Yotsutani, S.
Osaka J. Math.
20 (1983), 845-862

STEFAN PROBLEMS WITH THE UNILATERAL BOUNDARY CONDITION ON THE FIXED BOUNDARY III

SHOJI YOTSUTANI

(Received December 28, 1981)

Contents

0. Introduction

This paper is concerned with the following one-dimensional one-phase Stefan problems with the unilateral boundary condition on the fixed boundary: Given the data, l and $\phi(x)$, find two functions $s=s(t)$ and $u=u(x, t)$ such that the pair (s, u) satisfies

$$
(\mathrm{S}) \quad \begin{cases} (0.1) & u_{xx}-u_t = 0 & (0<x<s(t),\ t>0)\,, \\ (0.2) & u_x(0, t)\in\gamma(u(0, t)) & (t>0)\,, \\ (0.3) & u(s(t), t) = 0 & (t>0)\,, \\ (0.4) & u(x, 0) = \phi(x)\geqq 0 & (0\leqq x\leqq l)\,, \quad s(0) = l\geqq 0\,, \\ (0.5) & \dot{s}(t) = -u_x(s(t), t) & (t>0)\,. \end{cases}
$$

Here γ is a maximal monotone graph in $\boldsymbol{R}^2$ such that $\gamma^{-1}(0)\cap[0, \infty[$ is not an empty set. We put this assumption from the physical reasoning, that is, there is a kind of heater at the fixed boundary $x=0$. The physical meaning of this equation is explained in [6].

In the previous paper [6], the author proved the existence and uniqueness of the classical solution of (S). Left unanswered however were several questions as to the existence of the solution in the case where $l=0$ and $\gamma(0)$ is an empty set, and the asymptotic behavior of the solution. The present paper resolves these problems. Cannon & Hill [3] and Cannon & Primicerio [5] investigated the asymptotic behaviour of the solutions of one-phase Stefan problems with the linear boundary condition on the fixed boundary in detail. We will use these results efficiently for our problems. The main difficulties concerning our problems arise from the fact that the informations about $u(0, t)$ or $u_x(0, t)$ are not given different from the usual Stefan problems considered in [3] and [5].

The plan of the paper is as follows. In §1 we state main results. §2 collects several results concerning the moving boundary problem which is auxiliary for the original one. §3 collects properties of the solutions of the Stefan problem (S). In §4 we give the proof of Theorem 1 and Theorem 2. §5 collects several properties of the unilateral problems. §6 investigates the Stefan problem with Signorini boundary condition on the fixed boundary. In §7 we give the proof of Theorem 3. In §8 we give the proof of Theorem 4.

The author would like to express his gratitude to Professor H. Tanabe for his useful advice and encouragements.

1. Statements of main results

The assumptions required on the Stefan data are as follows.

(A) $\phi(x)$ is non-negative, bounded and continuous for a.e. $x \in [0, l]$.

REMARK 1.1. The assumption $\phi \geqq 0$ results from the physical background.

REMARK 1.2. If $l=0$ there is no ϕ. We do not need (A).

We introduce the notations,

$$D = \{(x, t);\ 0<x<s(t),\quad t>0\}\ ,$$
$$\bar{D} = \{(x, t);\ 0\leqq x\leqq s(t),\quad t\geqq 0\}\ ,$$
$$D^s = \{(x, t);\ 0<x\leqq s(t),\ t>0\}\ ,$$
$$Z = \{x\in[0, l];\ x \text{ is a discontinuous point of } \phi\}\times\{0\}\,.$$

DEFINITION 1.1. The pair (s, u) is a solution of the Stefan problem (S) if

i) $s(0)=l$, $s(t)>0$ for $t>0$, $s\in C([0, \infty[)\cap C^\infty(]0, \infty[)$,

ii) u is bounded on $\bar{D}$, $u\in C(\bar{D}-Z)\cap C^\infty(D^s)$,

$$\int_\sigma^T\int_0^{s(t)} u_t(x, t)^2 dxdt<\infty \qquad \text{for each} \quad 0<\sigma<T<\infty\ ,$$

iii) (0.1), (0.3), (0.4) and (0.5) hold,

iv) for a.e. $t>0$, $u_x(0, t)$ exists and satisfies (0.2).

REMARK 1.3. If $l=0$, we omit (0.4).

We put $H=\mathrm{Proj}_{\gamma^{-1}(0)}\, 0$, that is, H is the element of $\gamma^{-1}(0)$ which has the minimum absolute value.

This is well-defined and

$$H \geqq 0$$

by the assumption that γ is the maximal monotone graph in $\boldsymbol{R}^2$ such that $\gamma^{-1}(0) \cap [0, \infty[$ is not an empty set.

We will use the notations of the spaces of functions introduced in [6], if necessary.

We can now state existence and uniqueness theorems.

Theorem 1. *Let $l>0$ and $\phi(x)$ satisfies (A). Then there exists the unique solution (s, u) of (S) satisfying*

(1.1) $\quad s \in C([0, \infty[) \cap C^\infty(]0, \infty[)$, $s(t)$ *is non-decreasing in* t.

(1.2) $\quad 0 \leqq u \leqq \max(\|\phi\|_{L^\infty(0,l)},\ H) \quad$ *on* $\bar{D}$,

(1.3) $\quad |u(x', t)-u(x, t)| \leqq C_\sigma |x'-x| \quad$ *on* $\bar{D} \cap \{t \geqq \sigma\}$,

where C_σ is a positive constant depending on σ.

Theorem 2. *Let $l=0$. Suppose that γ satisfies the following assumption (Γ) in addition to the original one,*

$(\Gamma) \qquad \gamma(0) \subset]-\infty, 0[$.

Then there exists the unique solution (s, u) of (S) *satisfying* (1.1), (1.2) *and* (1.3), *where we define* $\|\phi\|_{L^\infty(0,0)}=0$.

REMARK 1.4. We needed that $D(\gamma) \ni 0$ and $\gamma(0) \subset]-\infty, 0[$ in [6, Theorem 3 and theorem 4]. However we do not need the assumption $D(\gamma) \ni 0$ in Theorem 2. In particular we may suppose that $\gamma(0)$ is an empty set. Hence Theorem 2 is an improvement of [6, Theorem 3 and Theorem 4].

REMARK 1.5. The assumption (Γ) is equivalent to the assumption $H>0$.

