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# DIFFERENTIABILITY OF FAMILIES OF THE FRACTIONAL POWERS OF SELF-ADJOINT OPERATORS ASSOCIATED WITH SESQUILINEAR FORMS

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## 1. Introduction

Let $H$ and $V$ be two Hilbert spaces, $V$ being densely and continuously embedded in $H$. Let $a(\cdot, \cdot)$ be a continuous symmetric sesquilinear form defined on $V\times V$ satisfying Gårding's inequality. Then, in the usual manner, $a(\cdot, \cdot)$ defines a positive definite self-adjoint operator $A$ in $H$, and the fractional power $A^{1/2}$ of $A$ takes $V$ as the definition domain. Thus, though the domain $\mathcal{D}(A)$ itself may depend on the sesquilinear form which defines $A$, the domain $\mathcal{D}(A^{1/2})$ always coincides with $V$. Making use of this fact, we may reduce an evolution equation of the second order to one of the first order in which the domain of the operator is independent of $t$.

In fact, let

$$d^2u/dt^2+A(t)u=f(t), \qquad 0\leqq t\leqq T \tag{1.1}$$

be an evolution equation in $H$, where, for each $t$, $A(t)$ is a positive definite self-adjoint operator in $H$ associated with a continuous symmetric sesquilinear form $a(t; \cdot, \cdot)$ on $V\times V$ satisfying Gårding's inequality. Assuming that $A(\cdot)^{1/2}$ is differentiable as a function with values in $\mathcal{L}_s(V, H)$, we set

$$\begin{cases} v_0=iA(t)^{1/2}u \\ v_1=du/dt\,. \end{cases}$$

Then (1.1) will be reduced to the following evolution equation

$$\frac{d}{dt}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}=\mathfrak{A}(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}+\mathfrak{B}(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}+F(t), \qquad 0\leqq t\leqq T \tag{1.2}$$

in the product space $\begin{matrix} H \\ \times \\ H \end{matrix}$, where

$$\mathfrak{A}(t)=\begin{pmatrix} 0 & iA(t)^{1/2} \\ iA(t)^{1/2} & 0 \end{pmatrix}, \quad \mathfrak{B}(t)=\begin{pmatrix} dA(t)^{1/2}/dt\,A(t)^{-1/2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.3}$$

It is clear that

$$\mathcal{D}(\mathfrak{A}(t)) = \begin{matrix} V \\ \times \\ V \end{matrix}, \quad \mathfrak{B}(t) \in \mathcal{L}\begin{pmatrix} H \\ \times \\ H \end{pmatrix}.$$

Since $\mathcal{D}(\mathfrak{A}(t))$ is independent of $t$, we are able to apply the results of linear evolution equations in [3], [4], [5] to solve (1.2), if $\mathfrak{A}(t)$ and $\mathfrak{B}(t)$ satisfy some further smoothness hypotheses. Indeed we know

**Theorem 1.1.** *Let $E$ and $F$ be two Banach spaces such that $F$ is densely and continuously embedded in $E$. Let $\{\mathfrak{A}(t)\}_{0 \leq t \leq T}$ be a family of the infinitesimal generators of linear contraction semi-groups on $E$ such that*

$$\mathcal{D}(\mathfrak{A}(t)) = F, \qquad 0 \leqq t \leqq T,$$

*and let $\{\mathfrak{B}(t)\}_{0 \leqq t \leqq T}$ be a family of strongly continuous bounded linear operators on $E$:*

$$\mathfrak{B} \in C([0, T]; \mathcal{L}_s(E)). \tag{1.4}$$

*If $\mathfrak{A}(t)$ satisfies*

$$\mathfrak{A} \in C^1([0, T]; \mathcal{L}_s(F, E)), \tag{1.5}$$

*and if $\mathfrak{B}(t)$ satisfies one of the following conditions*

$$\mathfrak{B} \in C([0, T]; \mathcal{L}_s(F)) \tag{1.6.1}$$

$$\mathfrak{B} \in C^1([0, T]; \mathcal{L}_s(F, E)), \tag{1.6.2}$$

*then there exists a unique evolution operator $\{\mathfrak{U}(t, s)\}_{0 \leqq s \leqq t \leqq T}$ for $\{\mathfrak{A}(t) + \mathfrak{B}(t)\}_{0 \leqq t \leqq T}$.*

Remarks concerning the proof of the theorem will be made in section 4.

Let us choose $E = \begin{matrix} H \\ \times \\ H \end{matrix}$, $F = \begin{matrix} V \\ \times \\ V \end{matrix}$ and take as $\mathfrak{A}(t)$, $\mathfrak{B}(t)$ the operators defined by (1.3). Then the conditions (1.5), (1.6.1) and (1.6.2) are equivalent to

$$A^{1/2} \in C^1([0, T]; \mathcal{L}_s(V, H)) \tag{1.7}$$

$$dA^{1/2}/dt\, A^{-1/2} \in C([0, T]; \mathcal{L}_s(V)) \tag{1.8.1}$$

$$dA^{1/2}/dt\, A^{-1/2} \in C^1([0, T]; \mathcal{L}_s(V, H)) \tag{1.8.2}$$

respectively. The first object of the present paper is to give a sufficient condition to be satisfied by the form $a(t; \cdot, \cdot)$ in order that (1.7) holds. Actually we shall prove in Theorem 2.2 that, if there exists a constant $1/2 < \rho \leqq 1$ such that

$$|(\partial/\partial t)a(t; u, v)| \leqq K_\rho \|A(t)^\rho u\|_H \|A(t)^{1-\rho} v\|_H, \qquad u \in \mathcal{D}(A(t)^\rho),\ v \in V \tag{1.9}$$

holds with some constant $K_\rho$ independent of $t$, then (1.7) is satisfied. Either

(1.8.1) or (1.8.2) is also to be verified. But we see that (1.8.1) jointed with (1.7) implies that $\mathcal{D}(A(t))$ is indepednent of $t$. Indeed, suppose that (1.8.1) holds. Since (1.8.1) is equivalent to

$$A^{1/2}dA^{1/2}/dt\,A^{-1}\in C([0,\ T];\ \mathcal{L}_s(H))\,,$$

it would follow from

$$A(t)\,dA(t)^{-1}/dt = -dA(t)^{1/2}/dtA(t)^{-1/2}-A(t)^{1/2}dA(t)^{1/2}/dtA(t)^{-1}$$

that

$$AdA^{-1}/dt\in C([0,\ T];\ \mathcal{L}_s(H))\,. \tag{1.10}$$

Then the result follows immediately from (1.10). Therefore we have to verify (1.8.2) in the general case where $\mathcal{D}(A(t))$ depends on $t$ (cf. [1]).

The second object is to prove that, if $H=L_2(\Omega)$, $V=H_1(\Omega)$ and $a(t;\cdot,\cdot)$ is of the form

$$a(t;\ u,\ v)=\int_\Omega\left\{\sum_{i,j=1}^{n}a_{ij}(t,\ x)\frac{\partial u}{\partial x_i}\,\frac{\partial\bar{v}}{\partial x_j}+ku\bar{v}\right\}dx+\int_{\partial\Omega}h(t,\ \sigma)u\bar{v}d\sigma\,,$$
$$u,\ v\in H_1(\Omega)\,,$$

then the sufficient condition (1.9) is verified. We shall show by Theorem 3.1 that (1.9) is satisfied with any $1/2<\rho<3/4$, estimating $(\partial/\partial t)a(t;\ u,\ v)$ by $\|u\|_{1+\theta}\|v\|_{1-\theta}$ $(0<\theta<1/2)$ and using the fact that $\mathcal{D}(A(t)^\alpha)(0<\alpha<1)$ is continuously embedded in $H_{2\alpha}(\Omega)$.

