A NOTE ON KÄHLER MANIFOLDS OF NONNEGATIVE HOLOMORPHIC BISECTIONAL CURVATURE

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Introduction

Let M be a Kähler manifold with Kähler metric g and D be a relatively compact domain of M with smooth boundary ∂D. We assume that for some $K, \Lambda \in \mathbf{R}$,

the holomorphic bisectional curvature of $M \geq 2K$ on D,

and

$$g(S_\xi(X), X) + g(S_\xi(JX), JX) \leq 2\Lambda,$$

where S_ξ is the second fundamental form of ∂D with respect to the unit inner normal vector field ξ and X is a unit tangent vector of M such that X and JX are both tangent to ∂D (J is the complex structure of M). In this note, we shall prove the following results.

Theorem 1. *Let M and D be as above. Then D is a Stein manifold if one of the following conditions holds:*
  (1) $K=0$ *and* $\Lambda<0$,
  (2) $K<0$, $-\sqrt{-K} \leq \Lambda < 0$, *and* $i(D) < C(K, \Lambda)$,
  (3) $K<0$ *and* $\Lambda < -\sqrt{-K}$,
where $i(D) = \max_{x \in D}$ dis $(\partial D, x)$ *and* $C(K, \Lambda) = 1/\sqrt{-K}$ arctanh $(-\Lambda\sqrt{-K})$.

Theorem 2. *Let M be a Kähler manifold of non-negative holomorphic bisectional curvature. Then the following assertions are true.*
  (1) *M has no exceptional set in the sense of Grauert.*
  (2) *If there is a proper holomorphic map $\tau: M \to \bar{M}$ from M onto a complex manifold $\bar{M}$ such that τ is biholomorphic on an open dense subset of M, then τ is globally biholomorphic on M.*

Corollary. *Let M be a Kähler manifold as in Theorem 2 and D be a relatively compact domain of M. Then D is a Stein manifold if one of the following conditions holds:*

(1) *D is strongly pseudoconvex*[1].

(2) *D is locally pseudoconvex*[2] *and moreover there is a neighborhood U of ∂D and a continuous strictly plurisubharmonic function λ on $U \cap D$.*

We should mention here the three papers by Elencwajg ([1]), Suzuki ([10]) and Greene and Wu ([3]). They proved that a domain D as in Theorem 1 is Stein if $K > 0$ and $\Lambda = 0$. More generally they showed the above corollary provided M has positive holomorphic bisectional curvature and D is locally pseudoconvex. The essential ingredient in their arguments is to show that $-\log \rho_{\partial D}$ ($\rho_{\partial D} = \text{dis}(\partial D, *)$) is strictly plurisubharmonic on D. As we see their proof for this fact, the Rauch's comparison theorem is implicitly used. Recently we have obtained a general and sharp Laplacian comparison theorem which generalizes the Rauch's comparison theorem (cf. [5]). Applying the technique used in the proof of our comparison theorem, we shall show that, if a domain D satisfies one of the three conditions in Theorem 1, there is a smooth function ψ on $(0, \infty)$ such that $\psi(\rho_{\partial D})$ is strictly plurisubharmonic everywhere on D, where $\rho_{\partial D} = \text{dis}(\partial D, *)$ (cf. Main lemma in Section 1). As for Theorem 2, the first assertion was proved by Suzuki ([10]) provided M has positive holomorphic bisectional curvature and the second assertion has been conjectured by Wu ([12]).

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1. Preliminary

Certain facts and notations from Riemannian and Kähler geometry will be needed. Let M be a Kähler manifold with Kähler metric g. We write M_x for the (real) tangent space of M at x and J for the complex structure of M. We denote by ∇ the Riemannian connection and by $\text{dis}(x, y)$ the distance between two points x and y in M. Now we recall the definition of holomorphic bisectional curvature (cf. [6: p. 372]). Let R be the Riemannian curvature tensor of g with the sign convention: if X and Y are an orthonormal basis of a plane π, then the sectional curvature of π is $R(X, Y, X, Y)$. Given two J-invariant planes π_1 and π_2 in M_x, the holomorphic bisectional curvature $H(\pi_1, \pi_2)$ is defined by

$$H(\pi_1, \pi_2) = R(X, JX, Y, JY),$$

where X (resp. Y) is a unit vector in π_1 (resp. π_2). Then by Bianchi's identity

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1) i.e., for any $x \in \partial D$, there is a neighborhood U of x in M and a continuous strictly plurisubharmonic function ϕ such that $D \cap U = \{x \in U : \phi(x) < 0\}$.

