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ON HYPOELLIPTIC OPERATORS WITH MULTIPLE CHARACTERISTICS OF ODD ORDER

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Introduction. In the recent paper [4], Hörmander has clarified Egorov's work [3] on sub-elliptic operators, by improving several points. The purpose of the present paper is to show that the method in [4] is applicable to certain pseudodifferential operators with multiple characteristics of odd order. Rubinstein [13], Wenston [15], [16], Popivanov [12], Menikoff [6] and Popivanov-Popov [17] independently treated some class of differential (or pseudodifferential) operators with double (or multiple, see [17]) characteristics satisfying the conditions similar to those given by Nirenberg-Treves [10] for operators of principal type.

It should be noted that, roughly speaking, operators considered in those papers can be reduced micro-locally to $D_{x_1}+ix_1{}^kD_{x_2}$ (k integer), which was studied by Mizohata [8]. On the other hand, the operator considered in the present paper can not be reduced only to Mizohata type everywhere in the sense of micro-local. At some point it will be reduced even to Egorov type $D_{x_1}+i(x_1{}^sD_{x_2}+x_1{}^ax_2{}^b|D_x|)$, where $|D_x|$ denotes the square root of $D_{x_1}^2+D_{x_2}^2+D_{x_3}^2$ and s, a, b are integers.

The plan of this paper is as follows. In Section 1 we state the assumptions and result. In Section 2 we reduce the proof of main theorem to "sub-elliptic estimate" for a localized operator whose symbol has a parameter $0<\lambda\leqq 1$ (see (2.37) and (3.4)). To prove this estimate, in Section 3 we show that we can use the same method as in [4]. The most part of Section 3 is devoted to show that the symbol of the localized operator satisfies inequalities similar to those in [4, Section 4]. In final section we prove the non-hypoellipticity of some operator in order to show the importance of the notion of modified-null-bicharacteristic curve, which is introduced in Section 1.

1. Assumptions and result

We say that $p(x,\xi)\in\mathcal{B}^\infty(R^n_x\times R^n_\xi)$ belongs to $\bar{S}^m$ when $p(x,\xi)$ is positively homogeneous of degree m in $|\xi|\geqq 1/2$. (Clearly $\bar{S}^m$ is the subset of $S^m=S^m_{1,0}$. We refer the definition of $S^m_{1,0}$ to Kumano-go [5, p. 50].) For a conic set $U\subset R^n_x\times R^n_\xi$ and $q(x,\xi)\in C^\infty(U)$ with positive homogeneous of degree m in $|\xi|\geqq 1/2$

we write $q(x, \xi)\in\bar{S}^m(U)$.

Let $p_1(x, \xi)$ belong to $\bar{S}^1$ and be real principal type, that is, be real valued and satisfy

$$d_{x\xi}p_1(x, \xi)\neq 0 \quad \text{on} \quad \Gamma = p_1^{-1}(0)\cap\{|\xi|\geqq 1/2\}\,. \tag{1.1}$$

Let l be an odd integer $\geqq 3$ and let $a(x, \xi)\in\bar{S}^{l-1}$ be complex valued and satisfy

$$\operatorname{Re} a\neq 0 \qquad \text{on } \Gamma \tag{1.2}$$

and moreover

$$H_{p_1} \operatorname{Re} a = 0 \qquad \text{on } \Gamma\,, \tag{1.3}$$

where H_{p_1} denotes the Hamilton vector field of p_1.

Then under certain conditions among p_1, Re a and Im a we shall discuss the hypoellipticity for a pseudodifferential operator L of order l which has the form

$$\begin{aligned} L &= P(x, D_x)+A(x, D_x) \qquad \text{in } R_x^n\,, \\ \sigma(P) &= (p_1(x, \xi))^l\,, \qquad \sigma(A) = a(x, \xi)\,. \end{aligned} \tag{1.4}$$

Here $\sigma(P)$ denotes the symbol of pseudodifferential operator $P(x, D_x)$.