As an application we shall consider in section 4 the Cauchy problem of a hyperbolic equation of the second order.

We here describe the notations which will be used throughout the paper. Let $E, F$ be two Banach spaces. $\mathcal{L}(E, F)$ denotes the space of all bounded linear operators from $E$ to $F$ with the operator norm $\|\cdot\|_{\mathcal{L}(E,F)}$. $\mathcal{L}_s(E, F)$ denotes the space $\mathcal{L}(E, F)$ equipped with the strong topology. $\mathcal{L}(E, E)$ (resp. $\mathcal{L}_s(E, E)$) will be abbreviated as $\mathcal{L}(E)$ (resp. $\mathcal{L}_s(E)$). $C^k([a, b];\ \mathcal{L}(E, F))$ (resp. $C^k([a, b];\ \mathcal{L}_s(E, F))$ is the set of all $k$-times continuously differentiable mapping from the interval $[a, b]$ to $\mathcal{L}(E, F)$ (resp. $\mathcal{L}_s(E, F)$). We shall write $C([a, b];\ \mathcal{L}(E, F))$ (resp. $C([a, b];\ \mathcal{L}_s(E, F)))$ instead of $C^0([a, b];\ \mathcal{L}(E, F))$ (resp. $C^0([a, b];\ \mathcal{L}_s(E, F)))$. Let $\Omega\subset\boldsymbol{R}^n$ be a region. $H_s(\Omega)$ $(s\geqq 0)$ denotes the usual Sobolev space and $\|\cdot\|_{s,\Omega}$ denotes its norm. We shall abbreviate $\|\cdot\|_{s,\Omega}$ as $\|\cdot\|_s$ if there is no fear of confusion. As usual we also use $L_2(\Omega)$ to denote the space $H_0(\Omega)$. The inner product of $L_2(\Omega)$ is denoted by $(\cdot,\ \cdot)$.

## 2. Differentiability of $A(t)^{1/2}$ (abstract results)

Let $H$ (resp. $V$) be a Hilbert space with the norm $|\cdot|$ (resp. $\|\cdot\|$) such that $V$ is densely and continuously embedded in $H$. Let $\{a(t;\ \cdot,\ \cdot)\}_{0\leqq t\leqq T}$ be a family of sesquilinear forms defined on $V\times V$. We assume that

a) $a(t;u,v)=\overline{a(t;v,u)}$, $u, v\in V$

b) $|a(t;u,v)|\leqq M_0\|u\|\,\|v\|$, $u, v\in V$

c) $a(t;u,u)\geqq\delta\|u\|^2$, $u\in V$

with some constants $M_0$ and $\delta>0$ independent of $t$.

Then, for each $0\leqq t\leqq T$, a closed linear operator $A(t)$ in $H$ is defined from $a(t;\cdot,\cdot)$ in the usual manner:

$$\begin{cases}\mathcal{D}(A(t))=\{u\in V;\ \textit{there exists } f\in H \textit{ such that}\\ \qquad\qquad\qquad a(t;u,v)=(f,v), \qquad v\in V\}\\ A(t)u=f;\end{cases}$$

owing to a) and c), $A(t)$ is a positive definite self-adjoint operator in $H$, $A(t)\geqq\delta$. It is also verified that

$$\mathcal{D}(A(t)^{1/2})=V$$

with equivalent norms (see [7], English translation, Theorem 2.2.3).

We also assume that, for each $u, v\in V, a(\cdot\,;u,v)$ is continuously differentiable in $t$ and the derivative $\dot{a}(\cdot\,;u,v)$ satisfies

d) $|\dot{a}(t;u,v)|\leqq M_1\|u\|\,\|v\|$, $u, v\in V$

e) $\lim_{t\to s}\sup_{\|v\|\leqq 1}|\dot{a}(t;u,v)-\dot{a}(s;u,v)|=0$, $u\in V$

with some constant $M_1$ independent of $t$.

Then we have

**Lemma 2.1.** *For each* $\lambda\geqq 0$

$$(\lambda+A(\cdot))^{-1}\in C^1([0,T];\mathcal{L}_s(H,V)) \tag{2.1}$$

*with the following inequalities*

$$\|(\partial/\partial t)(\lambda+A(t))^{-1}\|_{\mathcal{L}(H)}\leqq\frac{M_1}{\delta}\,\frac{1}{\lambda+\delta} \tag{2.2}$$

$$\|(\partial/\partial t)(\lambda+A(t))^{-1}\|_{\mathcal{L}(V)}\leqq\left(\frac{M_0}{\delta}\right)^{1/2}\frac{M_1}{\delta}\,\frac{1}{\lambda+\delta}\,. \tag{2.3}$$

Proof. From the equality

$$(A(t)^{-1}f-A(s)^{-1}f,g)=a(s;A(t)^{-1}f,A(s)^{-1}g)-a(t;A(t)^{-1}f,A(s)^{-1}g), \qquad f,g\in H$$

we obtain

$$((dA(t)^{-1}/dt)f,g)=-\dot{a}(t;A(t)^{-1}f,A(t)^{-1}g), \tag{2.4}$$

which, together with the hypothesis e), yields

$$dA^{-1}/dt \in C([0,\ T];\ \mathcal{L}_s(H,\ V)),$$

whence (2.1) for $\lambda=0$ follows. For $\lambda>0$, we shall repeat the same argument taking the form

$$a_\lambda(t;\ u,\ v)=a(t;\ u,\ v)+\lambda(u,\ v), \qquad u,\ v\in V$$

which defines $\lambda+A(t)$. (2.1) is then obtained from

$$(2.5) \qquad ((\partial/\partial t)(\lambda+A(t))^{-1}f,\ g)=-\dot{a}(t;\ (\lambda+A(t))^{-1}f,\ (\lambda+A(t))^{-1}g)$$

instead of (2.4).

Next we shall show the inequalities (2.2) and (2.3). (2.5) together with d) yields

$$\|(\partial/\partial t)(\lambda+A(t))^{-1}\|_{\mathcal{L}(H)}\leqq M_1\|(\lambda+A(t))^{-1}\|^2_{\mathcal{L}(H,V)}.$$

Then (2.2) follows from

$$\|(\lambda+A(t))^{-1}\|_{\mathcal{L}(H,V)}\leqq\left(\frac{1}{\delta(\lambda+\delta)}\right)^{1/2}.$$

(2.5) implies

$$(2.6) \qquad a_\lambda(t;\ (\partial/\partial t)(\lambda+A(t))^{-1}f,\ v)=-\dot{a}(t;\ (\lambda+A(t))^{-1}f,\ v), \qquad v\in V.$$

Taking $v=(\partial/\partial t)(\lambda+A(t))^{-1}f$ and using c), d), we obtain

$$\delta\|(\partial/\partial t)(\lambda+A(t))^{-1}f\|\leqq M_1\|(\lambda+A(t))^{-1}f\|.$$

Then (2.3) follows from

$$\|(\lambda+A(t))^{-1}\|_{\mathcal{L}(V)}\leqq\left(\frac{M_0}{\delta}\right)^{1/2}\frac{1}{\lambda+\delta}.$$

We may now state

**Theorem 2.2.** *In addition to the hypotheses* a)$\sim$e), *assume that there exists a constant* $1/2<\rho\leqq1$ *such that*

$$\text{f)} \qquad |\dot{a}(t;\ u,\ v)|\leqq K_\rho|A(t)^\rho u|\,|A(t)^{1-\rho}v|, \qquad u\in\mathcal{D}(A(t)^\rho),\ v\in V$$

