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ON THE UNION OF COMPACT STATISTICAL STRUCTURES

Dedicated to the memory of Professor Goro Ishii

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In this paper we give an example of two compact statistical structures whose union is not compact, thus answering a question of Pitcher in the negative. We also give necessary and sufficient condition for the union $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1 \cup \mathcal{P}_2)$ to be compact whenever $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ are compact. In the later section we show that when $\mathcal{P}_2$ consists of a single probability measure Q then compactness of $(\mathcal{X}, \mathcal{A}, \mathcal{P} \cup Q)$ is equivalent to non-existence of real valued measureable cardinals.

1. Introduction. A triplet $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ where $\mathcal{X}$ is a set, $\mathcal{A}$ a σ-algebra of subsets of $\mathcal{X}$ and $\mathcal{P}$ is a family of probability measures on $(\mathcal{X}, \mathcal{A})$ will be referred to as a statistical structure. Pitcher in [4] introduced the notion of compact statistical structures as a generalization of statistical structures dominated by a σ-finite measure. In the same paper Pitcher raised also the question "if $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ are both compact then is $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1 \cup \mathcal{P}_2)$ compact?". Kusama and Yamada ([3]) gave an example of compact statistical structures $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ whose union $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1 \cup \mathcal{P}_2)$ is not compact, thus answering the question of Pitcher in the negative. However it was pointed out by Diepenbrock in a letter to one of the authors of [3] that their example is not valid by showing that in their example $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1 \cup \mathcal{P}_2)$ is indeed compact. Diepenbrock ([1]) moreover gave a valid example, using sets of measurable cardinal again to show that union of compact statistical structures need not necessarily be compact.

In section 2 of this paper we give an example of two compact statistical structures whose union is not compact, without invoking measurebla cardinals. In the same section we give also a necessary and sufficient condition for the union of two compact statistical structures to be compact.

In section 3 we study the relationship between the existence of measurable cardinals and Pitcher's problem. It turns out that the Diepenbrock type examples can occur only if measurable cardinals exist. In the same section we also study a few other questions whose answers are closely connected with existence of measurable cardinals. The studies in section 3 were motivated by Diepenbrock's example and some of the results there are similar to those contained in Diepenbrock's thesis [1].

It is known that compactness of statistical structures is equivalent to weak domination ([2]), i.e. domination by a localizable measure. In this paper our results are described in terms of weakly dominated statistical structures rather than compact statistical structures. However in view of the equivalence mentioned above these terms are interchangeable.

2. Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a statistical structure.

DEFINITION. A measure m on $(\mathcal{X}, \mathcal{A})$ is said to *dominate* $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ if

(1) $\mathcal{P}\equiv m$ *i.e.* if $A\in\mathcal{A}$, $m(A)=0$ is equivalent to $P(A)=0$ for all P in $\mathcal{P}$, and

(2) dP/dm exists for each P in $\mathcal{P}$.

It is easy to see that any dominating measure m has the finite subset property, *i.e.* if $m(A)>0$ then there is $B\subset A$ such that $0<m(B)<\infty$.

DEFINITION. $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is said to be *weakly dominated* if there is a measure m which dominates it and further if $(\mathcal{X}, \mathcal{A}, m)$ is localizable.

For definition of localizable measure and the equivalence of compactness and weak domination we refer to [2]. In terms of weakly dominated statistical structures Pitcher's question can be rephrased as "if $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ are both weakly dominated then is $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1\cup\mathcal{P}_2)$ also weakly dominated?". Let m_2 and m_2 be localizable measures dominating $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ respectively. If n is a measure dominating $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1\cup\mathcal{P}_2)$ then so is any other measure n' which is equivalent to n and has the finite subset property. Also n with the finite subset property is localizable iff every measure n' which is equivalent to n and has the finite subset property is localizable. Hence to check for weak domination of $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1\cup\mathcal{P}_2)$ it is enough to verify that m_1+m_2 is localizable and also dominates $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1\cup\mathcal{P}_2)$.

EXAMPLE. Let $\mathcal{X}=[-1, 1]$, $\mathcal{A}=\{A\subset X$; there is a set B symmetric about 0 such that $A\Delta B$ is countable$\}$, $\mathcal{P}_1=\{\delta_x; x\in[0, 1]\}$, and $\mathcal{P}_2=\{\delta_x; x\in[-1, 0]\}$, where δ_x is the point measure at x. Then

a) $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ is weakly dominated.