2) i.e., for any $x \in \partial D$, there is a neighborhood U of x in M such that $U \cap D$ is a Stein manifold.

we have

$$H(\pi_1, \pi_2) = R(X, Y, X, Y) + R(X, JY, X, JY)$$

It will be helpful to use the concept of Hartogs of the modulus of pluri-subharmonicity of a (not necessarily smooth) function, defined as follows: Let $\phi: U \to R$ be a continuous function on an open subset U of M and x a point of U. Let $\{z_1, \cdots, z_m\}$ $(m = \dim_C M)$ be a holomorphic local coordinate system in a neighborhood V of x such that $\{(\partial/\partial z_1), \cdots, (\partial/\partial z_m)\}$ is a unitary frame at x. Let $X_0 = \sum_j a_j(\partial/\partial z_j)(x)$ be a complex unit tangent vector at x such that $X_0 = \frac{1}{2}(X - \sqrt{-1}JX)(X \in M_x, \|X\| = 1)$. Now we define an extended real number $W\phi(x; X_0)$ by

$$W\phi(x; X_0) = \lim_{r \to 0} \inf \frac{4}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \phi(\tau(\exp \sqrt{-1}\,\theta))d\theta - \phi(x) \right\},$$

where $\tau: \{\lambda \in C: |\lambda| < \varepsilon\} \to V$ is a holomorphic imbedding such that $z_i(\sigma(\lambda)) = a_i \lambda$ (cf. e.g., [1, 3, 10]) and define an extended real number $W\phi(x)$ by

$$W\phi(x) = \inf_{x_0} W\phi(x; X_0).$$

Then the following facts are known (cf. [1, 2, 3, 10, 11]).

Fact 1. *If ϕ is of class C^2 near x, we have*

$$W\phi(x; X_0) = 4 \sum_{i,j} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(x) a_i \bar{a}_j,$$

and furthermore

$$4 \sum_{i,j} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j = \nabla^2 \phi(X, X) + \nabla^2 \phi(JX, JX).$$

Fact 2. *ϕ is plurisubharmonic on U if and only if $W\phi \geq 0$ on U and ϕ is strictly plurisubharmonic on U if and only if there is a positive continuous function p on U such that $W\phi \geq p$.*

Fact 3. *Let $\tilde{\phi}$ be a continuous function near x such that $\tilde{\phi} \leq \phi$ near x and $\tilde{\phi}(x) = \phi(x)$. Then $W\tilde{\phi}(x; X_0) \leq W\phi(x; X_0)$.*

Fact 4. *Suppose there is a sequence of continuous functions $\{\phi_n\}$ converging uniformly to ϕ on U and suppose $W\phi_n \geq \varepsilon_n$, where $\{\varepsilon_n\}$ is a sequence of real numbers converging to ε, then $W\phi \geq \varepsilon$.*

Let N be a closed (imbedded) submanifold of M with real demension n $(0 \leq n \leq 2m-1)$. We write $H_y(N)$ $(y \in N)$ for the maximal J-invariant subspace of N_y (i.e., $H_y(N) = N_y \cap J(N_y)$). Let x be a point of $M \setminus N$. Suppose there is a geodesic $\sigma: [0, l] \to M$ such that $\sigma(l) = x$ and dis $(\sigma(t), N) = t$ for $t \in [0, l]$. We