*holds with some constant* $K_\rho$ *independent of* $t$. *Then* $A^{1/2}$ *is a strongly continuously differentiable function with values in* $\mathcal{L}_s(V,\ H)$:

$$A^{1/2}\in C^1([0,\ T];\ \mathcal{L}_s(V,\ H)).$$

Proof. We first note that the hypotheses d) and f) imply the similar inequality for any $1/2\leqq\nu\leqq\rho$:

**Lemma 2.3.** *For any* $1/2\leqq\nu\leqq\rho$

(2.7) $$|\dot{a}(t; u, v)| \leqq K_\nu |A(t)^\nu u|\, |A(t)^{1-\nu} v|, \qquad u \in \mathcal{D}(A(t)^\nu),\ v \in V$$

*with some constant $K_\nu$ determined by $K_\rho$ and $\nu$.*

Proof. Because of (2.4), f) implies

$$A(t)^\rho dA(t)^{-1}/dt\, A(t)^{1-\rho} \in \mathcal{L}(H)$$

with

$$\|A(t)^\rho dA(t)^{-1}/dt\, A(t)^{1-\rho}\|_{\mathcal{L}(H)} \leqq K_\rho\,.$$

Similarly d) implies

$$A(t)^{1/2} dA(t)^{-1}/dt\, A(t)^{1/2} \in \mathcal{L}(H)$$

with

$$\|A(t)^{1/2} dA(t)^{-1}/dt\, A(t)^{1/2}\|_{\mathcal{L}(H)} \leqq \frac{M_1}{\delta} = K_{1/2}\,.$$

Therefore, according to the Heinz inequality, we conclude that

$$A(t)^\nu dA(t)^{-1}/dt\, A(t)^{1-\nu} \in \mathcal{L}(H)$$

with

$$\|A(t)^\nu dA(t)^{-1}/dt\, A(t)^{1-\nu}\|_{\mathcal{L}(H)} \leqq K_\rho^{\frac{\nu-1/2}{\rho-1/2}} K_{1/2}^{\frac{\rho-\nu}{\rho-1/2}},$$

which conversely implies (2.7).

Generally, when $A$ is a positive definite self-adjoint operator, its fractional power is defined by means of the spectral resolution of $A$. But, in view of Lemma 2.1, the expression by the Dunford integral will often be convenient for our purposes.

According to this $A(t)^{-1/2}$ is written in the form

$$A(t)^{-1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (\lambda + A(t))^{-1} d\lambda\,.$$

Then our theorem will be equivalent to proving that *$A^{-1/2}$ is strongly continuously differentiable from $H$ to $V$*:

$$A^{-1/2} \in C^1([0, T]; \mathcal{L}_s(H, V))\,.$$

(2.1) jointed with (2.2) yields

$$A^{-1/2} \in C^1([0, T]; \mathcal{L}_s(H))$$

as well as

$$dA(t)^{-1/2}/dt = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (\partial/\partial t)(\lambda + A(t))^{-1} d\lambda\,.$$

Therefore, our next step of the proof will be to estimate the product $((dA(t)^{-1/2}/dt)f,\ A(t)^{1/2}g)$ by $|f|\,|g|$ taking $f \in H$ and $g \in V$ arbitrarily.

Owing to (2.5) this product is described as

$$\begin{aligned}&((dA(t)^{-1/2}/dt)f,\ A(t)^{1/2}g)\\ &= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\dot{a}(t;\ (\lambda+A(t))^{-1}f,\ A(t)^{1/2}(\lambda+A(t))^{-1}g)d\lambda\,.\end{aligned}$$

From f) we have

$$\begin{aligned}&|((dA(t)^{-1/2}/dt)f,\ A(t)^{1/2}g)|\\ &\leqq \frac{K_\rho}{\pi}\int_0^\infty \lambda^{-1/2}|A(t)^\rho(\lambda+A(t))^{-1}f|\,|A(t)^{3/2-\rho}(\lambda+A(t))^{-1}g|\,d\lambda\,.\end{aligned}$$

Therefore the desired estimate will follow from

**Lemma 2.4.** *For any* $0<\alpha,\ \beta<1$

$$\begin{aligned}&\int_0^\infty \lambda^{1-(\alpha+\beta)}|A(t)^\alpha(\lambda+A(t))^{-1}f|\,|A(t)^\beta(\lambda+A(t))^{-1}g|\,d\lambda\\ &\leqq L_{\alpha\beta}|f|\,|g|, \qquad f,\ g \in H\end{aligned}$$

*holds with some constant* $L_{\alpha\beta}$ *determined by* $\alpha$, $\beta$ *alone.*

Proof. It is obviously sufficient to show that

$$\int_0^\infty \lambda^{1-2\alpha}|A(t)^\alpha(\lambda+A(t))^{-1}f|^2 d\lambda \leqq L_\alpha|f|^2, \qquad f \in H$$

with some constant $L_\alpha$ determined by $\alpha$ alone. But this inequality is easily established with the aid of the spectral resolution (see [7], English translation, Theorem 4.7.2).

Since we may assume $1/2<\rho<1$ owing to Lemma 2.3, Lemma 2.4 yields

$$|((dA(t)^{-1/2}/dt)f,\ A(t)^{1/2}g)| \leqq \frac{K_\rho L_{\rho(3/2-\rho)}}{\pi}|f|\,|g|, \qquad f \in H,\ g \in V;$$

and hence we conclude that

$$dA(t)^{-1/2}/dt \in \mathcal{L}(H,\ V)$$

with

$$\|dA(t)^{-1/2}/dt\|_{\mathcal{L}(H,\ V)} \leqq \frac{K_\rho L_{\rho(3/2-\rho)}}{\pi\delta^{1/2}}\,. \tag{2.8}$$

Thus our final step is to verify

$$dA^{-1/2}/dt \in C([0,\ T];\ \mathcal{L}_s(H,\ V))\,.$$

But, since (2.1) together with (2.3) yields that

$$(dA^{-1/2}/dt)f \in C(([0, T]; V)$$

if $f \in V$, this follows from (2.8) and the density of $V$ in $H$.

Next, we further assume that, for each $u, v \in V$, $a(\cdot; u, v)$ is twice continuously differentiable and the second derivative $\ddot{a}(\cdot; u, v)$ satisfies

g) $|\ddot{a}(t; u, v)| \leqq M_2 \|u\| \|v\|, \qquad u, v \in V$

h) $\lim_{t \to s} \sup_{\|v\| \leqq 1} |\ddot{a}(t; u, v) - \ddot{a}(s; u, v)| = 0, \qquad u, v \in V$

with some constant $M_2$ independent of $t$.

Then we have

**Theorem 2.5.** *Under the hypotheses* a)~h), $A^{-1/2}$ *is a twice strongly continuously differentiable function with values in* $\mathcal{L}_s(V)$:

$$A^{-1/2} \in C^2([0, T]; \mathcal{L}_s(V)).$$

Proof. The assertion of the theorem is an immediate consequence of the following lemma.