The counting measure m_1 on [0, 1], i.e. $m_1(A)=\#(A\cap[0, 1])$ is localizable on $(\mathcal{X}, \mathcal{A})$ and dominates $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$.

b) $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ is weakly dominated.

The counting measure m_2 on $[-1, 0]$ is localizable and dominates $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$.

c) $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1 \cup \mathcal{P}_2)$ is not weakly dominated.

We will establish the above claim by showing that $(\mathcal{X}, \mathcal{A}, m_1+m_2)$ is not localizable. Suppose not, *i.e.* suppose $(\mathcal{X}, \mathcal{A}, m_1+m_2)$ is localizable.

Consider $\{F_x; x\in[0, 1]\}$ where $F_x=\{x\}$. Then $m_1+m_2(F_x)<\infty$ and hence $\{F_x; x\in[0, 1]\}$ has an essential supremum F in $\mathcal{A}$ with respect to m_1+m_2. This F has to be $[0, 1]$. For if $x\in[0, 1]$ then $m_1+m_2\{x\}>0$ and $F_x\subset F$ so $x\in F$. If $y\bar{\in}[0, 1]$ then $F-\{y\}$ is also a supremum of $\{F_x; x\in[0, 1]\}$. Also since $m_1+m_2\{y\}>0$ and $F\subset F-\{y\}$, $y\bar{\in}F$. Therefore $F=[0, 1]$. Now $[0, 1]$ is not in A and consequently m_1+m_2 is not localizable.

Next we give a necessary and sufficient condition for $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1 \cup \mathcal{P}_2)$ to be compact if $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ are both compact. We denote by $\mathcal{A}(m)=\{A\in\mathcal{A}; m(A)<\infty\}$ and by $\mathcal{A}_\sigma(m)=\{A\in\mathcal{A}; A$ is σ-finite w.r.t. $m\}$. For two measures m_1 and m_2 on $(\mathcal{X}, \mathcal{A})$ we write $m_1\perp m_2$ if there is a set T in $\mathcal{A}$ such that $m_1(T)=0$ and $m_2(X-T)=0$. m_1 has a Lebesgue decomposition n_1+n_2 with respect to m_2 if there are measures n_1, n_2 such that $n_1\perp m_2$, $n_2\ll m_2$ and $m_1=n_1+n_2$. Note that the decomposition if it exists is unique.

Lemma 1 ([1] Remark 1.4). *Let m_1 and m_2 have the finite subset property. Then m_1+m_2 has the finite subset property.*

Lemma 2 ([1] Lemma 3.1). *Suppose $m_1\ll m_2$ and m_1 has the finite subset property then $\mathcal{A}_\sigma(m_2)\subset\mathcal{A}_\sigma(m_1)$.*

Lemma 3. *Suppose m_1 has the Lebesgue decomposition n_1+n_2 with respect to m_2 and m_2 has the Lebesgue decomposition $n_1'+n_2'$ with respect to m_1 then there is a decomposition of $\mathcal{X}$ into sets A, B, C in $\mathcal{A}$ such that*

(1) $m_1(A)=0$, $m_2(C)=0$

(2) *On B $m_1\equiv m_2$.*

Proof. We have $m_1=n_1+n_2$ and $m_2=n_1'+n_2'$. There are then sets T_1, $T_2=\mathcal{X}-T_1$, S_1 and $S_2=\mathcal{X}-S_1$ such that $m_2(T_1)=0$, $n_1(T_2)=0$, $m_1(S_1)=0$ and $n_1'(S_2)=0$. We note that on $T_2 m_2$ dominates m_1 and on $S_2 m_1$ dominates m_2 and hence $m_1\equiv m_2$ on $T_2\cap S_2$. Now set $A=S_1$, $B=S_2\cap T_2$ and $C=T_1-S_1$. It is then easy to see that these sets constitute a partition of $\mathcal{X}$ and (1) $m_1(A)=0$ (2) $m_2(C)=0$ (3) $m_1\equiv m_2$ on B.

