau} \{U_{\alpha} \nu(\bar{X}_{\tau}) - H(M\phi; \bar{X}_{\tau})\} \mid \bar{\mathcal{B}}_{\sigma}] \\ & \leq e^{-\alpha \sigma} \{U_{\alpha} \nu(\bar{X}_{\sigma}) - H(M\phi; \bar{X}_{\sigma})\} \quad \bar{P}_x\text{-a.s. q.e. } x. \end{aligned}$$

So we have

$$\bar{E}_x \left[ \int_{\sigma}^{\tau} e^{-\alpha s} d\bar{A}_s + e^{-\alpha \tau} H(M\phi; \bar{X}_{\tau}) \mid \bar{\mathcal{B}}_{\sigma} \right] \geq e^{-\alpha \sigma} H(M\phi; \bar{X}_{\sigma}).$$

Accordingly it follows that

$$\bar{E}_x \left[ \int_{\sigma}^{\tau} e^{-\alpha s} d\bar{A}_s + e^{-\alpha \tau} M\phi(\bar{X}_{\tau}) \mid \bar{\mathcal{B}}_{\sigma} \right] \geq e^{-\alpha \sigma} H(M\phi; \bar{X}_{\sigma})$$

from  $H(M\phi; x) \leq M\phi(x)$  q.e.

Proof of Proposition 5.2. Let  $v \in \underline{V}_n, v = \{(\tau_k, \xi_k)_{k=1,2,\dots,n}, \tau_{n+1} = \infty\}$ , then

$$J_x^n(v) = E_x^{n+1} \left[ \int_0^{\infty} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i) \right]$$

$$\begin{aligned}
 &= E_x^{n+1} \left[ \int_0^{\tau_n} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i) + e^{-\alpha \tau_n} \int_0^\infty e^{-\alpha s} dA_s(\theta_{\tau_n} \omega_{n+1}) \right] \\
 &= E_x^{n+1} \left[ \int_0^{\tau_n} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i) + e^{-\alpha \tau_n} E_{\xi_n} \left[ \int_0^\infty e^{-\alpha s} dA_s \right] \right] \\
 &\geq E_x^n \left[ \int_0^{\tau_n} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^{n-1} e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i) + e^{-\alpha \tau_n} M \widetilde{U}_\alpha \nu(X_{\tau_n}(\omega_n)) \right] \\
 &= E_x^n \left[ \int_0^{\tau_{n-1}} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^{n-1} e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i) \right. \\
 &\quad \left. + E_x^n \left[ \int_{\tau_{n-1}}^{\tau_n} e^{-\alpha s} dA_{s-\tau_{n-1}}(\theta_{\tau_{n-1}} \omega_n) + e^{-\alpha \tau_n} M \widetilde{U}_\alpha \nu(X_{\tau_n}(\omega_n)) \mid \mathcal{B}_{\tau_{n-1}}^n \right] \right] \\
 &\geq E_x^n \left[ \int_0^{\tau_{n-1}} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^{n-1} e^{-\alpha \tau_i} k(X_{\tau_i}, \xi_i) + e^{-\alpha \tau_{n-1}} H(M \widetilde{U}_\alpha \nu; X_{\tau_{n-1}}) \right] \\
 &= E_x^{n-1} \left[ \int_0^{\tau_{n-1}} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^{n-1} e^{-\alpha \tau_i} k(X_{\tau_i}, \xi_i) + e^{-\alpha \tau_{n-1}} w_1(X_{\tau_{n-1}}) \right] \\
 &\geq E_x^1 \left[ \int_0^{\tau_1} e^{-\alpha s} dA_s + e^{-\alpha \tau_1} k(X_{\tau_1}, \xi_1) + e^{-\alpha \tau_1} w_{n-1}(X_{\tau_1}) \right] \\
 &\geq E_x^1 \left[ \int_0^{\tau_1} e^{-\alpha s} dA_s + e^{-\alpha \tau_1} M w_{n-1}(X_{\tau_1}) \right] \geq w_n(x) \quad \text{q.e.}
 \end{aligned}$$

In order to get the converse inequality take a sequence  $\{\tilde{\tau}_j\}_{j=1,2,\dots,n}$  of stopping times such that  $\tau_j$  minimizes

$$E_x \left[ \int_0^\tau e^{-\alpha s} dA_s + M_{k_j} w_{k_{j-1} \dots k_1}^{j-1}(X_\tau) \right].$$

Furthermore take a sequence of functions  $y_{k_j}(x), j=1, 2, \dots, n$  such that

$$M_{k_j} w_{k_{j-1} \dots k_1}^{j-1}(x) = w_{k_{j-1} \dots k_1}^{j-1}(y_{k_j}(x)) + k(x, y_{k_j}(x))$$