denote by $\boldsymbol{P}_{\sigma(t)}$ the parallel displacement along σ from $M_{\sigma(0)}$ onto $M_{\sigma(t)}$. Let $V_1(t)$ be the J-invariant plane spanned by $\dot{\sigma}(t)$ and $J\dot{\sigma}(t)$ and $V_2(t)$ the J-invariant subspace $\boldsymbol{P}_{\sigma(t)}(H_{\sigma(0)}(N))$ of $M_{\sigma(t)}$. Put $V_3(t) =$ the orthogonal complement of $V_1(t) + V_2(t)$ in $M_{\sigma(t)}$. Now we choose a continuous function k: $[0, l] \to \boldsymbol{R}$ and a real number Λ such that for any $t \in [0, l]$ and every J-invariant plane π in $M_{\sigma(t)}$,

$$(1.1) \qquad\qquad H(\pi, V_1(t)) \geqq 2k(t)$$

and for any $X \in H_{\sigma(0)}(N)$ of norm $||X|| = 1$,

$$(1.2) \qquad\qquad g(S_{\dot{\sigma}(0)}X, X) + g(S_{\dot{\sigma}(0)}JX, JX) \geqq 2\Lambda ,$$

where $S_{\dot{\sigma}(0)}$ is the second fundamental form of N with respect to $\dot{\sigma}(0)$. Using essentially the method of the Rauch comparison theorem, we shall now give an estimate for $W\rho$ in the next lemma, where $\rho = \mathrm{dis}\,(N, *)$. For this purpose, let us introduce the three solutions f_i $(i=1, 2, 3)$ of the equations:

$$(1.3) \qquad f_1'' + 2kf_1 = 0 \quad \text{with} \quad f_1(0) = 0 \quad \text{and} \quad f_1'(0) = 1;$$

$$(1.4) \qquad f_2'' + kf_2 = 0 \quad \text{with} \quad f_2(0) = 1 \quad \text{and} \quad f_2'(0) = \Lambda;$$

$$(1.5) \qquad f_3'' + kf_3 = 0 \quad \text{with} \quad f_3(0) = 0 \quad \text{and} \quad f_3'(0) = 1 .$$

Then we have the following

Main lemma. *For any nonincreasing C^2-function ψ on $[0, l]$, the distance function $\rho = \mathrm{dis}\,(N, *)$ satisfies*

$$(1.6) \qquad W\psi(\rho)(x; X_0) \geqq (\psi'' + \psi' f_1'/f_1)(\rho(x))||X_1||^2$$
$$+ 2(\psi' f_2'/f_2)(\rho(x))||X_2||^2 + 2(\psi' f_3'/f_3)(\rho(x))||X_3||^2 ,$$

where X_0 is a complex unit tangent vector at x such that $X_0 = \frac{1}{2}(X - \sqrt{-1} \cdot JX)$ $(X \in M_x, ||X|| = 1)$, and X_i $(i=1, 2, 3)$ are V_i-component of X, respectively.

Proof. Since there is no focal point of N along $\sigma|[0, l)$, we see that f_i $(i=1, 2, 3)$ are all positive on $(0, l)$. In fact, suppose $f_i(t_0) = 0$ for some i and $t_0 \in (0, l)$. Let X be a unit tangent vector at $\sigma(t_0)$ such that $X \in V_i(t_0)$. Let $\tilde{X}(t)$ $(0 \leqq t \leqq l)$ be the parallel vector field along σ such that $\tilde{X}(t_0) = X$. Set $\tilde{X}_\perp = \tilde{X} - g(\tilde{X}, \dot{\sigma})\dot{\sigma}$ and $(J\tilde{X})_\perp = J\tilde{X} - g(J\tilde{X}, \dot{\sigma})\dot{\sigma}$. Then since N has no focal point along $\sigma|[0, t_0]$, we see that the index forms $I_{(t_0, N)}(f_i\tilde{X}, f_i\tilde{X})$ and $I_{(t_0, N)}(f_iJ\tilde{X}, f_iJ\tilde{X})$ are both positive. On the other hand, by (1.1) and (1.2), we have