Theorem. *Let $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ be dominated by the localizable measures m_1 and m_2 respectively. Then $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1\cup\mathcal{P}_2)$ is weakly dominated iff m_1 has the Lebesgue decomposition with respect to m_2 and m_2 has the Lebesgue decomposition with respect to m_1.*

Proof. 'If part'.

By Lemma 3 there are sets A, B and C, pairwise disjoint and covering $\mathcal{X}$ such that $m_1(A)=0$, $m_2(C)=0$ and $m_1\equiv m_2$ on B. We shall show that m_1+m_2 is localizable and dominates $P_1\cup P_2$.

Let $\mathcal{F}=\{F_i;\, i\in I\}\subset\mathcal{A}(m_1+m_2)=\mathcal{A}(m_1)\cap\mathcal{A}(m_2)$. Consider $\mathcal{F}_1=\{F_i\cap A;\, i\in I\}$, $\mathcal{F}_2=\{F_i\cap B;\, i\in I\}$ and $\mathcal{F}_3=\{F_i\cap C;\, i\in I\}$. Let F_1 be m_2 essential supremum of $\mathcal{F}_1$, F_2 be m_2 essential supremum of $\mathcal{F}_2$ and F_3 be m_1 essential supremum of $\mathcal{F}_3$. Noting that $m_1\equiv m_2$ on B it can be easily seen that $(F_1\cap A)\cup(F_2\cap B)\cup(F_3\cap C_1)$ is m_1+m_2 essential supremum of $\mathcal{F}$. Hence m_1+m_2 is localizable.

Let $P_1\in\mathcal{P}_1$ and let $S_P=\{x;\, dP/dm_1(x)>0\}$. Then $P(S_P)=1$, $S_P\in\mathcal{A}_\sigma(m_1)$ and further on S_P, $\mathcal{P}\equiv m_1$. Since $S_P\cap B\in\mathcal{A}_\sigma(m_1)$ and every dominating measure has the finite subset property $S_P\cap B\in\mathcal{A}_\sigma(m_2)$ and hence $S_P\cap B\in\mathcal{A}_\sigma(m_1+m_2)$. Now define f as

$$\begin{aligned} f &= 0 \text{ on } A\,,\\ &= dP/d(m_1+m_2) \text{ on } S_P\cap B\,,\\ &= dP/dm_1 \text{ on } S_P\cap C\,, \text{ and}\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

It is then easy to see that $\int_A fd(m_1+m_2)=P(A)$ for all A in $\mathcal{A}$. A similar argument would show that $dQ/d(m_1+m_2)$ exists if $Q\in\mathcal{P}_2$. Hence m_1+m_2 dominates $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1\cup\mathcal{P}_2)$.

'Only if part'. Suppose $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1\cup\mathcal{P}_2)$ is weakly dominated. Then we can show that, by Lemma 1 and Lemma 2, m_1+m_2 is itself localizable and dominates $\mathcal{P}_1\cup\mathcal{P}_2$. Let $T_P=\{x;\, dP/d(m_1+m_2)(x)>0\}$, $\mathcal{F}_1=\{T_P;\, P\in\mathcal{P}_1\}$ and $\mathcal{F}_2=\{T_Q;\, Q\in\mathcal{P}_2\}$. There are by localizability of m_1+m_2 sets F_1 and F_2 such that $F_1=(m_1+m_2)$ essential supremum of $\mathcal{F}_1$ and $F_2=(m_1+m_2)$ essential supremum of $\mathcal{F}_2$. Let $A=F_1-(F_1\cap F_2)$, $B=F_1\cap F_2$ and $C=F_2-(F_1\cap F_2)$. Denote by m_{1A} the measure m_1 restricted to A i.e. the measure defined by $m_{1A}(E)=m_1(A\cap E)$, $E\in\mathcal{A}$. The measures m_{1B}, m_{2B} and m_{2C} are similarly defined.