First we assume that

(1.5) for any $(x_0, \xi_0)\in\Gamma$ there exist a conic neighborhood U of (x_0, ξ_0) and $q_0(x, \xi)\in\bar{S}^0(U)$ such that

$$H_{p_1}q_0 = 1 \quad \text{in } U \tag{1.6}$$

and for $j=1, \cdots, l-2$

$$H_{q_0}^j a = 0 \quad \text{on } \Gamma\cap U\,. \tag{1.7}$$

To state the second condition corresponding to (A) in Egorov [3], or (Ψ) in Nirenberg-Treves [10], we define the modified-null-bicharacteristic curve of p_1 through $(x_0, \xi_0)\in\Gamma$ by the curve

$$[-T, T]\ni t \mapsto (x(t), \xi(t))\,, \tag{1.8}$$

where $(x(t), \xi(t))$ is the solution to

$$\begin{aligned} dx/dt &= d_\xi(p_1+{}^l\!\sqrt{\operatorname{Re} a}\,) \\ d\xi/dt &= -d_x(p_1+{}^l\!\sqrt{\operatorname{Re} a}\,)\,, \quad (x(0), \xi(0)) = (x_0, \xi_0)\,, \end{aligned} \tag{1.9}$$

and ${}^l\!\sqrt{\operatorname{Re} a}$ denotes a unique real l power root of Re a. It follows from (1.3) that if $(x_0, \xi_0)\in\Gamma$, then $(x(t), \xi(t))\in\Gamma$. The right hand side of (1.9) are not homogeneous in ξ, so that the behavior of modified-null-bicharacteristic curve is not so. But we can define it on $[-T, T]$ for some $T>0$ uniformly if $(x_0, \xi_0)\in\Gamma$ varies in a compact conic set $(|\xi_0|>1)$, because $d_\xi{}^l\!\sqrt{\operatorname{Re} a}$ and $|\xi|^{-1}d_x{}^l\!\sqrt{\operatorname{Re} a}$

are $\mathcal{O}(|\xi|^{-1/l})$. Second condition is that

$$(1.10)\qquad \begin{cases} \text{for any } (x_0, \xi_0)\in\Gamma \text{ with } |\xi_0| \text{ sufficiently large,} \\ \text{if } \operatorname{Im} a(x(t_0), \xi(t_0))>0 \text{ for some } t_0\in[-T, T], \\ \text{then } \operatorname{Im} a(x(t), \xi(t))\geqq 0 \text{ for all } t\in(t_0, T], \\ \text{where } (x(t), \xi(t)) \text{ is the modified-null-bicharacteristic} \\ \text{curve of } p_1 \text{ through } (x_0, \xi_0). \end{cases}$$

For a multi-index $I=(i_1, \cdots, i_k)$ whose components i_j are 1 or 2, we use the following notations: $|I|=k$, $b(I)=$"the number of j such that $i_j=2$", $\mathcal{J}=\{I; b(I)\leqq l-1\}$, $\mu(I)=(l-1)|I|/(l-b(I))$. For a $\mu>0$ we put $\mathcal{J}_\mu=\{I\in\mathcal{J};\ \mu(I)\leqq\mu\}$. Set $Q_0=\{\mu(I); I\in\mathcal{J}\}$. Then we can write $Q_0=\{\mu_j; j=1,2,\cdots\}$ with an appropriate increasing sequence of rational numbers. For any $(x,\xi)\in\Gamma$, $\mu(x,\xi)$ denotes the rational $\mu_j\in Q_0$ such that $p_I(x, \xi)\neq 0$ for some $I\in\mathcal{J}_{\mu_j}$ and $p_I(x, \xi)=0$ for any $I\in\mathcal{J}_{\mu_{j-1}}$, where

$$(1.11)\qquad p_I(x, \xi) = H_{i_1}H_{i_2}\cdots H_{i_{k-2}}p_{i_k}(x, \xi)\,,$$

$H_1=H_{p_1}$, $H_2=H_{p_2}$ and $p_2(x, \xi)=\operatorname{Im} a(x, \xi)$. The third condition corresponding to (B) in Egorov [3] is that

$$(1.12)\qquad \begin{cases} \text{for all } (x, \xi)\in\Gamma \text{ there exists some } \mu\in Q_0 \text{ such} \\ \text{that } \mu(x, \xi)\leqq\mu<\infty\,. \end{cases}$$

In what follows we denote the norm in Sobolev space H^s by $\|\cdot\|_s$. We write $\|\cdot\|=\|\cdot\|_0$.