$$I_{(t_0, N)}(f_i\tilde{X}, f_i\tilde{X}) + I_{(t_0, N)}(f_i J\tilde{X}, f_i J\tilde{X})$$
$$= g(S_{\dot{\sigma}(0)}(f_i\tilde{X})(0), (f_i\tilde{X})(0)) + \int_0^{t_0} ||\nabla_{\dot{\sigma}}(f_i X_\perp)||^2 - R(f_i\tilde{X}_\perp, \dot{\sigma}, f_i\tilde{X}_\perp, \dot{\sigma})dt$$
$$+ g(S_{\dot{\sigma}(0)}(f_iJ\tilde{X})(0), (f_iJ\tilde{X})(0)) + \int_0^{t_0} ||\nabla_{\dot{\sigma}}(f_i(J\tilde{X})_\perp)||^2 - R(f_i(J\tilde{X})_\perp, \dot{\sigma}, f_i(J\tilde{X})_\perp, \dot{\sigma})dt$$

$$\leq 2f_i(0)^2\Lambda + \int_0^{t_0} \delta_i (f_i')^2 - 2kf_i^2 dt$$
$$= 2f_i(0)^2\Lambda + \delta_i f_i(t_0)f_i'(t_0) - \delta_i f_i(0)f_i'(0)$$
$$= \delta_i f_i(t_0)f_i'(t_0) = 0,$$

where $\delta_i = 1$ if $i=1$ and $\delta_i = 2$ if $i=2$ or 3. This is a contradiction. Thus f_i ($i=1, 2, 3$) are all positive on $(0, l)$. Now suppose they are all positive on $(0, l]$. Let $\dot{B}_\varepsilon$ be a (sufficiently small) open ball with radius ε in M_x such that the exponential map $\exp_x$ restricted to $\dot{B}_\varepsilon$ induces a diffeomorphism between $\dot{B}_\varepsilon$ and its image $B_\varepsilon(x)$. For any tangent vector $X \in M_x$, we denote by $\tilde{X}(t)$ ($0 \leq t \leq l$) the parallel vector field along σ such that $\tilde{X}(l) = X$. Let $\alpha: [0, l] \times \dot{B}_\varepsilon \to M$ be a $2m$-parameter N-variation of σ such that (i) $\alpha(t, 0) = \sigma(t)$, (ii) $\alpha(l, X) = \exp_x X$, (iii) $(\partial\alpha/\partial s)(t, sX)|_{s=0} = (f_1(t)/f_1(l))\tilde{X}_1(t) + (f_2(t)/f_2(l))\tilde{X}_2(t) + (f_3(t)/f_3(l))\tilde{X}_3(t)$, (iv) $\alpha(0, X) \subset N$ and (v) $\alpha(t, s\dot{\sigma}(l)) = \sigma(t + (f_1(t)/f_1(l))s)$. (We see by e.g., [7: p. 25] that such a $2m$-parameter N-variation α of σ exists, taking a sufficiently small ε if necessarily.) Now define a smooth mapping $\beta: B_\varepsilon(x) \to R$ by $\beta(\exp_x X) =$ the length of the curve $t \mapsto \alpha(t, X)$. Then $\rho = \text{dis}(N, *) \leq \beta$ on $B_\varepsilon(x)$ and $\rho(x) = \beta(x)$. Hence we see by Facts 1 and 3 that

$$W\psi(\rho)(x; X_0) \geq W\psi(\beta)(x; X_0)$$
$$= \nabla^2\psi(\beta)(X, X) + \nabla^2\psi(\beta)(JX, JX)$$
$$= \psi''\{d\beta(X)^2 + d\beta(JX)^2\} + \psi'\{\nabla^2\beta(X, X) + \nabla^2\beta(JX, JX)\}$$
$$= \psi''\|X_1\|^2 + \psi'\{\nabla^2\beta(X, X) + \nabla^2\beta(JX, JX)\}.$$

Therefore in order to prove (1.6), it suffices to show that for any $X \in \dot{B}_\varepsilon$,