REMRAK 1.6. The assumption (Γ) guarantees that the solid melts. For example, if $\gamma \equiv 0$, then the solid could not melt.

We describe the asymptotic behavior, as $t \to \infty$, of the solution (s, u) of (S). We suppose that the assumptions of Theorem 1 and Theorem 2 are satisfied for γ and the data ϕ.

Theorem 3. *Let $H>0$ $(l \geqq 0)$. Then we have*

(1.4) $\lim_{t\to\infty} u(x, t) = H$ *uniformly on any compact subset of* $[0, \infty[$,

(1.5) $\lim_{t\to\infty} s(t)/\sqrt{t} = \beta$,

where β *is given by the unique solution of*

(1.6) $\sum_{n=1}^{\infty} [n!/(2n)!]\beta^{2n} = H$.

Theorem 4. *Let* $H=0$ $(l>0)$. *Then it follows that*

(1.7) $\lim_{t\to\infty} u(x, t) = 0$ *uniformly on* $[0, s^*]$,

(1.8) $s^* = \lim_{t\to\infty} s(t)$ *exists,*

(1.9) $l \leqq s^* \leqq l + \int_0^l \phi(x)dx$.

REMARK 1.7. We define $u(x, t)=0$ for $x \geqq s(t)$, $t \geqq 0$.

REMARK 1.8. We see also that the condition (Γ) (i.e. $H>0$) is necessary for us to guarantee the melting of the solid in case $l=0$ from Theorem 4. In fact, we can not expect that $s(t)>0$ for $t>0$ if $H=0$ and $l=0$ by (1.9).

2. Moving boundary problems

We investigate the moving boundary problems which are auxiliary for the Stefan problems.

Consider the following moving boundary problem: Given the data $\phi(x)$ and the given non-decreasing function $s(t) \in C([0, \infty[) \cap C^{0,1}(]0, \infty[)$, that is positive for $t>0$, find a function $u=u(x, t)$ such that

$$
(\mathrm{M}) \quad \begin{cases} (2.1) \quad u_{xx}-u_t = 0 & (0<x<s(t), \quad t>0), \\ (2.2) \quad u_x(0, t) \in \gamma(u(0, t)) & (t>0), \\ (2.3) \quad u(s(t), t) = 0 & (t>0), \\ (2.4) \quad u(x, 0) = \phi(x) & (0<x<s(0) \equiv l). \end{cases}
$$

Here is γ a maximal monotone graph in $\boldsymbol{R}^2$ such that $\gamma^{-1}(0)$ is not an empty set. We put $H = \mathrm{Prof}_{\gamma^{-1}(0)}\, 0$.

REMARK 2.1. If $l=0$, we omit (2.4).

DEFINITION 2.1. $u=u(x, t)$ is the solution of the moving boundary problem (M) if

i) u is bounded on $\bar{D}$. $u \in C(\bar{D}-Z) \cap C^\infty(D)$,

(2.5) $\int_\sigma^T \int_0^{s(t)} u_t(x, t)^2 dxdt < +\infty$ for each $0<\sigma<T<\infty$, where D, $\bar{D}$ and Z are the sets defined in §1,

ii) (2.1), (2.3) and (2.4) hold,

iii) for a.e. $t>0$, $u_x(0, t)$ exists and satisfies (2.2).

REMARK 2.2. The above definition is slightly different from [6, Definition 9.1] (see (2.5)). However it is easily seen that [6, Proposition 10.1] and [6, Proposition 10.2] hold in this case.

Proposition 2.1. *Let $l>0$ with $\phi(x)$ which is bounded and continuous for a.e. $x\in(0, l]$, or $l=0$. Then there exists the unique solution of the moving boundary problem (M). In addition*

$$(2.6)\qquad |u(x, t)|\leqq \max(\|\phi\|_{L^\infty(0,l)}, |H|)\qquad on\quad \bar{D},$$

$$(2.7)\qquad |u(x', t)-u(x, t)|\leqq C_\sigma|x'-x|^{1/2}\qquad on\quad \bar{D}\cap\{t\geqq\sigma\}\ ,$$

where C_σ is a constant depending on σ, and we define $\|\phi\|_{L^\infty(0,l)}=0$ when $l=0$.

Proof. We get the uniqueness by [6, Proposition 10.1]. We know the existence when $l>0$ from [6, Proposition 9.1]. We shall show the existence when $l=0$. We put $s^m(t)=s(t)+1/m$. Hence the sequence $\{s^m\}_m$ is a monotonically decreasing one of increasing continuous functions in t, and $s^m(t)\to s(t)$ uniformly on any compact subset of $[0, \infty[$ as $m\to\infty$. Let u^m be the solution of (M) corresponding to the initial data 0 and the moving boundary s^m. Thus each u^m is constructed by the finite difference method stated in [6, §9]. Hence we have

$$(2.8)\qquad |u^m(x, t)|\leqq|H|\qquad (0\leqq x\leqq s^m(t),\quad t\geqq 0)\,,$$

using the proof of [6, Lemma 5.1].

Let us fix $0<\sigma<T<\infty$. We can regard u^m as the solution of the moving boundary problem (M) with the initial time σ, the initial data $u^m(\cdot, \sigma)$ and the moving boundaıy $s^m(t)$ $(\sigma\leqq t\leqq T)$. Therefore we have

$$\int_\sigma^T\int_0^{s^m(t)}(t-\sigma)(u_t^m)^2\,dxdt+(1/2)(t-\sigma)\int_0^{s^m(t)}u_x^m(x, t)^2dx\leqq C(\sigma, T, s(\sigma))$$
$$(\sigma\leqq t\leqq T)$$

using (2.8), $s^m(\sigma)\geqq s(\sigma)>0$ and [6, Lemma 9.2], where $C(\sigma, T, s(\sigma))$ is a positive constant depending only on σ, T and $s(\sigma)$. Hence we have

$$(2.9)\qquad \int_\sigma^T\int_0^{s^m(t)}(u_t^m)^2\,dxdt\leqq C_{\sigma,T}$$

$$(2.10)\qquad \int_0^{s^m(t)}u_x^m(x, t)^2\,dx\leqq C_{\sigma,T}\qquad (\sigma\leqq t\leqq T)$$

for any $0<\sigma\leqq T<\infty$. Thus we have easily

$$|u^m(x', t') - u^m(x, t)| \leqq C'_{\sigma,T}[|x'-x|^{1/2} + |t'-t|^{1/4}]$$
$$(0 \leqq x' \leqq x < \infty, \quad \sigma \leqq t' \leqq t \leqq T)$$