**Lemma 2.6.** *For each* $\lambda \geqq 0$

$$(\lambda + A(\cdot))^{-1} \in C^2([0, T]; \mathcal{L}_s(H, V)) \tag{2.9}$$

*with the inequality*

$$\|(\partial^2/\partial t^2)(\lambda + A(t))^{-1}\|_{\mathcal{L}(V)} \leqq \left(\frac{M_0}{\delta}\right)^{1/2} \left\{2\left(\frac{M_1}{\delta}\right)^2 + \frac{M_2}{\delta}\right\} \frac{1}{\lambda + \delta}. \tag{2.10}$$

Proof. The first assertion (2.9) follows from h). The second inequality (2.10) is obtained from

$$\begin{aligned} &a_\lambda(t; (\partial^2/\partial t^2)(\lambda + A(t))^{-1} f, v) \\ &= -2\dot{a}(t; (\partial/\partial t)(\lambda + A(t))^{-1} f, v) - \ddot{a}(t; (\lambda + A(t))^{-1} f, v) \end{aligned}$$

which follows from (2.6).

Therefore, as a corollary we conclude:

**Corollary 2.7.**

$$dA^{1/2}/dt A^{-1/2} \in C^1([0, T]; \mathcal{L}_s(V, H)).$$

Proof. Because of

$$dA(t)^{1/2}/dt A(t)^{-1/2} = -A(t)^{1/2} dA(t)^{-1/2}/dt, \qquad 0 \leqq t \leqq T,$$

this is a direct consequence of Theorem 2.2 and Theorem 2.5.

## 3. Differentiability of $A(t)^{1/2}$ (concrete results)

Let $\Omega$ be a region in $\boldsymbol{R}^n$ with the infinitely differentiable compact boundary $\partial\Omega$, and let $[0, T]$ be a closed interval. We take $H=L_2(\Omega)$, $V=H_1(\Omega)$ and set

$$a(t; u, v) = \int_\Omega \left\{\sum_{i,j=1}^n a_{ij}(t, x)\frac{\partial u}{\partial x_i}\frac{\partial \bar{v}}{\partial x_j} + ku\bar{v}\right\}dx + \int_{\partial\Omega} h(t, \sigma)u\bar{v}d\sigma\,,$$
$$u, v\in H_1(\Omega)\,,$$

where $a_{ij}$ is a real-valued funciton defined on $[0, T]\times\bar{\Omega}$, $h$ is a real-valued function defined on $[0, T]\times\partial\Omega$, and $k$ is a real number.

We would like to prove that $a(t; \cdot, \cdot)$ satisfies all the conditions a)~h) mentioned in section 2, assuming that

1) $a_{ij}\in\mathscr{B}^2([0, T]\times\bar{\Omega})$

2) $a_{ij}(t, x)=a_{ji}(t, x)$

3) *there exists a constant* $\delta'>0$ *such that*

$$\sum_{i,j=1}^n a_{ij}(t, x)\xi_i\xi_j\geqq\delta'|\xi|^2, \qquad \xi\in\boldsymbol{R}^n$$

4) $h\in\mathscr{B}^2([0, T]\times\partial\Omega)$,

where $\mathscr{B}^2([0, T]\times\bar{\Omega})$ (resp. $\mathscr{B}^2([0, T]\times\partial\Omega)$) is the set of all twice continuously differentiable functions defined on $[0, T]\times\bar{\Omega}$ (resp. $[0, T]\times\partial\Omega$) with bounded derivatives up to the second order.

Then it is easy to see that, if $k$ is sufficiently large, the hypotheses 1)~4) imply a)~h) except f). Thus the only thing to verify is that:

**Theorem 3.1.** *For any* $1/2<\rho<3/4$

$$|\dot{a}(t; u, v)|\leqq K_\rho\|A(t)^\rho u\|_0\|A(t)^{1-\rho}v\|_0\,, \qquad u\in\mathscr{D}(A(t)^\rho),\ v\in H_1(\Omega)$$

*holds with some constant* $K_\rho$ *independent of* $t$.

Proof. We first note that in the present case $A(t)$ can be precisely described as a differential operator with the domain in $H_2(\Omega)$. Actually we have

**Lemma 3.2.** *Let*

$$A(t, x; D) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(t, x)\frac{\partial}{\partial x_j}\right)+k\,,$$

*and let*

$$B(t, \sigma; D) = \sum_{i,j=1}^n a_{ij}(t, \sigma)\nu_i(\sigma)\frac{\partial}{\partial x_j}+h(t, \sigma)\,,$$

*where* $\nu(\sigma)=(\nu_1(\sigma), \cdots, \nu_n(\sigma))$ *denotes the outer normal vector at* $\sigma\in\partial\Omega$. *Then* $A(t)$ *coincides with*:

$$\begin{cases} \mathcal{D}(A(t)) = \{u \in H_2(\Omega);\ B(t, \sigma; D)u = 0 \text{ on } \partial\Omega\} \\ A(t)u = A(t, x; D)u\,. \end{cases}$$

For the proof see, for example, [6].

From this fact we derive the following inequality.

**Proposition 3.3.** *For any* $0<\theta<1/2$

$$|\dot{a}(t;\, u, v) - (d(t)A(t)u, v)| \leqq C_\theta \|u\|_{1+\theta,\Omega} \|v\|_{1-\theta,\Omega}\,, \quad u \in \mathcal{D}(A(t)),\ v \in H_1(\Omega) \tag{3.1}$$

*holds with some constant* $C_\theta$ *independent of* $t$ *and with some real-valued function* $d(t) \in \mathcal{B}^1(\Omega)$ *such that*

$$\sup_{0 \leqq t \leqq T} |d(t)|_{\mathcal{B}^1(\Omega)} < \infty, \tag{3.2}$$

*where* $|\cdot|_{\mathcal{B}^1(\Omega)}$ *denotes the norm of the space* $\mathcal{B}^1(\Omega)$.

Proof. $\dot{a}(t;\, u, v)$ is written in the form

$$\dot{a}(t;\, u, v) = \sum_{i,j=1}^{n} \int_\Omega \dot{a}_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} dx + \int_{\partial\Omega} \dot{h}(t, \sigma) u \bar{v} d\sigma\,,$$

where

$$\dot{a}_{ij}(t, x) = \frac{\partial}{\partial t} a_{ij}(t, x)\,, \quad \dot{h}(t, \sigma) = \frac{\partial}{\partial t} h(t, \sigma)\,.$$

According to the trace theorem, the inequality

$$\|\cdot\|_{s-1/2,\partial\Omega} \leqq C_1 \|\cdot\|_{s,\Omega} \tag{3.3}$$

holds for any $s>1/2$, hence we obtain

$$\begin{aligned} \left| \int_{\partial\Omega} \dot{h} u \bar{v} d\sigma \right| &\leqq C_2 \|u\|_{0,\partial\Omega} \|v\|_{0,\partial\Omega} \\ &\leqq C_3 \|u\|_{1+\theta,\Omega} \|v\|_{1-\theta,\Omega} \end{aligned}$$

with any $-1/2<\theta<1/2$. In this section $C_1, C_2, \cdots$ denote constants determined by $a_{ij}$, $h$, $\Omega$ and $\theta$, and hence they are independent of $u$, $v$ and $t$.

Next, we would like to estimate the integral

$$\int_\Omega \dot{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} dx\,, \qquad 1 \leqq i, j \leqq n\,. \tag{3.4}$$

But the right hand side of (3.1) suggests that integration by parts in a certain sense is required. Therefore it may be convenient to change (3.4) to a sum of integrals in the space $\boldsymbol{R}^n$ or $\boldsymbol{R}^n_+$ introducing a system of local neighborhoods and a partition of unity on $\bar{\Omega}$.