(1.7) $\quad \nabla^2\beta(X, X) + \nabla^2\beta(JX, JX)$
$$\leq (f_1'/f_1)(l)\|X_1\|^2 + 2(f_2'/f_2)(l)\|X_2\|^2 + 2(f_3'/f_3)(l)\|X_3\|^2.$$

In fact, fix any $X \in \dot{B}_\varepsilon$. Since the both sides of (1.7) are invariant under the rotations of $V_1(l)$, we may assume $g(X, J\dot{\sigma}(l)) = 0$. Put $Z(t) = (f_1(t)/f_1(l))\tilde{X}_1(t) + (f_2(t)/f_2(l))\tilde{X}_2(t) + (f_3(t)/f_3(l))\tilde{X}_3(t)$ and $Z_\perp(t) = Z(t) - g(Z(t), \dot{\sigma}(t))\dot{\sigma}(t)$. Since the restriction of α to $[0, l] \times \{sX: -\varepsilon < s < \varepsilon\}$ is an N-variation of σ which induces a vector field Z such that $Z(0) \in N_{\sigma(0)}$, we see by the second variational formula of arc length that

(1.8) $\quad \nabla^2\beta(X, X) = g(S_{\dot{\sigma}(0)}Z(0), Z(0)) + \int_0^l \|\nabla_{\dot{\sigma}}Z_\perp\|^2 - R(Z_\perp, \dot{\sigma}, Z_\perp, \dot{\sigma})dt$

$$= (1/f_2(l))^2 g(S_{\dot{\sigma}(0)}X_2(0), X_2(0)) + \int_0^l \|\nabla_{\dot{\sigma}}Z_\perp\|^2 - R(Z, \dot{\sigma}, Z, \dot{\sigma})dt.$$

Similarly we see that

(1.9) $\quad \nabla^2\beta(JX, JX) = (1/f_2(l))^2 g(S_{\dot{\sigma}(0)}JX_2(0), JX_2(0))$

$$+ \int_0^l \|\nabla_{\dot{\sigma}}JZ\|^2 - R(JZ, \dot{\sigma}, JZ, \dot{\sigma})dt.$$

Therefore by the assumptions (1.1) and (1.2), we have

$$\nabla^2\beta(X,\,X)+\nabla^2\beta(JX,\,JX)$$

$$\leqq 2\Lambda/f_2(l)^2+(1/f_1(l)^2)\int_0^l (f_1')^2-2kf_1^2dt||X_1||^2+2(1/f_2(l)^2)$$

$$\times\int_0^l (f_2')^2-kf_2^2dt||X_2||^2+2(1/f_3(l)^2)\int_0^l (f_3')^2-kf_3^2dt||X_3||^2$$

$$\leqq (f_1'/f_1)(l)||X_1||^2+2(f_2'/f_2)(l)||X_2||^2+2(f_3'/f_3)(l)||X_3||^2\,.$$

This shows inequality (1.7). Now we assume $f_1(l)f_2(l)f_3(l)=0$. Then we choose a family $\{k_\delta\}_{\delta>0}$ of continuous functions on $[0,\,l]$ such that $k_\delta<k$ and $\lim_{\delta\to 0} k_\delta=k$. Let $f_{i,\delta}$ $(i=1,\,2,\,3)$ be, respectively, the solutions of the equations (1.3), (1.4) and (1.5) defined by k_δ. Then the Strum's theorem tells us that $f_{i,\delta}>f_i$ $(i=1,\,2,\,3)$ so that $f_{i,\delta}$ $(i=1,\,2,\,3)$ are all positive on $(0,\,l]$. Therefore by the preceding argument, we see that

$$(1.10)\qquad W\psi(\rho)(x;\,X_0)\geqq(\psi''+\psi'f_{1,\delta}'/f_{1,\delta})(l)||X_1||^2+(\psi'f_{2,\delta}'/f_{2,\delta})(l)||X_2||^2$$
$$+(\psi'f_{3,\delta}'/f_{3,\delta})(l)||X_3||^2(l=\rho(x))\,.$$

Since the right-hand side of (1.10) tends to that of (1.6) as $\delta\to 0$, we see that (1.6) holds also in the case when $f_1(l)f_2(l)f_3(l)=0$. This completes the proof of our main lemma.