using (2.9), (2.10) and [6, Lemma 16.4], where we define $u^m(x, t) = 0$ $(x \geqq s^m(t), t \geqq 0)$. Consequently there exists a subsequence of $\{u^m\}_m$ (which we denote again by the same symbol), and a function $u \in C(G)$, $G = [0, \infty[\times]0, \infty[$ such that

$$u^m \to u \quad \text{uniformly on any compact subset of } G$$

by Ascoli-Arzelà's theorem. We shall examine that $u|_{\bar{D}}$ is the solution of (M). We get (2.3) using $u^m(s^m(t), t) = 0$ $(t > 0)$. We note $u^m_{xx} - u^m_t = 0$ $(0 < x < s^m(t), t > 0)$. Thus it is easily seen that $u_{xx} - u_t = 0$ $(0 < x < s(t), t > 0)$ in the distribution's sense. Hence we have $u \in C^\infty(D) \cap C(\bar{D} \cap \{t > 0\})$ and (2.1) in view of the well-known result concerning the heat equation. We get also easily

$$\int_\sigma^T \int_0^{s(t)} u_t^2 \, dxdt \leqq C_{\sigma,T} \qquad (0 < \sigma < T < \infty).$$

We shall show (2.2). Let us fix $0 < \sigma < T < \infty$. It follows from [6, (8.10)] that

$$\int_\sigma^T \int_0^{s^m(t)} u^m_x (w - u^m)_x \, dxdt + \int_\sigma^T \theta(\eta) dt - \int_\sigma^T \theta(u^m(0, t)) dt$$
$$\geqq - \int_\sigma^T \int_0^{s^m(t)} u^m_{xx}(w - u^m) dxdt .$$

where $\eta \in D(\theta)$, θ is a lower semicontinuous convex θ function from $\boldsymbol{R}$ into $]-\infty, \infty]$ such that $\theta \not\equiv +\infty$, $\theta \geqq 0$, $\theta(H) = 0$ and $\partial\theta = \gamma$, and

$$w(x) = \begin{cases} \eta(1 - x/s(\sigma)) & \text{for} \quad 0 \leqq x \leqq s(\sigma), \\ 0 \quad \text{for} \quad x > s(\sigma). \end{cases}$$

Hence it is easily seen from $u^m_{xx} = u^m_t$, $u^m \to u$ as $m \to \infty$, (2.9) and (2.10) that

$$\int_\sigma^T \int_0^{s(t)} u_x (w - u)_x \, dxdt + \int_\sigma^T \theta(\eta) dt - \int_\sigma^T \theta(u(0, t)) dt$$
$$\geqq - \int_\sigma^T \int_0^{s(t)} u_{xx}(w - u) dxdt .$$

Therefore we get (2.2) by the arguments used in the last part of the proof of [6, Lemma 8.4]. Consequently we have shown that u is the solution of (M).

We shall show (2.6) and (2.7). We get (2.6) noting [6, Lemma 9.1 and 5.1] if $l > 0$, and (2.8) if $l = 0$. Let us fix $\sigma > 0$. Then $s(\sigma) > 0$. We regard u as the solution of the moving boundary problem with the initial time σ and the initial data $u(x, \sigma)$. Thus $u(x, t)$ $(t \geqq \sigma)$ is constructed by the finite difference method stated in [6, §9]. Hence it follows from [6, Lemma 9.2, 7.4] and (2.6) that for $t \geqq \sigma$

$$
\begin{aligned}
&(t-\sigma)\int_0^{s(t)} u_x(x, t)^2 dx \\
&\leqq (t-\sigma)\int_0^{s(\sigma)} v_x(x)^2 dx + \int_0^{s(\sigma)} (u(x, \sigma)-v(x))^2 dx \\
&\leqq (t-\sigma)\int_0^{s(\sigma)} v_x^2 dx + \int_0^{s(\sigma)} (|H| + \|\phi\|_{L^\infty(0,l)} + \|v\|_{L^\infty(0,s(\sigma))})^2 dx\,,
\end{aligned}
$$

where $v(x)$ is a linear function such that $v(0)=H$ and $v(s(\sigma))=0$. Consequently we have

$$
\int_0^{s(t)} u_x(x, t)^2 dx \leqq C'_\sigma \qquad (t \geqq \sigma)\,,
$$

where C'_σ is a constant depending on σ. Thus we get (2.7) using the Schwarz's inequality. q.e.d.

We shall state the comparison theorems. We introduce the following notations.

$$
\begin{aligned}
&\gamma^+(v) = \max\{z;\, z\in\gamma(v)\}, \quad \gamma^-(v) = \min\{z;\, z\in\gamma(v)\} \qquad \text{if} \quad v\in D(\gamma)\,, \\
&\gamma^+(v) = \gamma^-(v) = +\infty \qquad \text{if} \quad v\notin(\gamma) \quad \text{and} \quad v\geqq \sup D(\gamma)\,, \\
&\gamma^+(v) = \gamma^-(v) = -\infty \qquad \text{if} \quad v\notin D(\gamma) \quad \text{and} \quad v\leqq \inf D(\gamma)\,.
\end{aligned}
$$

Proposition 2.2. *For $i=1, 2$, let u_i be the solution of the moving boundary problem (M) corresponding to the moving boundary $s_i(t)$, the initial condition $\phi_i(x)$ and the boundary condition*

(2.11) $\qquad u_{ix}(0, t)\in\gamma_i(u_i(0, t)) \qquad (t>0)\,.$

Suppose that

(2.12) $\qquad s_1(t)\leqq s_2(t) \qquad (t\geqq 0)\,, \quad \phi_1(x)\leqq\phi_2(x) \qquad (0<x<l)\,,$

(2.13) $\qquad u_1(s_1(t), t)\leqq u_2(s_1, (t), t) \qquad (t>0)\,,$

(2.14) $\qquad \gamma_1^+(v)\geqq\gamma_2^-(v) \qquad$ *for any* $\quad v\in\boldsymbol{R}\,.$

Then we have

(2.15) $\qquad u_1(x, t)\leqq u_2(x, t) \qquad (0\leqq x\leqq s_1(t), \quad t\geqq 0)\,.$

Proof. We get

$$
\begin{aligned}
(2.16)\qquad &(u_{1x}(0, t)-u_{2x}(0, t))((u_1(0, t))-u_2(0, t))^+ \\
&\geqq(\gamma_1^-(u_1(0, t))-\gamma_2^+(u_2(0, t)))(u_1(0, t)-u_2(0, t))^+\geqq 0 \qquad (\text{a.e. } t>0)\,.
\end{aligned}
$$

In fact, if $u_1>u_2$, let $u_1>\xi>u_2$. Then we have

$$
\gamma_1^-(u)\geqq\gamma_1^+(\xi)\geqq\gamma_2^-(\xi)\geqq\gamma_2^+(u_2)\,.
$$

Hence we have (2.15) from (2.12), (2.13), (2.16) and [6, Lemma 10.1]. q.e.d.