Let $\{U_k\}_{0\leqq k\leqq l}$ be a finite open covering of $\bar{\Omega}$ such that $U_0\subset\Omega$ and that, for each $1\leqq k\leqq l$, there exists an infinitely differentiable mapping $\pi_k$ from $U_k$ to

$$V_k=\{y;\, y=(y',y_n),\ |y'|<1,\ -1<y_n<1\}$$

such that $\pi_k^{-1}$ is also an infinitely differentiable mapping from $V_k$ to $U_k$, $\pi_k$ mapping $U_k\cap\Omega$ to $V_k\cap \boldsymbol{R}^n_+$ and $U_k\cap\partial\Omega$ to $V_k\cap\{y_n=0\}$ with the condition that

$$(3.5)\qquad (\partial y_n/\partial x_1,\cdots,\partial y_n/\partial x_n)_{|x=\sigma}=-\nu(\sigma)\,,\qquad \sigma\in U_k\cap\partial\Omega\,.$$

Let $\{\phi_k\}_{0\leqq k\leqq l}$ be a partition of unity such that, for each $0\leqq k\leqq l$, $\phi_k$ is an infinitely differentiable non-negative function with the support in $U_k$ and that

$$\sum_{k=0}^{l}\phi_k^2(x)=1\qquad\text{on } \bar{\Omega}\,.$$

Then we can write (3.4) in the form

$$(3.6)\quad \int_\Omega \dot{a}_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\bar{v}}{\partial x_j}dx=\sum_{k=0}^{l}\int_\Omega \dot{a}_{ij}\phi_k^2\frac{\partial u}{\partial x_i}\frac{\partial\bar{v}}{\partial x_j}dx$$
$$=\sum_{k=0}^{l}\int_\Omega \dot{a}_{ij}\phi_k\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}(\phi_k\bar{v})\,dx-\sum_{k=0}^{l}\int_\Omega \dot{a}_{ij}\phi_k\frac{\partial u}{\partial x_i}\frac{\partial\phi_k}{\partial x_j}\bar{v}dx\,.$$

It is clear that for any $0\leqq k\leqq l$

$$\left|\int_\Omega \dot{a}_{ij}\phi_k\frac{\partial u}{\partial x_i}\frac{\partial\phi_k}{\partial x_j}\bar{v}dx\right|\leqq C_4\|u\|_{1,\Omega}\|v\|_{0,\Omega}\,.$$

We shall first consider the case where $k=0$ in (3.6). According to Parseval's theorem we have

$$\left|\int_\Omega \dot{a}_{ij}\phi_0\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}(\phi_0\bar{v})dx\right|=\left|\int_{\boldsymbol{R}^n}\mathcal{F}\left[\dot{a}_{ij}\phi_0\frac{\partial u}{\partial x_i}\right]\xi_j\overline{\mathcal{F}[\phi_0 v]}d\xi\right|,$$

therefore with any $0<\theta<1$

$$\leqq\left\{\int_{\boldsymbol{R}^n}|\xi_j|^{2\theta}\left|\mathcal{F}\left[\dot{a}_{ij}\phi_0\frac{\partial u}{\partial x_i}\right]\right|^2 d\xi\right\}^{1/2}\left\{\int_{\boldsymbol{R}^n}|\xi_j|^{2(1-\theta)}|\mathcal{F}[\phi_0 v]|^2 d\xi\right\}^{1/2}$$
$$\leqq\left\|\dot{a}_{ij}\phi_0\frac{\partial u}{\partial x_i}\right\|_{\theta,\boldsymbol{R}^n}\|\phi_0 v\|_{1-\theta,\boldsymbol{R}^n}\,,$$

whence

$$\leqq C_5\|u\|_{1+\theta,\Omega}\|v\|_{1-\theta,\Omega}\,.$$

When $1\leqq k\leqq l$, $\pi_k$ yields

$$\int_\Omega \dot{a}_{ij}\phi_k\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}(\phi_k\bar{v})dx=\sum_{p,q=1}^{n}\int_{\boldsymbol{R}^n_+}b_{pq}\phi_k\frac{\partial u}{\partial y_p}\frac{\partial}{\partial y_q}(\phi_k\bar{v})dy\,,$$

where

$$b_{pq}=a_{ij}J_k\frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j},\quad J_k=\frac{\partial(x_1,\cdots,x_n)}{\partial(y_1,\cdots,y_n)}.$$

If $q\neq n$, using the partial Fourier transform $\mathcal{F}'$ in the variables $y'$, we have

$$\left|\int_{R_+^n}b_{pq}\phi_k\frac{\partial u}{\partial y_p}\frac{\partial}{\partial y_q}(\phi_k\bar{v})dy\right|=\left|\int_0^\infty\int_{R^{n-1}}b_{pq}\phi_k\frac{\partial u}{\partial y_p}\frac{\partial}{\partial y_q}(\phi_k\bar{v})dy'dy_n\right|$$
$$=\left|\int_0^\infty\int_{R^{n-1}}\mathcal{F}'\left[b_{pq}\phi_k\frac{\partial u}{\partial y_p}\right]\xi_q\overline{\mathcal{F}'[\phi_k v]}d\xi'dy_n\right|,$$

therefore with any $0<\theta<1$

$$\leqq\int_0^\infty\left\|b_{pq}\phi_k\frac{\partial u}{\partial y_p}(y_n)\right\|_{\theta,R^{n-1}}\|\phi_k v(y_n)\|_{1-\theta,R^{n-1}}dy_n$$
$$\leqq\left\|b_{pq}\phi_k\frac{\partial u}{\partial y_p}\right\|_{L_2(0,\infty\,;\,H_\theta(R^{n-1}))}\|\phi_k v\|_{L_2(0,\infty\,;\,H_{1-\theta}(R^{n-1}))},$$

applying the inequality

$$\|\cdot\|_{L_2(0,\infty\,;\,H_s(R^{n-1}))}\leqq\|\cdot\|_{s,R_+^n}$$

which is valid for any $s\geqq0$, we obtain

$$\leqq C_6\left\|b_{pq}\phi_k\frac{\partial u}{\partial y_p}\right\|_{\theta,R_+^n}\|\phi_k v\|_{1-\theta,R_+^n},$$

whence

$$\leqq C_7\|u\|_{1+\theta,\Omega}\|v\|_{1-\theta,\Omega}.$$

When $q=n$, by integration by parts in the variable $y_n$, we have

$$\int_{R_+^n}b_{pn}\phi_k\frac{\partial u}{\partial y_p}\frac{\partial}{\partial y_n}(\phi_k\bar{v})dy=-\int_{R_+^n}\frac{\partial}{\partial y_n}(b_{pn}\phi_k)\frac{\partial u}{\partial y_p}\phi_k\bar{v}dy$$

$$-\int_{R_+^n}b_{pn}\phi_k\frac{\partial^2u}{\partial y_n\partial y_p}\phi_k\bar{v}dy\tag{3.7}$$

$$-\int_{R^{n-1}}b_{pn}\phi_k\frac{\partial u}{\partial y_p}\phi_k\bar{v}(y',0)dy'.\tag{3.8}$$

It is easy to observe that for any $1\leqq p\leqq n$

$$\left|\int_{R_+^n}\frac{\partial}{\partial y_n}(b_{pn}\phi_k)\frac{\partial u}{\partial y_p}\phi_k\bar{v}dy\right|\leqq C_8\|u\|_{1,\Omega}\|v\|_{0,\Omega}.$$