Put

$$\hat{\tau}_1 = \hat{\tau}_{i-1} + \tilde{\tau}_{n+1-i}(\theta_{\tau_{i-1}} \omega_i), \quad \hat{\tau}_1 = \tilde{\tau}_n(\omega_1)$$

and

$$\hat{\xi}_1 = y_{k_{n+1-i}}(X_{\hat{\tau}_i}(\omega_i)).$$

Then  $\hat{v} = \{(\hat{\tau}_i, \hat{\xi}_i)_{i=1,2,\dots,n}, \hat{\tau}_{i+1} = \infty\} \in \underline{V}_n$

and

$$\begin{aligned}
 w_{k_n \dots k_1}^n(x) &= E_x \left[ \int_0^{\tilde{\tau}_n} e^{-\alpha s} dA_s + e^{-\alpha \tilde{\tau}_n} M_{k_n} w_{k_{n-1} \dots k_1}^{n-1}(X_{\tilde{\tau}_n}) \right] \\
 &= E_x \left[ \int_0^{\tilde{\tau}_n} e^{-\alpha s} dA_s + e^{-\alpha \tilde{\tau}_n} \{w_{k_{n-1} \dots k_1}^{n-1}(y_{k_n}(X_{\tilde{\tau}_n})) + k(X_{\tilde{\tau}_n}, y_{k_n}(X_{\tilde{\tau}_n}))\} \right] \\
 &= E_x^1 \left[ \int_0^{\hat{\tau}_1} e^{-\alpha s} dA_s + e^{-\alpha \hat{\tau}_1} k(X_{\hat{\tau}_1}, y_{k_n}(X_{\hat{\tau}_1})) \right]
 \end{aligned}$$

$$\begin{aligned}
 &+ e^{-\omega \hat{\tau}_1} E_{\hat{\xi}_1} \left[ \int_0^{\hat{\tau}_{n-1}} e^{-\alpha s} dA_s + e^{-\omega \hat{\tau}_{n-1}} M_{k_{n-1}} w_{k_{n-2} \dots k_1}^{n-2} (X_{\hat{\tau}_{n-1}}) \right] \\
 &= E_x^{n+1} \left[ \int_0^\infty e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^n e^{-\omega \hat{\tau}_i} k(X_{\hat{\tau}_i}, \hat{\xi}_i) \right] \\
 &\geq u_n^*(x) \quad \text{q.e..}
 \end{aligned}$$

Therefore making use of Lemma 5.3 we conclude that

$$w_n(x) \geq u_n^*(x) \quad \text{q.e..}$$

Proof of Theorem 2. By Propositions 5.1 and 5.2  $u_n^*$  is a quasi-continuous modification of the solution  $u_n$  of the variational inequality (1.5) for each  $n$ . Since  $u_n$  converges to the maximum solution  $u$  of the QVI (1.2) in  $\mathcal{E}_\omega$ -norm:  $\mathcal{E}_\omega(u_n - u, u_n - u) \rightarrow 0, n \rightarrow \infty$  we have

$$u_n^*(x) \rightarrow u(x) \quad \text{q.e., } n \rightarrow \infty$$

taking a sub-sequence if necessary. On the other hand it holds that

$$u_n^*(x) \downarrow u^*(x) \quad \text{q.e., } n \rightarrow \infty$$

by the next lemma. This completes the proof of Theorem 2.

**Lemma 5.5.** *It holds that  $u_N^*(x) \downarrow u^*(x)$  q.e.,  $N \rightarrow \infty$ .*

Proof. For each  $\varepsilon > 0$  there exists  $v = v(x) = \{(\tau_i, \xi_i)_{i=1}^\infty\} \in \underline{V}$  such that

$$u^*(x) \geq \lim_{n \rightarrow \infty} E_x^n \left[ \int_0^{\tau_n} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^n e^{-\omega \tau_i} k(X_{\tau_i}, \xi_i) \right] - \varepsilon$$

So for any  $N$  it holds that

$$u^*(x) \geq E_x^N \left[ \int_0^{\tau_N} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^N e^{-\omega \tau_i} k(X_{\tau_i}, \xi_i) \right] - \varepsilon$$

Put  $v^N = \{(\tau_i, \xi_i)_{i=1,2,\dots,N}, \tau_{N+1} = \infty\}$ , then  $v^N \in \underline{V}_N$ . Therefore from

$$E_x^{N+1} \left[ \int_0^{\tau_{N+1}} e^{-\alpha s} d\underline{A}_s + \sum_{i=1}^{N+1} e^{-\omega \tau_i} k(X_{\tau_i}, \xi_i) \right] \geq u_N^*(x)$$

and  $u_N^*(x) \geq u^*(x)$  it follows that

$$|u_N^*(x) - u^*(x)| \leq E_x^{N+1} \left[ \int_{\tau_N}^\infty e^{-\alpha s} d\underline{A}_s \right] + 2\varepsilon.$$

Since  $\tau_N \rightarrow \infty$  as  $N \rightarrow \infty$  we obtain

$$\lim_{N \rightarrow \infty} u_N^*(x) = u^*(x) \quad \text{q.e..}$$

$u_N^*(x) \geq u_{N+1}^*(x)$  q.e. is obvious.



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