The following fact was proved by Greene and Wu (cf. [3: Theorem 1 (A)). Since we shall show the second assertion of Theorem 2 using this fact, we shall give the proof for it in this place for the convenience to the reader.

Theorem 3 (Greene and Wu). *Let M be a Kähler manifold and D be a locally pseudoconvex domain in M. Then if M has non-negative holomorphic bisectional curvature, there is a neighborhood U of ∂D in M such that $-\log\rho$ is plurisubharmonic on $U\cap D$, where $\rho=$dis $(\partial D,\,*)$.*

Proof. To prove the theorem, it suffices to show that given $y\in\partial D$, there is a neighborhood V of y in M such that $-\log\rho$ is plurisubharmonic on $V\cap D$. Let V_1 be a neighborhood of y in M such that $V_1\cap D$ is Stein so that $V_1\cap D$ can be approximated from within by strictly pseudoconvex domains $\{D_n\}$ with smooth boundary ∂D_n. For each D_n, set $\rho_n=$dis $(\partial D_n,\,*)$. Let $\varepsilon>0$ be so small that $B_{2\varepsilon}(y)\equiv\{x\in M:$ dis $(x,\,y)<2\varepsilon\}$ is relatively compact, geodesically convex and contained in V_1. Then ρ_n converges to ρ uniformly on $B_\varepsilon(y)\cap D$ as $n\to\infty$ $(B_\varepsilon(y)\equiv\{x\in M:$ dis $(x,\,y)<\varepsilon\})$. Therefore it suffices to prove that $-\log\rho_n$ is plurisubharmonic on $B_\varepsilon(y)\cap D_n$. In fact, for any $x\in B_\varepsilon(y)\cap D_n$, there is a geodesic $\sigma_n:[0,\,l_n]\to M$ such that $\sigma_n(l_n)=x$ and $\rho_n(\sigma_n(t))=t$ for $t\in[0,\,l_n]$. Applying the above main lemma $(k=0,\,\Lambda=0)$ to ρ_n, we see that $W(-\log\rho_n)(x)\geqq 0$, since $f_1(t)=t$ and $f_2(t)=1$. This shows that $-\log\rho_n$ is

plurisubharmonic on $B_\varepsilon(y) \cap D_n$ by Fact 2. This completes the proof of Theorem 3.

2. Proofs of the results in Introduction

Throughout this section, we keep the notations in Introduction.

Proof of Theorem 1. We first remark that for any $x \in D$, there is a geodesic $\sigma: [0, l] \to M$ such that $\sigma(l) = x$ and $\mathrm{dis}\,(\partial D, \sigma(t)) = t$ for $t \in [0, l]$, since the closure of D is compact (cf. e.g., the first few lines in the proof of Corollary (2.44) in [5]). Let f_1 and f_2 be, respectively, the solutions of the equations: $f_1'' + 2Kf_1 = 0$ with $f_1(0) = 0$ and $f_1'(0) = 1$, and $f_2'' + Kf_2 = 0$ with $f_2(0) = 1$ and $f_2'(0) = \Lambda$. Set $\psi(t) = \int_t^a 1/f_1^2$ $(0 < a < \inf \{s > 0: f_1(s) \leq 0\} \leq +\infty)$. Then by Main lemma, we see that the distance function $\rho = \mathrm{dis}\,(\partial D, *)$ satisfies

$$(2.1) \qquad W\psi(\rho)(x) \geq \min \{f_1'/f_1^3, -f_2'/(f_1^2 f_2)\}\,(\rho(x))$$