Let us estimate (3.7) and (3.8). If $p\neq n$ in (3.7), we shall repeat the same argument as in the case where $q\neq n$, and hence we conclude that

$$\left|\int_{R^n_+} b_{pn}\phi_k \frac{\partial^2 u}{\partial y_p \partial y_n}\phi_k \bar{v}\,dy\right| \leqq C_9 \|u\|_{1+\theta,\Omega}\|v\|_{1-\theta,\Omega}$$

with any $0<\theta<1$. If $p \neq n$ in (3.8), we have

$$\int_{R^{n-1}} b_{pn}\phi_k \frac{\partial u}{\partial y_p}\phi_k \bar{v}(y', 0)\,dy' = -\int_{R^{n-1}} b_{pn}\frac{\partial \phi_k}{\partial y_p}u\phi_k \bar{v}\,dy' \tag{3.9}$$
$$+\int_{R^{n-1}} b_{pn}\frac{\partial}{\partial y_p}(\phi_k u)\phi_k \bar{v}\,dy' .$$

Since

$$\left|\int_{R^{n-1}} b_{pn}\frac{\partial \phi_k}{\partial y_p}u\phi_k \bar{v}\,dy'\right| \leqq \left\|\frac{\partial \phi_k}{\partial y_p}u\right\|_{0,R^{n-1}}\|b_{pn}\phi_k v\|_{0,R^{n-1}}$$
$$\leqq C_{10}\|u\|_{1+\theta,\Omega}\|v\|_{1-\theta,\Omega}$$

with any $-1/2<\theta<1/2$, it suffices to estimate (3.9). According to Parseval's theorem it follows that

$$\left|\int_{R^{n-1}} b_{pn}\frac{\partial}{\partial y_p}(\phi_k u)\phi_k \bar{v}\,dy'\right| = \left|\int_{R^{n-1}} \xi_p \mathcal{F}'[\phi_k u]\overline{\mathcal{F}'[b_{pn}\phi_k v]}\,d\xi'\right| ,$$

therefore with any $0<\theta<1/2$

$$\leqq \left\{\int_{R^{n-1}} |\xi_p|^{1+2\theta}|\mathcal{F}'[\phi_k u]|^2 d\xi'\right\}^{1/2}\left\{\int_{R^{n-1}} |\xi_p|^{1-2\theta}|\mathcal{F}'[b_{pn}\phi_k v]|^2 d\xi'\right\}^{1/2}$$
$$\leqq \|\phi_k u\|_{1/2+\theta,R^{n-1}}\|b_{pn}\phi_k v\|_{1/2-\theta,R^{n-1}} ,$$

the trace theorem (3.3) then yields

$$\leqq C_{11}\|\phi_k u\|_{1+\theta,R^n_+}\|b_{pn}\phi_k v\|_{1-\theta,R^n_+}$$
$$\leqq C_{12}\|u\|_{1+\theta,\Omega}\|v\|_{1-\theta,\Omega} .$$

Thus we have obtained the desired estimates of (3.7) and (3.8) in the case where $p \neq n$.

There remain two integrals now:

$$\int_{R^{n-1}} b_{nn}\phi_k \frac{\partial u}{\partial y_n}\phi_k \bar{v}\,dy', \quad \int_{R^n_+} b_{nn}\phi_k \frac{\partial^2 u}{\partial y_n^2}\phi_k \bar{v}\,dy .$$

Since $u \in \mathcal{D}(A(t))$, Lemma 3.2 implies

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+ku = A(t)u \qquad \text{in } \Omega ,$$

$$\sum_{i,j=1}^{n} a_{ij}\nu_i \frac{\partial u}{\partial x_j}+hu = 0 \qquad \text{on } \partial\Omega ,$$

therefore in the local coordinates $\partial^2 u/\partial y_n^2$ is written in the form

$$\frac{\partial^2 u}{\partial y_n^2} = \sum_{(p,q)\neq(n,n)} \tilde{b}_{pq}\frac{\partial^2 u}{\partial y_p \partial y_q} + A_k(t, y; D)u + d_k A(t)u \qquad \text{in } V_k \cap \boldsymbol{R}_+^n, \tag{3.10}$$

where

$$\tilde{b}_{pq} = \left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j}\right)d_k, \qquad (p, q)\neq(n, n),$$

$$d_k = -\left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial y_n}{\partial x_i}\frac{\partial y_n}{\partial x_j}\right)^{-1},$$

and $A_k$ is a differential operator of the first order; and in view of (3.5) $\partial u/\partial y_n$ in the form

$$\frac{\partial u}{\partial y_n} = \sum_{p=1}^{n-1} b_p \frac{\partial u}{\partial y_p} - d_k h u \qquad \text{on } V_k \cap \{y_n = 0\}, \tag{3.11}$$

where

$$b_p = \left(\sum_{i,j=1}^{n} a_{ij}\nu_i \frac{\partial y_p}{\partial x_j}\right)d_k, \qquad 1 \leqq p \leqq n-1.$$

(3.11) yields

$$\left|\int_{\boldsymbol{R}^{n-1}} b_{nn}\phi_k \frac{\partial u}{\partial y_n}\phi_k \bar{v}\,dy'\right| \leqq \sum_{p=1}^{n-1}\left|\int_{\boldsymbol{R}^{n-1}} b_{nn}b_p\phi_k \frac{\partial u}{\partial y_p}\phi_k \bar{v}\,dy'\right| + \left|\int_{\boldsymbol{R}^{n-1}} b_{nn}d_k h\phi_k^2 u\bar{v}\,dy'\right|,$$

then, as was already verified,

$$\leqq C_{13}\|u\|_{1+\theta,\Omega}\|v\|_{1-\theta,\Omega}$$

with any $0<\theta<1/2$. On the other hand (3.10) yields

$$\int_{\boldsymbol{R}_+^n} b_{nn}\phi_k \frac{\partial^2 u}{\partial y_n^2}\phi_k \bar{v}\,dy = \sum_{(p,q)\neq(n,n)}\int_{\boldsymbol{R}_+^n} b_{nn}\tilde{b}_{pq}\phi_k \frac{\partial^2 u}{\partial y_p \partial y_q}\phi_k \bar{v}\,dy$$
$$+\int_{\boldsymbol{R}_+^n} b_{nn}\phi_k(A_k u)\phi_k \bar{v}\,dy + \int_{\boldsymbol{R}_+^n} b_{nn}d_k\phi_k(A(t)u)\phi_k \bar{v}\,dy.$$

But it is now easy to see that

$$\left|\int_{\boldsymbol{R}_+^n} b_{nn}\phi_k^2(A_k u)\bar{v}\,dy\right| \leqq C_{14}\|u\|_{1,\Omega}\|v\|_{0,\Omega}$$

and that

$$\left|\int_{\boldsymbol{R}_+^n} b_{nn}\tilde{b}_{pq}\phi_k \frac{\partial^2 u}{\partial y_p \partial y_q}\phi_k \bar{v}\,dy\right| \leqq C_{15}\|u\|_{1+\theta,\Omega}\|v\|_{1-\theta,\Omega}$$

with any $0<\theta<1$ because of $(p, q)\neq(n, n)$. Finally, since the last integral is equal to

$$(b_{nn}d_k\phi_k^2J_k^{-1}(A(t)u),\ v)\,,$$

we shall take $d(t)$ as

$$\begin{aligned} d(t) &= \sum_{k=1}^{l}\Big(\sum_{i,j=1}^{n}\dot{a}_{ij}J_k\frac{\partial y_n}{\partial x_i}\frac{\partial y_n}{\partial x_j}\Big)d_k\phi_k^2J_k^{-1} \\ &= \sum_{k=1}^{l}\Big(\sum_{i,j=1}^{n}\dot{a}_{ij}\frac{\partial y_n}{\partial x_i}\frac{\partial y_n}{\partial x_j}\Big)\Big(\sum_{i,j=1}^{n}a_{ij}\frac{\partial y_n}{\partial x_i}\frac{\partial y_n}{\partial x_j}\Big)^{-1}\phi_k^2\,. \end{aligned}$$

Then it is obvious that $d(t)$ belongs to $\mathscr{B}^1(\Omega)$ and satisfies (3.2).