Theorem 1.1. *Let* (1.1)–(1.3) *hold and let* $L(x,D_x)$ *in* (1.4) *satisfy* (1.5)–(1.7), (1.10) *and* (1.12). *Then for any compact set* K *of* R^n *there exists a constant* C_K *such that*

$$(1.13)\qquad \|u\|_{l-2+\sigma'}+\|P'u\|_\sigma\leqq C_K(\|Lu\|+\|u\|),\ u\in C_0^\infty(K)\,,$$

where $\qquad \sigma(P') = (p_1(x, \xi))^{l-1},\ \sigma = (l-1)/l\mu,\ \sigma' = \sigma+1/l\,.$

REMARK 1. It follows from (1.13) that P is hypoelliptic. See Oleinik-Radkevich [11] and Morimoto [9, Theorem 2.2]. Theorem 2.2 of [9] is stated only for differential operators but its proof is also applicable to pseudo-differential operators.

REMARK 2. In differential operators we have the following example:

$$D_{x_1}^5+D_{x_2}^4+D_{x_3}^4+i(x_1^2x_2(D_{x_2}^4+D_{x_3}^4)+x_1^3D_{x_2}^2(D_{x_2}^2+D_{x_3}^2)) \text{ in } R_x^3\,.$$

All conditions of Theorem 1.1 are satisfied. Specially, the condition (1.10) is satisfied at $(0, \xi_0)$ with $\xi_0=(0, 0, \xi_{03})$ as follows; the sign of Im a changes from $-$

to $+$ along the modified-null-bicharacteristic curve of p_1 through $(0, \xi_0)$. At the point $(0, \xi_0)$ this example must be reduced to Egorov type. Note that Egorov's operator of principal type is not the differential operator but the pseudodifferential operator. (See the introduction of [3].) For this example we have $\sigma=1/25$.

REMARK 3. The condition (1.10) is delicated and necesasry in general for the hypoellipticity of P. Indeed, the operator with replaced Re $a=\xi_2^4+\xi_3^4$ in the above example by $(\xi_2-\xi_3)^4+\xi_3^4$ satisfies all conditions except (1.10), which is violated at $(0, \xi_0)$ with $\xi_0=(0, 0, \xi_{03})$ and $\xi_{03}>0$. Furthermore, we have

Proposition 1.2. *Differential operator* $\tilde{L}$

$$
\tag{1.14} \begin{aligned} \tilde{L} = & D_{x_1}^5+(D_{x_2}-D_{x_4})^4+D_{x_3}^4 \\ & +i(x_1^2x_2(D_{x_2}^4+D_{x_3}^4)+x_1^3D_{x_2}^2(D_{x_2}^2+D_{x_3}^2)) \quad \text{in} \quad R_x^3 , \end{aligned}
$$

is not hypoelliptic at the origin.

2. Reduction to localized operator

Let $h(x)\in C_0^\infty(R_x^n)$ be 1 for $|x|<1/2$ and vanish for $|x|>1$. Set $h_\varepsilon(x)=h(x/\varepsilon)$ for a small $\varepsilon>0$. For a $f(x, \xi)\in \bar{S}^m$ and $\gamma=(x_0, \bar{\xi}_0)\in R^n\times S^{n-1}$ we introduce a pseudodifferential operator $F_{\gamma,\varepsilon,\lambda}$ with a parameter $0<\lambda\leqq 1$ and a small $\varepsilon>0$ as follows:

$$
\tag{2.1} \begin{aligned} & F_{\gamma,\varepsilon,\lambda}(y, D_y)v \\ & = \lambda^{-2m}\int e^{iy\eta}h_\varepsilon(\lambda y)f(x_0+\lambda y,\ \bar{\xi}_0+\lambda\eta)h_\varepsilon(\lambda\eta)\hat{v}(\eta)d\eta , \\ & \qquad v\in S_y , \qquad d\eta = (2\pi)^{-n}d\eta , \end{aligned}
$$

where $\hat{v}$ denotes the Fourier transform of v. Obviously, for a fixed $\varepsilon>0$, $\{\lambda^{2m}\sigma(F_{\gamma,\varepsilon,\lambda})\,(y, \eta);\ 0<\lambda\leqq 1\}$ is a bounded set of $S_{0,0}^0$. Furthermore we obtain for a sufficiently small $\varepsilon>0$

$$
\tag{2.2} \begin{aligned} & (F_{\gamma,\varepsilon,\lambda}v)\,(\lambda^{-1}(x-x_0)) \\ & = e^{-i\lambda^{-2}x\cdot\bar{\xi}_0}h_\varepsilon(x-x_0)f(x, D_x)h_\varepsilon(\lambda^2D_x-\bar{\xi}_0)u(x) , \end{aligned}
$$

where $\hat{u}(\xi) = \lambda^n\hat{v}(\lambda(\xi-\lambda^{-2}\,\bar{\xi}_0))\exp\,(ix_0(\lambda^{-2}\,\bar{\xi}_0-\xi))$.