We shall show that $m_1=m_{1A}+m_{1B}$ is the Lebesgue decomposition of m_1 with respect to m_2. Towards this first note that $m_1(\mathcal{X}-F_1)=0$. For if $P\in\mathcal{P}_1$, $P(\mathcal{X}-F_1)=P(T_P-F_1)=0$. Hence $m_1(C)=0$. A similar argument shows that $m_2(\mathcal{X}-F_2)=0$. Next we shall show that $m_1\equiv m_2$ on B. Suppose $E\subset B$ and $m_1(E)>0$. Then $0<(m_1+m_2)(E)=(m_1+m_2)(E\cap F_2)=(m_1+m_2)(F_2-E^c)$. Hence there exists a $P\in\mathcal{P}_2$ such that $0<(m_1+m_2)(T_P-E^c)=(m_1+m_2)(T_P\cap E)$. So $P(T_P\cap E)>0$ which implies $m_2(E)>0$. Therefore $m_1\ll m_2$ on B. Similarly $m_2\ll m_1$ on B. So we have proved that $m_1=m_{1A}+m_{1B}$ is the Lebesgue decomposition of m_1 with respect to m_2 such that $m_{1A}\perp m_2$ and $m_{1B}\ll m_2$. A similar argument will establish that $m_2=m_{2C}+m_{2B}$ is the Lebesgue decomposition of

m_2 with respect to m_1. This completes the proof of the theorem.

3. In this section we study some ramifications of Pitcher's problem. Our results were motivated by Diepenbrock's example which appears here as proposition 3.1. Since the questions considered in this section have close connection with the existence of real valued measurable cardinals we begin by defining sets of real valued measurable cardinals.

DEFINITION. Let Z be a set and $\mathcal{P}(Z)$ the set of all subsets of Z. Z is said to be of *real valued measurable cardinal*, if there is a probability measure λ on $(Z, \mathcal{P}(Z))$ such that $\lambda\{z\}=0$ for all z in Z.

It has already been shown in Section 2 that even if $(\mathcal{X}, \mathcal{A}, \mathcal{P}_1)$ and $(\mathcal{X}, \mathcal{A}, \mathcal{P}_2)$ are both compact, yet $(\mathcal{X}, \mathcal{A}_1, \mathcal{P} \cup \mathcal{P}_2)$ need not be compact. We now look at the special situation when $\mathcal{P}_2$ consists of a single probability measure Q. The question then becomes "if $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is compact and if Q is a probability measure on $(\mathcal{X}, \mathcal{A})$ then is $(\mathcal{X}, \mathcal{A}, \mathcal{P} \cup Q)$ compact?". The answer to this question is provided by the following two propositions.

Proposition 3.1 ([1] Section 10). *Let $\mathcal{X}$ be of RVMC. Then there is a family of probability measures $\mathcal{P}$ and a single probability measure Q on $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$ such that*

(1) $(\mathcal{X}, \mathcal{P}(\mathcal{X}), \mathcal{P})$ *is compact,*

(2) $(\mathcal{X}, \mathcal{P}(\mathcal{X}), \mathcal{P} \cup Q)$ *is not compact.*

Proof. Take $\mathcal{P}$ to be $\{\delta_x; x \in \mathcal{X}\}$ and Q to be the continuous probability measure on $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$.

REMARK. In Diepenbrock's example if m_1 is the counting measure m_1+Q is localizable however Q does not have a density with respect to m_1+Q. In terms of the theorem of section 2, $Q=Q$ is a Lebesgue decomposition of Q with respect to m_1 i.e. Q has no singular component with respect to m_1. However m_1 does not have a Lebesgue decomposition with respect to Q.

Proposition 3.2. *Suppose $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is a statistical structure and Q a probability measure on $(\mathcal{X}, \mathcal{A})$ such that*

(1) $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ *is compact, and*

(2) $(\mathcal{X}, \mathcal{A}, \mathcal{P} \cup Q)$ *is not compact.*

Then $Z=\mathcal{P}(\mathcal{X})$ is of real valued measurable cardinal.

Proof. Let $\boldsymbol{A}=\{A \in \mathcal{A}; P(A)=0$ for all P in $\mathcal{P}$ and $Q(A)>0\}$. Let A_0 be a set in $\boldsymbol{A}$ satisfying

$$Q(A_0) = \sup \{Q(A); A \in \boldsymbol{A}\}$$

if $\boldsymbol{A}$ is not empty, and take $A_0=\phi$ if $\boldsymbol{A}$ is empty.

(1) $Q(A_0)<1$.

Suppose not, then it can be seen easily that $(X, A, \mathcal{P}\cup Q)$ is compact.

(2) Let A_1 denote $\mathcal{X}-A_0$. By (1) $Q(A_1)>0$.