Thus we have demonstrated Proposition 3.3.

From (3.1) we have

$$|\dot{a}(t;\ u,\ v)| \leqq |(d(t)A(t)u,\ v)| + C_\theta\|u\|_{1+\theta}\|v\|_{1-\theta} \qquad u\in\mathscr{D}(A(t)),\ v\in H_1(\Omega)\,.$$

But, according to the Heinz inequality, the inequality

$$\|A(t)^{1/2}d(t)v\|_0 \leqq C_{16}\|A(t)^{1/2}v\|_0 \qquad v\in\mathscr{D}(A(t)^{1/2})$$

which follows from (3.2) implies

$$\|A(t)^{1-\rho}d(t)v\|_0 \leqq C_{17}\|A(t)^{1-\rho}v\|_0 \qquad v\in\mathscr{D}(A(t)^{1-\rho})$$

for any $1/2\leqq\rho\leqq 1$, hence it follows that

$$\begin{aligned} |(d(t)A(t)u,\ v)| &= |(A(t)^\rho u,\ A(t)^{1-\rho}d(t)v)| \\ &\leqq C_{17}\|A(t)^\rho u\|_0\|A(t)^{1-\rho}v\|_0 \end{aligned}$$

for any $1/2\leqq\rho\leqq 1$.

Therefore, we complete the proof of the theorem with $\rho=(1+\theta)/2$, if we show

**Lemma 3.4.** *For any* $0\leqq\alpha\leqq 1$, $\mathscr{D}(A(t)^\alpha)$ *is continuously embedded in* $H_{2\alpha}(\Omega)$:

$$\text{(3.12)} \qquad \|u\|_{2\alpha} \leqq C_\alpha\|A(t)^\alpha u\|_0\,, \qquad u\in\mathscr{D}(A(t)^\alpha)\,,$$

*with some constant* $C_\alpha$ *independent of* $t$.

Proof. Lemma 3.2 jointed with the *a priori* estimates of elliptic operators yields

$$\mathscr{D}(A(t))\subset H_2(\Omega)$$

with the inequality

$$\|u\|_2 \leqq C_{18}\|A(t)u\|_0\,, \qquad u\in\mathscr{D}(A(t))\,,$$

which shows that (3.12) is valid when $\alpha=1$. (3.12) is trivial when $\alpha=0$; $\mathscr{D}(A(t)^0)=L_2(\Omega)=H_0(\Omega)$. Then, since $\mathscr{D}(A(t)^\alpha)$ (resp. $H_{2\alpha}(\Omega)$) is obtained as the intermediate space between $\mathscr{D}(A(t))$ and $\mathscr{D}(A(t)^0)$ (resp. $H_2(\Omega)$ and $H_0(\Omega)$), (3.12) will follow for any $0<\alpha<1$ from the interpolation theorem applied to the identity mapping on $L_2(\Omega)$ (see [6]).

## 4. Application

Let us consider the Cauchy problem of a hyperbolic equation

$$(4.1)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}=\displaystyle\sum_{i=1}^n a_i(t,x)\frac{\partial^2 u}{\partial x_i\partial t}+\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i(t,x)\frac{\partial u}{\partial x_i} & \\ \qquad\qquad +c(t,x)u+f(t,x) & \text{in } (0,T)\times\Omega \\ \displaystyle\sum_{i,j=1}^n a_{ij}(t,\sigma)\nu_i(\sigma)\frac{\partial u}{\partial x_j}+h(t,\sigma)u=0 & \text{on } [0,T]\times\partial\Omega \\ u(0,x)=u_0(x) \qquad \text{in } \Omega & \\ \dfrac{\partial u}{\partial t}(0,x)=u_1(x) \qquad \text{in } \Omega & \end{cases}$$

(cf. [2]), where $a_i$ is a real-valued function defined on $[0,T]\times\bar{\Omega}$ and $a_{ij}$, $h$ are real-valued functions satisfying the hypotheses 1)~4). We also assume that $a_i$, $b_i$ and $c$ satisfy

5) $a_i\in\mathscr{B}^1([0,T]\times\bar{\Omega})$

6) $\sum_{i=1}^n a_i(t,\sigma)\nu_i(\sigma)\leqq 0$ *on* $[0,T]\times\partial\Omega$

7) $b_i$, $c\in\mathscr{B}^1([0,T]\times\Omega)$.

The equation (4.1) is rewritten in the form

$$\frac{\partial^2 u}{\partial t^2}=\left(\sum_{i=1}^n a_i(t,x)\frac{\partial}{\partial x_i}-k_1\right)\frac{\partial u}{\partial t}+\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)-k_2u$$
$$+k_1\frac{\partial u}{\partial t}+\sum_{i=1}^n\left(b_i(t,x)-\sum_{j=1}^n\frac{\partial a_{ij}}{\partial x_j}(t,x)\right)\frac{\partial u}{\partial x_i}+(c(t,x)+k_2)u+f(t,x)$$

with some constant $k_1$ such that

$$(4.2)\qquad k_1\geqq -1/2\sum_{i=1}^n\frac{\partial a_i}{\partial x_i}(t,x) \qquad \text{in } [0,T]\times\Omega$$

and with some sufficiently large constant $k_2$.

Therefore, if we define operators $A(t)$, $B(t)$ and $C(t)$ as follows:

$$\begin{cases} \mathscr{D}(A(t))=\left\{u\in H_2(\Omega);\ \displaystyle\sum_{i,j=1}^n a_{ij}(t,\sigma)\nu_i(\sigma)\frac{\partial u}{\partial x_j}+h(t,\sigma)u=0 \ \textit{on}\ \partial\Omega\right\} \\ A(t)u=-\displaystyle\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)+k_2u\,, \end{cases}$$

$$\begin{cases} \mathcal{D}(B(t)) = H_1(\Omega) \\ B(t)u = \sum_{i=1}^{n} a_i(t, x)\dfrac{\partial u}{\partial x_i} - k_1 u\,, \end{cases}$$

$$\begin{cases} \mathcal{D}(C(t)) = H_1(\Omega) \\ C(t)u = \sum_{i=1}^{n}\left(b_i(t, x) - \sum_{j=1}^{n}\dfrac{\partial a_{ij}}{\partial x_j}(t, x)\right)\dfrac{\partial u}{\partial x_i} + (c(t, x)+k_2)u\,; \end{cases}$$

then the problem (4.1) may be interpreted as the Cauchy problem of the evolution equation

$$(4.3)\quad \begin{cases} \dfrac{d^2u}{dt^2} = B(t)\dfrac{du}{dt} - A(t)u + k_1\dfrac{du}{dt} + C(t)u + f(t)\,, \qquad 0 \leqq t \leqq T \\ u(0) = u_0 \\ \dfrac{du}{dt}(0) = u_1 \end{cases}$$

in $L_2(\Omega)$.