Proposition 2.3. *Let u be the solution of (M). Suppose that $H\geqq 0$ and*

$$u(x, 0) = \phi(x)\geqq 0 \qquad (0<x<s(0) = l)$$

especially when $l>0$. Then we have

$$0\leqq u\leqq \max(H, ||\phi||_{L^\infty(0,l)}) \qquad \text{on} \quad \bar{D},$$

where we define $||\phi||_{L^\infty(0,l)}=0$ when $l=0$.

Proof. See [6, Proposition 10.2]. q.e.d.

We consider the following moving boundary problem (M_D) with Dirichlet boundary condition on the fixed boundary.

$$(M_D) \quad \begin{cases} (2.17) \quad u_{xx}-u_t = 0 & (0<x<s(t), \quad t>0)\,, \\ (2.18) \quad u(0, t) = f(t) & (t>0)\,, \\ (2.19) \quad u(s(t), t) = 0 & (t>0)\,, \\ (2.20) \quad u(x, 0) = \phi(x) & (0<x<s(0)\equiv l)\,, \end{cases}$$

where $f(t)\in C(]0, \infty[)$.

REMARK 2.3. The solution of (M_D) is defined by the way analogous to Definition 2.1.

Proposition 2.4. *For $i=1, 2$, let u_i be the solution of (M_D) corresponding to the moving boundary $s_i(t)$, the initial condition $\phi_i(x)$ and the boundary condition*

$$(2.21) \qquad u_i(0, t) = f_i(t)\,.$$

Suppose that

$$(2.22) \qquad s_1(t)\leqq s_2(t) \quad (t\geqq 0)\,, \quad \phi_1(x)\leqq \phi_2(x) \qquad (0<x<l)\,,$$

$$(2.23) \qquad u_1(s_1(t), t)\leqq u_2(s_2(t), t) \qquad (t>0)\,,$$

$$(2.24) \qquad f_1(t)\leqq f_2(t) \qquad (t>0)\,.$$

Then we have

$$(2.25) \qquad u_1(x, t)\leqq u_2(x, t) \qquad (0\leqq x\leqq s_1(t), \quad t>0)\,.$$

Proof. We get

$$(2.26) \qquad \begin{aligned} &(u_{1x}(0, t)-u_{2x}(0, t))(u_1(0, t)-u_2(0, t))^+ \\ &= (u_{1x}(0, t)-u_{2x}(0, t))(f_1(t)-f_2(t))^+ = 0 \qquad (\text{a.e. } t>0)\,. \end{aligned}$$

Hence we have (2.25) from (2.22), (2.23), (2.26) and [6, Lemma 10.1]. q.e.d.

3. Properties of the solutions of (*S*)

We return to the general situations of the Stefan problem (S). First we state useful results concerning the reformations of the Stefan's condition.

Proposition 3.1. *Let* (s, u) *be a solution of* (S). *Then it follows that for any* $\sigma, t \in]0, \infty[$

$$s(t) = s(\sigma) + \int_0^{s(\sigma)} u(x, \sigma)dx - \int_0^{s(t)} u(x, t)dx - \int_\sigma^t u_x(0, \tau)d\tau, \tag{3.1}$$

$$s(t)^2 = s(\sigma)^2 + 2\int_0^{s(\sigma)} xu(x, \sigma)dx - 2\int_0^{s(t)} xu(x, t)dx + 2\int_\sigma^t u(0, \tau)d\tau. \tag{3.2}$$

Proof. See [6, Lemma 11.1 and 11.2]. q.e.d.

Consider two sets $\{l_i, \phi_i\}$ $(i=1, 2)$ of Stefan data. If $l_i>0$ we require that ϕ_i satisfies (A), and if $l_i=0$ there is no ϕ_i.

Proposition 3.2. *For* $i=1, 2$, *let* (s_i, u_i) *be solutions of* (S) *corresponding to the data* $\{l_i, \phi_i\}$ *and the boundary condition* $u_{ix}(0, t) \in \gamma_i(u_i(0, t))$. *Suppose that*

$$0 \leqq l_1 \leqq l_2, \quad \phi_1(x) \leqq \phi_2(x) \qquad (0<x<l_1), \tag{3.3}$$

$$\gamma_1^+(v) \geqq \gamma_2^-(v) \qquad \text{for any} \quad v \in \boldsymbol{R}. \tag{3.4}$$

Then we have

$$s_1(t) \leqq s_2(t) \qquad (t>0), \tag{3.5}$$

$$u_1 \leqq u_2 \qquad (0 \leqq x \leqq s_1(t), \quad t>0). \tag{3.6}$$

Proof. We note that the existence of the solution of (S) under the assumption $l_i>0$ and (A) is shown by [6, Theorem 1]. Thence we can show the conclusion using Proposition 2.2, Proposition 2.3 and the proof of [6, Lemma 12.1]. q.e.d.

Proposition 3.3. *The solution of* (S) *is uniquely determined.*

Proof. If (s_1, u_1) and (s_2, u_2) are two solutions of the Stefan problem (S) then $s_1=s_2$ by Proposition 3.2. Hence $u_1=u_2$ by Proposition 2.2. q.e.d.

We state the results concerning the Stefan problem with the Drichlet boundary condition on the fixed boundary. The following proposition is essentially due to [2, Theorem 6] and [4, Result 2], so we omit the proof.

Proposition 3.4. *For* $i=1, 2$, *let* (s_i, u_i) *be the solution of the Stefan problem corresponding to the data* $\{l_i, \phi_i\}$ *and the Dirichlet boundary condition* $u_i(0, t)= f_i(t)$, *where* $f_i(t) \geqq 0$ *and* $f_i(t) \in C(]0, \infty[)$. *Suppose that*

$$0 \leqq l_1 \leqq l_2, \quad \phi_1(x) \leqq \phi_2(x) \qquad (0<x<l_1), \tag{3.7}$$

(3.8) $\quad f_1(t) \leqq f_2(t) \qquad (t>0)$.