for every $x \in D$. Since each of the three conditions in Theorem 1 implies that $f_1' > 0$ and $f_2' < 0$ on $(0, i(D))$, we see that the right-hand side of (2.1) is positive for every $x \in D$. Therefore we see by Fact 2 that $\psi(\rho)$ is a strictly plurisubharmonic and exhaustion function on D. Thus D is a Stein manifold (cf. [8]). This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose M admits an exceptional analytic set E. Here we call E an exceptional analytic set of M if there exists an analytic space Y and a holomorphic mapping $\pi: M \to Y$ such that $\pi(E) = \{y\}$ and $\pi: M \backslash E \cong Y \backslash \{y\}$ (cf. [4: p. 284]). Then it is clear that there is a strongly pseudoconvex neighborhood D of E whose boundary is smooth and whose closure is compact. Since D is a Stein manifold by Theorem 1, the existence of E is absurd. This proves the first assertion of Theorem 2. Now we shall show the second assertion. Set $Z = \{x \in M: \mathrm{rank}_x \tau < m\}$. Suppose Z is not empty, that is, Z is an analytic hypersurface of M. Then by the proper mapping theorem $\tau(Z)$ is an analytic subset of $\bar{M}$. Set $M_0 = M \backslash Z$. Since τ is biholomorphic on an open dense subset of M, we see that $\tau^{-1}(\tau(Z)) = Z$ and τ is biholomorphic on M_0. Set $\rho = \mathrm{dis}\,(Z, *)$. Then by Theorem 3, we see that there is an open neighborhood U of $Z\,(= \partial M_0)$ such that $-\log \rho$ is plurisubharmonic on $U \backslash Z$, since M_0 is locally pseudoconvex. Therefore $\phi = (-\log \rho) \circ \tau^{-1}: \tau(M_0) \to R$ is plurisubharmonic on an open subset $\tau(U \backslash Z) = \tau(U) \backslash \tau(Z)$. Now suppose there is an irreducible component V of $\tau(Z)$ such that $\dim_C V < m - 1$. Then it is well known that ϕ has a plurisubharmonic extention to an open set $\tau(U \backslash Z) \cup \Re(V)$, where $\Re(V)$ is the regular points of V. This is absurd, since $\phi(y)$ tends to $+\infty$ as $y \to \Re(V)$. Therefore we see that $\tau(Z)$ is an analytic

hypersurface of M. Thus $\tau: Z \to \tau(Z)$ is a surjective holomorphic mapping between complex spaces of pure dimension $m-1$. Then the same argument in the proof of Lemma 2 in [12] shows that Z must be empty and hence $\tau: M \to \bar{M}$ is biholomorphic. This completes the proof of the second assertion of Theorem 2.

Proof of Corollary. The first assertion is a direct consequence of Theorem 2 (1) and Theorem 1 in [8]. Now we shall show the second assertion. Set $\rho = \mathrm{dis}\,(\partial D, *)$ and $\phi = \exp \lambda - \log \rho$. Then since $-\log \rho$ is plurisubharmonic on $V \cap D$ for some neighborhood V of ∂D, ϕ is strictly plurisubharmonic on $V \cap U \cap D$. Set $D_a = \{x \in D : \phi(x) < a\}$ $(a \in \mathbf{R})$. Then by the first assertion of the corollary, we see that D_a is a Stein manifold for large a and further D_a is Runge in $D_{a'}$, for large a and a' $(a' > a)$ (cf. [8: Theorem II]). Therefore a theorem of K. Stein tells us that D is a Stein manifold (cf. [9]). This completes the proof of Corollary.

Now we shall show a proposition which slightly extends the assertion (c) of Theorem C in [11]. Let M be a noncompact, complete Kähler manifold and a point $0 \in M$ be fixed. Let $\{C_t\}_{t \in I}$ be a family of closed subsets of M indexed by a subset I of $\mathbf{R}$. Assume $e_t \equiv \mathrm{dis}\,(C_t, 0) \to \infty$ as $t \to \infty$. The family of functions $\eta_t: M \to \mathbf{R}$ defined by $\eta_t(x) = e_t - \mathrm{dis}\,(x, C_t)$ is Lipschitz continuous (with Lipschitz constant 1) and also satisfies $|\eta_t(x)| \leq \mathrm{dis}\,(x, 0)$ (by the triangle inequality). It is thus an equi-continuous family uniformly bounded on compact sets. By Ascoli's theorem, a subsequence of $\{\eta_t\}$, to be denoted by η_n, converges to a continuous function $\eta: M \to R$, the convergence being uniform on compact sets of M. Then we have the following