Now since $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is compact there is a localizable measure m on $(\mathcal{X}, \mathcal{A})$ such that $m\equiv\mathcal{P}$. We note that $m(A_0)=0$ and that on A_1, m dominates Q. Now consider the statistical structure $(A_1, \mathcal{A}_1, \mathcal{P})$ where $\mathcal{A}_1$ is the σ-field $\mathcal{A}$ restricted to A_1 Let $\{E_\gamma;\ \gamma\in\Gamma\}$ denote a maximal decomposition of $(A_1, \mathcal{A}_1, m)$ satisfying

(a) $0<m(E_\gamma)<\infty$,

(b) $\gamma_1\neq\gamma_2\Rightarrow m(E_{\gamma1}\cap E_{\gamma2})=0$.

Since m dominates Q on A_1, $Q(E_{\gamma1}\cap E_{\gamma2})=0$ whenever $\gamma_1\neq\gamma_2$. Hence the set $\{\gamma;\ Q(E_\gamma)>0\}$ is at most countable, say $\gamma_1, \gamma_2, \cdots$. We now claim that $Q(A_1-\bigcup_{i=1}^{\infty}E_{\gamma i})>0$. For if $Q(A_1)=Q(\bigcup_{i=1}^{\infty}E_{\gamma_i})$, then we can construct Lebesgue decomposition for m with respect to Q (and Q with respect to m) on each E_{γ_i} and put them together to obtain a decomposition of m with respect to Q (and Q with respect to m). This would, by theorem of section 2, entail $(\mathcal{X}, \mathcal{A}, \mathcal{P}\cup Q)$ to be compact. Hence $Q(A_1-\bigcup_{i=1}^{\infty}E_{\gamma_i})>0$.

(3) We will now construct a continuous finite measure on $(\Gamma_1, \mathcal{P}(\Gamma_1))$, where $\Gamma_1=\Gamma-\{\gamma_1, \gamma_2, \cdots\}$. For $E\subset\Gamma_1$ define $\lambda(E)=Q(m\text{-ess sup}\{E_\gamma;\ \gamma\in E\})$. Since on A_1 m dominates Q, λ is well defined. Further $\lambda(\gamma)=0$ for $\gamma\in\Gamma_1$ and $\lambda(\Gamma_1)=Q(A_1-\bigcup_{i=1}^{\infty}E_{\gamma_i})>0$.

(4) Since $E_\gamma\in\mathcal{A}$, and $\mathcal{A}\subset\mathcal{P}(\mathcal{X})$, card.$(\Gamma_1)\leqq$ card. $\mathcal{P}(\mathcal{X})$. Therefore $\mathcal{P}(\mathcal{X})$ is of real valued measurable cardinal.

Proposition 3.1 and 3.2 can be combined to give the following theorem.

Theorem 3.1. *The following are equivalent.*

(1) *If $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is compact then so is $(\mathcal{X}, \mathcal{A}, \mathcal{P}\cup Q)$.*

(2) *There does not exist a real valued measurable cardinal.*

Let $(\mathcal{X}, \mathcal{A}, m)$ be a localizable measure space and Q be a probability measure on $(\mathcal{X}, \mathcal{A})$ such that m dominates Q. It is then known that there is a local density of Q with respect to m, i.e. there is a function such that

$$\int_A f\,dm = Q(A) \text{ for all } A \text{ in } \mathcal{A} \text{ such that } m(A)<\infty.$$

However there may not exist a global density *i.e.* there may not exist any function for which the above integral equation is satisfied for all A in $\mathcal{A}$. In fact argument essentially similar to that Propositions 3.1 and 3.2 yields the following theorem.

Theorem 3.2. *The following are equivalent.*

(1) *If* $(\mathscr{X}, \mathscr{A}, m)$ *is localizable and* Q *is a probability measure such that* $Q \ll m$, *then* Q *has a global density with respect to* m.

(2) *There does not exist a real valued measurable cardinal.*

Let $(\mathscr{X}, \mathscr{A}, m)$ be a measure space with the finite subset property. It is then known that $(\mathscr{X}, \mathscr{A}, m)$ has an almost disjoint decomposition, *i.e.* there is a family of sets $\{E_\gamma;\ \gamma \in \Gamma\}$ in $\mathscr{A}$ satisfying

(1) $0 < m(E_\gamma) < \infty$,

(2) $\gamma_1 \neq \gamma_2 \Rightarrow m(E_{\gamma_1} \cap E_{\gamma_2}) = 0$, and

(3) $m(A) = \sum\{m(A \cap E_\gamma);\ \gamma \in \Gamma\}$ for all A in $\mathscr{A}$.