Before proceeding to the problem (4.3), we verify some properties of the operators $A(t)$, $B(t)$ and $C(t)$. According to Lemma 3.2, $A(t)$ is associated with the form

$$a(t;\, u,\, v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial u}{\partial x_i}\frac{\partial \bar{v}}{\partial x_j} + k_2 u\bar{v} \right\} dx + \int_{\partial\Omega} h(t, \sigma) u\bar{v}\, d\sigma$$

on $H_1(\Omega)\times H_1(\Omega)$. Hence the results obtained in sections 2,3 are applicable to $A(t)$; in paticular, Theorem 2.2, Corollary 2.7 and Theorem 3.1 yield that

**Lemma 4.1.**

$$A^{1/2},\ (dA^{1/2}/dt)A^{-1/2} \in C^1([0, T];\ \mathcal{L}_s(H_1(\Omega),\ L_2(\Omega)))\,.$$

We next have

**Lemma 4.2.**

$$(4.4)\qquad \operatorname{Re}(B(t)u,\, u) \leqq 0\,, \qquad u \in H_1(\Omega)\,.$$

Proof. By integration by parts we obtain

$$\begin{aligned} 2\operatorname{Re}(B(t)u,\, u) &= 2\operatorname{Re}\left\{ \sum_{i=1}^{n}\int_\Omega a_i\frac{\partial u}{\partial x_i}\bar{u}\,dx - k_1\int_\Omega |u|^2 dx \right\} \\ &= \int_{\partial\Omega}\left(\sum_{i=1}^{n} a_i\nu_i\right)|u|^2 d\sigma - \int_\Omega \left(\sum_{i=1}^{n}\frac{\partial a_i}{\partial x_i} + 2k_1\right)|u|^2 dx\,, \end{aligned}$$

hence the lemma follows from 6) and (4.2).

Finally, we easily verify

**Lemma 4.3.**

$$B,\ C\in\mathcal{C}^1([0,\ T];\ \mathcal{L}_s(H_1(\Omega),\ L_2(\Omega)))\,.$$

Now, in view of Lemma 4.1, we set

$$\begin{cases} v_0=iA(t)^{1/2}u \\ v_1=\dfrac{du}{dt}\,. \end{cases}$$

Then (4.3) is reduced to the Cauchy problem of the evolution equation

$$(4.3)\qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}=\mathfrak{A}(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}+\mathfrak{B}(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}+\begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \qquad 0\leqq t\leqq T \\ \begin{pmatrix} v_0(0) \\ v_1(0) \end{pmatrix}=\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}$$

in the product space $\begin{matrix} L_2(\Omega) \\ \times \\ L_2(\Omega) \end{matrix}$ ,where

$$\mathfrak{A}(t)=\begin{pmatrix} 0 & iA(t)^{1/2} \\ iA(t)^{1/2} & B(t) \end{pmatrix}, \quad \mathfrak{B}(t)=\begin{pmatrix} (dA(t)^{1/2}/dt)A(t)^{-1/2} & 0 \\ -iC(t)A(t)^{-1/2} & k_1 \end{pmatrix}.$$

It is obvious from Lemma 4.1 and Lemma 4.3 that

$$\mathcal{D}(\mathfrak{A}(t))=\begin{matrix} H_1(\Omega) \\ \times \\ H_1(\Omega) \end{matrix}, \quad \mathfrak{B}\in\mathcal{C}\left([0,\ T];\ \mathcal{L}_s\begin{pmatrix} L_2(\Omega) \\ \times \\ L_2(\Omega) \end{pmatrix}\right).$$

We are now able to apply Theorem 1.1 to solve (4.5) with

$$E=\begin{matrix} L_2(\Omega) \\ \times \\ L_2(\Omega) \end{matrix}, \quad F=\begin{matrix} H_1(\Omega) \\ \times \\ H_1(\Omega) \end{matrix}.$$

Indeed Lemma 4.1 and Lemma 4.3 also imply that

$$\begin{aligned} &\mathfrak{A}\in\mathcal{C}^1([0,\ T];\ \mathcal{L}_s(F,\ E)) \\ &\mathfrak{B}\in\mathcal{C}^1([0,\ T];\ \mathcal{L}_s(F,\ E))\,. \end{aligned}$$

Thus the only thing to verify is that, for each $0\leqq t\leqq T$, $\mathfrak{A}(t)$ generates a contraction semi-group on $E$. But, as may be well known, this assertion is equivalent to:

**Proposttion 4.4.** *For each* $0\leqq t\leqq T$, $\mathfrak{A}(t)$ *is a maximal dissipative operator in* $E$.

Proof. For any $\begin{pmatrix} u \\ v \end{pmatrix} \in F$, we have

$$(\mathfrak{U}(t)\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) = i(A(t)^{1/2}u, v) + i(A(t)^{1/2}v, u) + (B(t)v, v)$$
$$= 2i \operatorname{Re}(A(t)^{1/2}u, v) + (B(t)v, v).$$

Hence it follows from (4.4) that

$$\operatorname{Re}(\mathfrak{A}(t)\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) \leqq 0, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathfrak{A}(t)),$$

which shows that $\mathfrak{A}(t)$ is dissipative. We verify that $\mathfrak{A}(t)$ is maximal from the fact that $\mathfrak{A}(t)$ has the bounded inverse

$$\mathfrak{A}(t)^{-1} = \begin{pmatrix} A(t)^{-1/2}B(t)A(t)^{-1/2} & -iA(t)^{-1/2} \\ -iA(t)^{-1/2} & 0 \end{pmatrix}$$

on $E$.

We conclude this section with noting that Theorem 1.1 is established by making use of the theorem of Kato and Kobayasi. For this theorem see [4] or [5]. We shall here follow the notations in [5].

Proof of Theorem 1.1. It is sufficient to prove that $\{-(\mathfrak{A}(t)+\mathfrak{B}(t))\}_{0 \leqq t \leqq T}$ satisfies three hypotheses of the above theorem stated in [5]. By an elementary calculation, (I) is verified from the hypothesis that $\mathfrak{A}(t)$ is the generator of a contraction semi-group and from (1.4). (II) is obvious from (1.4) and (1.5). In the case where the conditoin (1.6.1) holds, we shall take

$$S(t) = 1 + \mathfrak{A}(t), \qquad 0 \leqq t \leqq T.$$

Then (III) follows from (1.5) and (1.6.1). In the case where (1.6.2) holds, we shall take

$$S(t) = \beta + \mathfrak{A}(t) + \mathfrak{B}(t), \qquad 0 \leqq t \leqq T$$

with some constant $\beta > \sup_{0 \leqq t \leqq T} \|\mathfrak{B}(t)\|_{\mathcal{L}(E)}$. Then (III) follows from (1.5) and (1.6.2). Thus we have completed the proof of the theorem.

## Bibliography

[1] M. Ikawa: *Mixed problems for hyperbolic equations of second order*, J. Math. Soc. Japan **20** (1968), 580–608.

[2] M. Ikawa: *On the mixed problem for hyperbolic equations of second order with the Neumann boundary condition*, Osaka J. Math. **7** (1970), 203–223.

[3] T. Kato: *Linear evolution equations of "hyperbolic" type*, J. Fac. Sci. Univ. Tokyo, Sec. I, **17** (1970), 241–258.

[4] K. Kobayashi: *On a theorem for linear evolution equations of hyperbolic type,* J. Math. Soc. Japan **31** (1979), 647–654.

[5] A. Yagi: *Remarks on proof of a theorem of Kato and Kobayashi on linear evolution equations,* Osaka J. Math. **17** (1980), 233–244.

[6] J.L. Lions & E. Magenes: Problèmes aux limites non homogènes et applications I, Dunod, Paris, 1968.

[7] H. Tanabe: Evolution equations, Iwanami, Tokyo, 1975 (in Japanese). English translation: Equations of evolution, Pitman, London, 1979.

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