Then we have

(3.9) $\quad s_1(t) \leqq s_2(t) \qquad (t>0)$,

(3.10) $\quad u_1 \leqq u_2 \qquad (0 \leqq x \leqq s_1(t), \quad t>0)$.

In particular if $\phi_1(x) \equiv \phi_2(x) \equiv 0$, $f_1(t) \equiv f_2(t)$, *then we have*

$$s_2(t) \leqq s_1(t) + (l_2^2 - l_1^2)^{1/2}.$$

The following result is due to [2, p. 13].

Proposition 3.5. *Let* (s, u) *be the unique solution of the Stefan problem corresponding to the data* $l=0$ *and the Dirichlet boundary conditoin* $u(0, t) \equiv \rho > 0$ $(t>0)$. *Then*

$$s(t) = \beta t^{1/2},$$

where β *is the unique solution of* $\rho = \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \beta^{2n}$.

The aymptotic behavior of the solution of the Stefan problem with the Dirichlet boundary condition is shown in [3]. We state [3, Theorem 7].

Proposition 3.6. *Let* (s, u) *be the solution of the Stefan problem corresponding to the data* $\{l, \phi\}$ *and the Dirichlet boundary condition* $u(0, t) = f(t)$, *where* $f(t) \geqq 0$ *and* $f(t) \in C(]0, \infty[)$. *Suppose that*

$$\lim_{t \to \infty} f(t) = \rho > 0.$$

Then we have

$$\lim_{t \to \infty} s(t)/\sqrt{t} = \beta,$$

where β *is the unique solution of* $\rho = \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \beta^{2n}$.

4. Proof of Theorem 1 and Theorem 2

In this section we give the proof of Theorem 1 and Theorem 2.

Proof of Theorem 1. We can get the existence and the uniqueness of the solution of (S) from [6, Theorem 1 and Theorem 2]. We have (1.1) and (1.2) using (0.5) and Proposition 2.3. We shall show (1.3). Let us fix $\sigma > 0$. Since $u \in C^{\infty}(D^s)$ and

$$\int_{\sigma/2}^{\sigma} \int_0^{s(t)} u_{xx}(x, t)^2 dx dt < \infty,$$

there exists a time $t_0 \in \,]\sigma/2, \sigma]$ such that $u(\cdot, t_0)$ satisfies the assumption (A.2) and (H.1) introduced in [6, §6 and §8]. We regard (s, u) as the solution of (S) with the initial time t_0 and the initial data $\{s(t_0), u(\cdot, t_0)\}$. Thus (s, u) is constructed by the finite difference method stated in [6, §2]. Hence we see that there exists a constant C'_{t_0} depending on t_0 such that

$$|u(x, t)-u(x', t)| \leqq C'_{t_0}|x-x'| \qquad (t \geqq t_0)$$

from [6, Lemma 6.2]. Therefore we get (1.3). q.e.d.

We shall prove Theorem 2. We note that we know the existence of the solution of (S) under the assumption that $l=0$, $D(\gamma)\supset[0, H]$ and $\gamma(0)\subset]-\infty, 0[$ by [6, Theorem 3]. Hence we may treat the case $D(\gamma)\not\ni 0$.

We define a maimxal monotone graph $\hat{\gamma}$ in $\boldsymbol{R}^2$ by

$$(4.1) \qquad \hat{\gamma}(v) = \begin{cases} \{\max(z, -1);\ z\in\gamma(v)\} & \text{for } \ v\in D(\gamma), \\ -1 \quad \text{for} \quad v\leqq \inf D(\gamma). \end{cases}$$

Hence $D(\hat{\gamma})\supset[0, H]$, $\hat{\gamma}(0)\subset]-\infty, 0[$ and $\hat{\gamma}^+(v)\geqq\gamma^-(v)$ for any $v\in\boldsymbol{R}$. Let (ρ, v) be the solution of (S) with the initial data $\rho(0)=0$ and the boundary condition $v_x(0, t)\in\hat{\gamma}(v(0, t))$. This is well-defined, and we get $\rho(t)\in C^{0,1}([0, \infty[)\cap C^\infty(]0, \infty[)$, $\rho(0)=0$ and $\rho(t)>0$ for $t>0$ by [6, Theorem 3].

Consider the sequence $\{(s^m, u^m)\}_m$ of the solution of the Stefan problem (S) corresponding to the data $\{(1/m, 0)\}_m$. The sequence $\{s^m\}$ is a monotonically decreasing one of increasing functions in t by Proposition 3.2 and Theorem 1. The sequence $\{u^m\}$ is a monotonically decreasing sequence of continuous functions by Proposition 3.2. Set $s(t)=\lim_{m\to\infty} s^m(t)$ and $u(x, t)=\lim_{m\to\infty} u^m(x, t)$. We shall show that (s, u) is the solution of (S).

Lemma 4.1. *For any $m\in\boldsymbol{N}$ we have*

$$(4.2) \qquad 0\leqq u^m(x, t)\leqq H \qquad (0\leqq x\leqq s^m(t), \quad t\geqq 0),$$

$$(4.3) \qquad \rho(t)\leqq s^m(t)\leqq \beta t^{1/2}+m^{-1/2} \qquad (t\geqq 0)$$

where β is the unique solution of $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}\beta^{2n}=H$

Proof. We have (4.2) by Theorem 1 easily. We get $\rho\leqq s^m$ by Proposition 3.2. We shall show the right side of (4.3). Let (ρ_m, v_m) be the solution of the Stefan problem with the initial condition $\{1/m, 0\}$ and the Dirichlet boundary condition $v_m(0, t)\equiv H$. Then we have $s^m(t)\leqq\rho_m(t)$ using (4.2) and Proposition 3.4. On the other hand we get $\rho_m(t)\leqq\beta t^{1/2}+m^{-1/2}$ by Proposition 3.4 and 3.5. Hnece we complete the proof. q.e.d.