Now suppose $\mathscr{B}$ is a subfield of $\mathscr{A}$ such that $(\mathscr{X}, \mathscr{B}, m)$ has the finite subset property and further that $\{F_\gamma;\ \gamma \in \Gamma\}$ is a decomposition of $(\mathscr{X}, \mathscr{B}, m)$. Is $\{F_\gamma;\ \gamma \in \Gamma\}$ also a decomposition for $(\mathscr{X}, \mathscr{A}, m)$? Or more specifically for all A in $\mathscr{A}$ is it true that $m(A) = \sum\{m(A \cap F_\gamma);\ \gamma \in \Gamma\}$? It is easy to construct examples where the answer is in the negative. Professor Morimoto had asked us whether the answer is in the affirmative if we further assume that $(\mathscr{X}, \mathscr{B}, m)$ is localizable. The following example and proposition answer Morimoto's question.

EXAMPLE. Let Z be a set of real valued measurable cardinal and let λ be the continuous probability measure on $(Z, \mathscr{P}(Z))$. Let $\mathscr{X} = \{0\} \times Z \cup \{1\} \times Z$, $\mathscr{A} = \mathscr{P}(X)$, $m = C + \lambda$, where C is the counting measure on $\{0\} \times Z$ and λ the continuous probability measure on $\{1\} \times Z$. Let $\mathscr{B} = \{B \subset \mathscr{X};\ (0, z) \in B \Leftrightarrow (1, z) \in B$ for all $z \in Z\}$. Then $F_z = \{(0, z), (1, z)\}\ ;\ z \in Z\}$ is a decomposition of $(\mathscr{X}, \mathscr{B}, m)$. Further $(\mathscr{X}, \mathscr{B}, m)$ is localizable. However $m(\{1\} \times Z) = 1 \neq \sum\{m(\{1\} \times Z \cap F_z);\ z \in Z\} = 0$.

Proposition 3.3. *Suppose* $(\mathscr{X}, \mathscr{A}, m)$ *has the finite subset property and* $(\mathscr{X}, \mathscr{B}, m)$ *is localizable. If there is a decomposition* $\{F_\gamma;\ \gamma \in \Gamma\}$ *of* $(\mathscr{X}, \mathscr{B}, m)$ *which is not a decomposition of* $(\mathscr{X}, \mathscr{A}, m)$ *then* $Z = \mathscr{P}(\mathscr{X})$ *is of real valued measurable cardinal.*

Proof. There is a set A in $\mathscr{A}$ such that $m(A) \neq \sum\{m(A \cap F_\gamma);\ \gamma \in \Gamma\}$. We can assume without loss of generality that $m(A \cap F_\gamma) = 0$ for all γ and also that $0 < m(A) < \infty$. We now define a measure on $(\Gamma, \mathscr{P}(\Gamma))$ by

$$\lambda(E) = m[(\text{ess-sup}\{F_\gamma;\ \gamma \in E\}) \cap A],\ E \subset \Gamma\ .$$

It is then easily seen that $\lambda(\gamma) = 0$ and $\lambda(\Gamma)$ is positive and finite. As before, since $\text{Card}\,\Gamma \leq \text{Card}\,\mathscr{P}(\mathscr{X})$, $Z = \mathscr{P}(\mathscr{X})$ is of real valued measurable cardinal.

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References

[1] F.R. Diepenbrock: *Charakterisierung einer allgemeineren Bedingung als Dominiertheit mit Hilfe von lokalisierten Massen*, Thesis: University of Münster, 1971.
[2] J.K. Ghosh, H. Morimoto and S. Yamada: *Neyman factorization and minimality of pairwise sufficient subfields*, Ann. Statist. **9** (1981), 514–530.
[3] T. Kusama and S. Yamada: *On compactness of the statistical structure and sufficiency*, Osaka J. Math. **9** (1972), 11–18.
[4] T.S. Pitcher: *A more general property than domination for sets of probability measures*, Pacific J. Math. **15** (1965), 597–611.

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