Proposition. *Let M be a complete noncompact Kähler manifold. Suppose the holomorphic bisectional curvature is nonnegative everywhere on M and positive outside a compact set A of M. Then η is strictly plurisubharmonic on M for each η as above.*

REMARK. It has been proved in the theorem cited above that e^η is strictly plurisubharmonic on M.

Proof. We fix any point x of M. By the assumption, we can find a small neighborhood U of x and positive numbers α, β and γ which have the following properties: (1) $\mathrm{dis}\,(y, z) < \alpha$ for any $y \in U$ and $z \in A$; (2) $\alpha > 3\gamma$ and $3\beta\gamma^2 < 1$; (3) for any distance minimizing geodesic $\sigma: [0, l] \to M$ such that $\sigma(l) \in U$ and $l > \alpha$,

(2.2)   the holomorphic bisectional curvature at $\sigma(t) \geq k_l(t)$,

where k_l is a continuous function on $[0, l]$ defined by $k_l(t) = 0$ for $t \in [0, l-\alpha]$, $k_l(t) = \beta/\gamma \cdot (t - (l-\alpha))$ for $t \in [l-\alpha, l-\alpha+\gamma]$, $k_l(t) = \beta$ for $t \in [l-\alpha+\gamma, l-\alpha+2\gamma]$,

$k_l(t) = -\beta/\gamma \cdot (t - (l - \alpha + 2\gamma))$ for $t \in [l - \alpha + 2\gamma, l - \alpha + 3\gamma]$ and $k_l(t) = 0$ for $t \in [l - \alpha + 3\gamma, l]$. Let $f_l \colon [0, l] \to \mathbf{R}$ $(l > \alpha)$ be the solution of the equation: $f_l'' + k_l f_l = 0$ with $f_l(0) = 0$ and $f_l'(0) = 1$. Then by simple computations, we see that there are positive constants $\delta = \delta(\alpha, \beta, \gamma)$ and $l_0 = l_0(\alpha, \beta, \gamma)$, depending only on α, β and γ, such that

(2.3) $$(f_l'/f_l)(l) \leqq -\delta < 0$$

for any $l \geqq l_0$. Let y be a point of U and y_n be a point of C_n such that dis $(y, y_n) = $ dis (y, C_n). Then since $\eta_n \geqq e_n - $ dis $(y_n, *)$, and $\eta_n(y) = e_n - $ dis (y_n, y) we see by Fact 3 that

(2.4) $$W\eta_n(y) \geqq W(e_n - \text{dis}\,(y_n, *))(y).$$

We claim that for large n, the right-hand side of $(2.4) \geqq \delta > 0$. In fact, let $\sigma_n \colon [0, l_n] \to M$ be a geodesic such that $\sigma_n(y_n) = y\,(\in U)$ and dis $(\sigma_n(t), y) = t$ for $t \in [0, l_n]$. Then by (2.2), Main lemma and (2.3), we have

$$W(e_n - \text{dis}\,(y_n, *))\,(y) \geqq -(f_{l_n}'/f_{l_n})\,(l_n)$$
$$> \delta\,(> 0),$$

for large n so that $l_n \geqq l_0(\alpha, \beta, \gamma)$. This shows that for large n and any $y \in U$,

$$W\eta_n(y) > \delta\,(> 0).$$

Therefore by Fact 4, we see that η satisfies

$$W\eta \geqq \delta\,(> 0)$$

on U. Thus η is strictly plurisubharmonic on U and hence on M. This completes the proof of the proposition.

REMARK. During the submission of this paper to the journal, the author received the reprint [13] which contained among other things some extentions of Theorem 1 (1), Theorem 2 (1) and Corollary (2), respectively.

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