Lemma 4.2. $s\in C([0, \infty[)\cap C^{0,1}(]0, \infty[)$, $s(t)>0$ *for* $t>0$, *and as* $m\to\infty$

$$(4.4) \qquad s^m(t)\to s(t) \text{ uniformly on any compact subset of }]0, \infty[.$$

Proof. We have by (4.3)

$$\rho(t)\leqq s(t)\leqq\beta t^{1/2}\qquad(t\geqq 0)\,.$$

Hence $s(t)$ is continuous at $t=0$ and $s(t)>0$ for $t>0$. It follows from $s^m\geqq\rho$, $u^m\leqq H$, Proposition 2.3 and Proposition 2.4 arguing as in §13 of [6] that

$$0\leqq u^m(x,\,t)\leqq K_\sigma(s^m(t)-x)\qquad(0\leqq x\leqq s^m(t),\quad t\geqq\sigma)\,,$$

where $K_\sigma=H/\rho(\sigma)$. Since $u^m(s^m(t),\,t)=0$, we have

$$0\leqq -u_x^m(s^m(t),\,t)\leqq K_\sigma\qquad(t\geqq\sigma)\,.$$

Hence we have for all $m\in\boldsymbol{N}$,

$$0\leqq\dot{s}^m(t)\leqq K_\sigma\qquad(t\geqq\sigma)\,.$$

Consequently we get $s\in C^{0,1}(]0,\,\infty[)$ and (4.4) using Ascoli-Arzela's theorem.

q.e.d.

Lemma 4.3. *$(s,\,u)$ is the solution of (S).*

Proof. We see that $u\in C(\bar{D}-\{0,\,0\})\cap C^\infty(D)$ and u satisfies (0.1), (0.2), (0.3) using Lemma 4.2 and the proof of Proposition 2.1. We can investigate the Stefan's condition (0.5) and $u\in C^\infty(D^s)$ using the arguments of the last part of [6, §13]. q.e.d.

Consequently we have shown the existence of the solution of (S) when $l=0$. We can show (1.1), (1.2), (1.3) easily. The uniqueness is obtained from Proposition 3.3. Thus we complete the proof of Theorem 2.

5. Unilateral problem

We investigate the asymptotic behavior of the solution of a parabolic unilateral problem (P_k) which are useful to get the several estimates for the solution of (S).

$$(P_k)\qquad\begin{cases} v_{xx}-v_t=0 & (0<x<k,\quad t>0)\,,\\ v_x(0,\,t)\in\gamma(v(0,\,t)) & (t>0)\,,\\ v(k,\,t)=0 & (t>0)\,,\\ v(x,\,0)=\psi(x) & (0<x<k)\,.\end{cases}$$

Here γ is a maximal monotone graph in $\boldsymbol{R}^2$ such that $\gamma^{-1}(0)$ is not the empty set, and k is a positive parameter. We put $H=\mathrm{Proj}_{\gamma^{-1}(0)}\,0$.

Lemma 5.1. *Let ψ be bounded and continuous for a.e. $x\in[0,\,k]$. Then there exists a unique solution $v^k(x,\,t)$ of (P_k). In addition*

(5.1) $\quad |v^k| \leqq \max(|H|, \|\psi\|_{L^\infty(0,k)}) \qquad (0\leqq x\leqq k, \quad t>0)$,

(5.2) $\quad |v^k(x', t)-v^k(x, t)| \leqq C_{k,\sigma}|x'-x|^{1/2} \qquad (0\leqq x'\leqq x\leqq k, \quad t\geqq\sigma)$,

where $C_{k,\sigma}$ is a constant depending on k and σ.

Proof. It is obvious from Proposition 2.1. q.e.d.

We introduce an elliptic unilateral problem (E_k) corresponding to (P_k),

$$(E_k) \qquad \begin{cases} (5.3) \quad w_{xx} = 0 \qquad (0<x<k), \\ (5.4) \quad w_x(0)\in\gamma(w(0)), \\ (5.5) \quad w(k) = 0. \end{cases}$$

Lemma 5.2. *There exists a unique solution $w^k(x)$ of (E_k) such that $w^k(x)=a_k(1-x/k)$, where $a_k=(I+k\gamma)^{-1}(0)$.*

Proof. $w^k(x)$ is a linear function by (5.3). Hence we can put $w^k(x)=a_k(1-x/k)$. Substituting w^k in (5.4), we have $-a_k/k\in\gamma(a_k)$. Hence $a_k=(I+k\gamma)^{-1}(0)$. q.e.d.

The following result is essentially due to [1, Theorem 3.9]. However we give the proof of it for the sake of completeness.

Proposition 5.1. *Let $v^k(x, t)$ and $w^k(x)$ be the solution of (P_k) and (E_k) respectively. Then we have*

$$\lim_{t\to\infty} v^k(x, t) = w^k(x) \qquad \textit{uniformly on} \quad [0, k].$$

Proof. We put $k'=k-d$, $a=2k^{-2}$, $u=u^k$ and $v=v^k$. We have

$$\begin{aligned} 2^{-1}\frac{d}{dt}\int_d^{k'} (v(x, t)-w(x))^2dx &= \int_d^{k'} (v-w)(v_t-w_t)dx \\ &= \int_d^{k'} (v-w)(v_{xx}-w_{xx})dx \\ &= [(v-w)(v_x-w_x)]_d^{k'} - \int_d^{k'} (v_x-w_x)^2dx \end{aligned}$$

by $v_{xx}=v_t$ and $w_{xx}=w_t\,(=0)$. Hence we get

$$\begin{aligned} &\frac{d}{dt}\{e^{at}2^{-1}\int_d^{k'} (v(x, t)-w(x))^2dx\} \\ &= e^{at}[(v-w)(v_x-w_x)]_d^{k'} \\ &\quad -e^{at}\int_d^{k'} (v_x-w_x)^2dx+e^{at}2^{-1}a\int_d^{k'} (v-w)^2dx. \end{aligned}$$

Integrate over $[\sigma, t]$ and let $d\to 0$, then we have

$$e^{at}2^{-1}\int_0^k (v(x, t)-w(x))^2dx-e^{a\sigma}2^{-1}\int_0^k (v(x, \sigma)-w(x))^2dx \leqq 0 ,$$

where we used $v(k, t)=w(k)=0$, $v_x(0, t)\in\gamma(v(0, t))$ (a.e. t), $w(0)\in\gamma(w(0))$, the monotonicity of γ, (1.3) and the Poincare's inequality $\int_0^k (v_x-w_x)^2dx \geqq a\int_0^k (v-w)^2dx$. Letting $\sigma\to 0$, we obtain

$$\int_0^k (v(x, t)-w(x))^2dx \leqq e^{-at}\int_0^k (\psi(x)-w(x))^2dx .$$

Thus $v(\cdot, t)\to w(\cdot)$ in $L^2(0, k)$ as $t\to\infty$. Hence we get the conclusion using (5.1), (5.2) and Ascoli-Arzelà's theorem. q.e.d.

We prepare a simple lemma concerning the maximal monotone graph and state a useful proposition.

Lemma 5.3. $\lim_{k\to\infty}(I+k\gamma)^{-1}(0)=H$

Proof. We put $a_k=(I+k\gamma^{-1})(0)$. It follows from $\gamma(a_k)\ni -a_k/k$, $\gamma(H)\ni 0$ and the monotonicity of γ that

(5.6) $\quad a_k(a_k-H)\leqq 0$.

Hence we get $|a_k|\leqq|H|$. Thus there exists a real number a^* and the subsequence $\{a_{k_j}\}_j$ such that $a_{k_j}\to a^*$ as $j\to\infty$. Therefore we have $\gamma(a^*)\ni 0$ using $\gamma(a_k)\ni -a_k/k$ and the closedness of γ. Consequently $a^*\in\gamma^{-1}(0)$ and $|a^*|\leqq|H|$. Thus $a^*=H$ by the definition of H. Hence we get $a_k\to H$ without taking the subsequence. q.e.d.

Proposition 5.2. $\lim_{k\to\infty} w^k(x)=H$ $(0<x<\infty)$

Proof. It is obvious from $w^k(x)=a_k-(a_k/k)x$ $(0\leqq x\leqq k)$ and $\lim_{k\to\infty} a_k=H$. q.e.d.

We state a simple lemma related to Lemma 5.3.

Lemma 5.4. *Let* $H>0$. *Then for* $k>0$

$$0<(I+k\gamma)^{-1}(0)\leqq H .$$

Proof. we put $a_k=(I+k\gamma)^{-1}(0)$. We can get (5.6). Hence we have $0\leqq a_k\leqq H$. Assume that $a_k=0$. Then we get $\gamma(0)\ni 0$, which is in contradiction to $H>0$. q.e.d.

Lemma 5.5. *Let* $H=0$. *Then for* $k>0$

$$(I+k\gamma)^{-1}(0) = 0$$

Proof. It is obvious from $\gamma(0)\ni 0$. q.e.d.

6. Signorini type's problem

We investigate the asymptotic behavior of the solution of a Stefan problem with a Signorini type's boundary condition on the fixed boundary.

Let us fix $H>0$. We introduce a maximal monotone graph γ_H in $\boldsymbol{R}^2$ such that $\gamma_H(v)=]-\infty, 0]$ for $v=H$, $=0$ for $v>H$. We call the boundary condition $u_x(0, t)\in\gamma_H(u(0, t))$ of Signorini type. We shall show the following proposition.

Proposition 6.1. *Let (s^H, u^H) be the solution of the Stefan problem (S) with $\gamma=\gamma_H$. Then there exists $t^*\geqq 0$ such that*

$$u^H(x, t)\leqq H \qquad (t\geqq t^*)\,.$$

We give several lemmas to prove the proposition above. We put $l^1=l+1$ and $\psi^1(x)=-M(x-l_1)$ $(0\leqq x\leqq l^1)$, where $M=\max(\|\phi\|_{L^\infty(0,l)}, H)$. Let (s, u) be the solution of the Stefan problem (S) with $\gamma=\gamma_H$ corresponding to the initial data $\{l^1, \psi^1\}$. We get the following lemma by Proposition 3.2.

Lemma 6.1. $s^H\leqq s.$ $u^H\leqq u.$

Lemma 6.2. $\lim_{t\to\infty} s^H(t) = \lim_{t\to\infty} s(t) = \infty.$

Proof. We get $u^H(0, t)\geqq H$ $(t>0)$ from $u_x^H(0, t)\in\gamma_H(u^H(0, t))$. Hence we have the conclusion using Proposition 3.4, Proposition 3.6 and Lemma 6.1.

q.e.d.

We estimate u instead of u^H, since we can get several estimates of the former owing to the simplicity of the initial data.

Lemma 6.3. $u_x(x, t)\leqq 0 \qquad (0<x<s(t), \quad t\geqq 0).$

Proof. We note that the initial data $\{l^1, \psi^1\}$ satisfies the condition (H.1) in [6, §8]. Hence (s, u) is constructed as the limit of the solutions $\{(s_n, u_j^n)\}$ of the difference equation introduced in [6, §2]. Consequently we shall show $u_{jx}^n\leqq 0$ to get the conclusion. It is easily seen that $(u_x^{nj})_{x\bar{x}}-(u_{jx}^n)_{\bar{t}}=0$, $u_{jx}^0=-M\leqq 0$, $u_{0x}^n(\in\gamma_H(u_0^n))\leqq 0$, and $u_{J_n-1,x}^n=u_{J_n\bar{x}}^n\leqq 0$ by [6, Lemma 5.2]. Hence we have $u_{jx}^n\leqq 0 (0\leqq j\leqq J_n, n\geqq 0)$ using the maximum principle. q.e.d.

Lemma 6.4. *There exists $t^*\geqq 0$ such that*

$$u(0, t^*) = H\,.$$

Proof. We note that $u(0, t)\geqq H$ $(t>0)$ by $u_x(0, t)\in\gamma_H(u(0, t))$, Assume that $u(0, t)>H$ (for any $t>0$). Then $u_x(0, t)=0$ (a.e. $t>0$). Thus we get $s(t)\leqq l'+\int_0^{l'}\psi(x)dx$ $(t\geqq 0)$ using (3.1). This is in contradiction to Lemma 6.2.

q.e.d.

Lemma 6.5. $u(x, t)\leqq H \qquad (t\geqq t^*).$

Proof. We get $u(x, t^*)\leqq H$ by Lemma 6.3 and 6.4. We regard (s, u) as the solution of (S) with $\gamma=\gamma_H$ corresponding to the initial time t^* and the initial data $\{s(t^*), u(\cdot, t^*)\}$. Hence we have the conclusion by $u(\cdot, t^*)\leqq H$ and (1.2). q.e.d.

Thus we complete the proof of Proposition 6.1 by Lemma 6.1 and 6.5.

7. Proof of Theorem 3

We shall show the proof of Theorem 3. We assume that $H>0$ in this section.

Let (s, u) be the solution of (S).

Lemma 7.1. $\lim_{t\to\infty} s(t)=\infty$

Proof. Let us fix $\sigma>0$. Then we have

$$s(t)\geqq s(\sigma)\equiv k>0 \qquad (t\geqq\sigma).$$

We regard $u(x, t)$ $(t\geqq\sigma)$ as the solution of the moving boundary problem (M) with the moving boundary $s(t)$ $(t\geqq\sigma)$ corresponding to the initial time σ and the initial data $u(\cdot, \sigma)$. Let $v^k(x, t)$ $(0\leqq x\leqq s(\sigma), t\geqq\sigma)$ be the solution of the parabolic unilateral problem (P_k) corresponding to the initial time σ and the initial data $u(\cdot, \sigma)$. Thus it follows from Proposition 2.2 that

$$(7.1) \qquad u(x, t)\geqq v^k(x, t) \qquad (0\leqq x\leqq k, \quad t\geqq\sigma).$$

Hence we get by Proposition 5.1

$$(7.2) \qquad \liminf_{t\to\infty} u(x, t)\geqq w^k(x) \qquad (0\leqq x\leqq k),$$

where $w^k(x)$ is the solution of the elliptic unilateral problem (E_k). In particular we have

$$(7.3) \qquad \liminf_{t\to\infty} u(0, t)\geqq w^k(0) = (I+k\gamma)^{-1}(0)>0$$

by Lemma 5.2 and Lemma 5.4. Consequently we obtain the conclusion using (7.3) and Proposition 3.4 and 3.5.

Lemma 7.2. $\liminf_{t\to\infty} u(x, t)\geqq H \qquad (0<x<\infty).$

Proof. It follows from Lemma 7.1 that for any $k>l$ there exists $\sigma>0$ such that $s(\sigma)=k$. Thus we get by repeating the proof of Lemma 7.1,

$$(7.2) \qquad \liminf_{t\to\infty} u(x, t)\geqq w^k(x) \qquad (0\leqq x\leqq k).$$

Hence we get the conclusion by Proposition 5.2. q.e.d.

We shall show the estimate from above.

Lemma 7.3. $\limsup_{t\to\infty} u(x, t) \leqq H \qquad (0<x<\infty)$.

Proof. Let (s^H, u^H) be the solution of the Stefan problem (S) with $\gamma=\gamma_H$, where γ_H is the maximal monotone graph defined in §6. It follows from Proposition 3.2 and the definition of γ_H that

$$u(x, t) \leqq u^H(x, t) \qquad (0 \leqq x \leqq s(t), \quad t \geqq 0)\,.$$

Hence we get the conlcusion by Proposition 6.1. q.e.d.

Consequently we get (1.4) using Lemma 7.2, Lemma 7.3, (1.2), (1.3) and Ascoli-Arzelà's theorem. We obtain (1.5) from (1.4) and Proposition 3.6. Thus we complete the proof of Theorem 3.

8. Proof of Theorem 4

We shall show the proof of Theorem 4. We assume that $H=0$ in this section.

Let (s, u) be the solution of (S).

Lemma 8.1. $s(t) \leqq l + \int_0^l \phi(x)dx \qquad (t \geqq 0)\,.$

Proof. We shall show

(8.1) $\qquad u_x(0, t) \geqq 0 \qquad (\text{a.e. } t>0)\,.$

We note $u \geqq 0$ by (1.2). Consider the case $u(0, t)>0$. We have (8.1) by virtue of (0.2) and $H=0$. For the case $u(0, t)=0$, we get (8.1) by $u(x, t) \geqq 0$ $(x>0)$. Thus we obtain the conclusion using $u \geqq 0$, (8.1) and (3.1). q.e.d.

Lemma 8.2. *There exists s^* such that*

$$\lim_{t\to\infty} s(t) = s^*\,,$$
$$l \leqq s^* \leqq l + \int_0^l \phi(x)dx\,.$$

Proof. Since $s(t)$ is increasing in t, we get the conclusion by Lemma 8.1. q.e.d.

Lemma 8.3.

$$\lim_{t\to\infty} u(x, t) = 0 \qquad \textit{uniformly on} \quad [0, s^*]\,,$$

where we define $u=0$ $(s(t)\leqq x\leqq s^*,\ t>0)$.

Proof. We put $k=s^*$. Let v^k be the solution of the parabolic unilateral problem (P_k) corresponding to the initial data $\phi(x)$. Then it follows from Lemma 8.2 and Proposition 2.2 that

$$(8.2)\qquad 0\leqq u(x,t)\leqq v^k(x,t)\qquad (0\leqq x\leqq s^*)\,.$$

On the other hand it follows from Proposition 5.1 that

$$(8.3)\qquad \lim_{t\to\infty} v^k(x,t)=w^k(x)\qquad \text{uniformly on}\quad [0,k]\,,$$

where $w^k(x)$ is the solution of the elliptic unilateral problem (E_k). We see that $w^k(x)\equiv 0$, since $(I+k\gamma)^{-1}(0)=0$ holds by Lemma 5.5. Consequently we get the conclusion from (8.2) and (8.3). q.e.d.

Thus we complete the proof of Theorem 4 by Lemma 8.2 and 8.3.

References

[1] H. Brézis: Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert, Math. Studies, 5, North Holland, 1973.

[2] J.R. Cannon and C.D. Hill: *Existence, uniqueness, stability, and monotone dependence in a Stefan problem for the heat equation,* J. Math. Mech. **17** (1967), 1–20.

[3] J.R. Cannon and C.D. Hill: *Remarks on a Stefan problem,* J. Math. Mech. **17** (1967), 433–442.

[4] J.R. Cannon, C.D. Hill and M. Primicerio: *The one-phase Stefan problem for the heat equation with boundary temperature specification,* Arch. Rational Mech. Anal. **39** (1970), 270–274.

[5] J.R. Cannon and M. Primicerio: *Remarks on the one-phase Stefan problem for the heat equation with the flux prescribed on the fixed boundary*, J. Math. Anal. Appl. **35** (1971), 361–373.

[6] S. Yotsutani: *Stefan problems with the unilateral boundary condition on the fixed boundary* I, Osaka J. Math. **19** (1982), 365–403.

[7] S. Yotsutani: *Stefan problems with the unilateral boundary condition on the fixed boundary* II, Osaka J. Math. **20** (1983), 803–844.

Department of Applied Science
Faculty of Engineering
Miyazaki University
Miyazaki 880
Japan