Lemma 2.1. *For any compact set* $K\subset R^n$ *there exists a constant* C_K *such that* (1.13) *holds if and only if for any* $\gamma=(x_0, \bar{\xi}_0)\in R^n\times S^{n-1}$ *one can find positives* $\varepsilon_i=\varepsilon_{i,\gamma}(i=1,2,3,\ \varepsilon_1<\varepsilon_2<\varepsilon_3)$ *and a constant* C_γ *so that for any* $0<\lambda\leqq 1$ *the following estimate holds*;

$$
\tag{2.3} \begin{aligned} & \lambda^{-2(l-2+\sigma')}||H_{\varepsilon_1,\lambda}\,v||+\lambda^{-2\sigma}||P'_{\gamma,\varepsilon_1,\lambda}\,v|| \\ & \leqq C_\gamma(||(P_{\gamma,\varepsilon_2,\lambda}+A_{\gamma,\varepsilon_2,\lambda})v||+||P'_{\gamma,\varepsilon_3,\lambda}\,v|| \\ & +\lambda^{-2(l-2+\sigma)}||v||),\ \ v\in S_y , \end{aligned}
$$

where $H_{\varepsilon,\lambda}, P_{\gamma,\varepsilon,\lambda}, P'_{\gamma,\varepsilon,\lambda}$ *and* $A_{\gamma,\varepsilon,\lambda}$ *are defined by* (2.1) *with* f *replaced by* $1, p_1{}^l, p_1{}^{l-1}$ *and* a, *respectively.*

Proof. In view of (2.2) it is not difficult to see the necessity of (2.3). We only show the sufficiency. The proof is the same way as in [4, p. 143], except the appearance of the second term in left hand side of (1.13) or (2.3). Note that for any $\xi_0(|\xi_0|=1)$ and any small $\varepsilon>0$ and any real s

$$\begin{aligned}(2.4)\qquad C^{-1}\|h_\varepsilon(\lambda D_y)v\| &\leqq \|(\xi_0+\lambda D_y)^s h_\varepsilon(\lambda D_y)v\| \\ &\leqq C\|h_\varepsilon(\lambda D_y)v\|\,,\quad v\in S_y\,,\end{aligned}$$

holds for some $C=C_{s,\varepsilon}$ since $C^{-1}\leqq|\xi_0+\xi|^s\leqq C$ on supp $h_\varepsilon(\xi)$. Substituting $\hat{v}(\eta)=h_{\varepsilon_0}(\lambda\eta)\hat{u}(\lambda^{-1}\eta+\lambda^{-2}\xi_0)\exp(ix_0(\lambda^{-1}\eta+\lambda^{-2}\xi_0))$ for $u\in C_0^\infty(K)$ and some $\varepsilon_0>\varepsilon_3$ into (2.3), we obtain by means of (2.2) and (2.4)

$$\begin{aligned}&\|h_1(x-x_0)h_1(\lambda^2D_x-\xi_0)|D_x|^{l-2+\sigma'}u\| \\ &\quad+\|h_1(x-x_0)P'(x,D_x)|D_x|^\sigma h_1(\lambda^2D_x-\xi_0)u\| \\ &\leqq C(\|h_2(x-x_0)\,(P(x,D_x)+A(x,D_x))h_2(\lambda^2D_x-\xi_0)u\| \\ &\quad+\|h_3(x-x_0)P'(x,D_x)h_3(\lambda^2D_x-\xi_0)u\| \\ &\quad+\|h_0(\lambda^2D_x-\xi_0)|D_x|^{l-2+\sigma}u\|)\,,\end{aligned}$$

where $h_j=h_{\varepsilon_j}$. Here we used the fact that, for a fixed $\varepsilon>0$, $\{\lambda^{-2}[h_\varepsilon(\lambda y), h_\varepsilon(\lambda D_y)];\ 0<\lambda\leqq 1\}$ is a bounded set of of $S^0_{0,0}$. Since $[P, h_2(\lambda^2D_x-\xi_0)]$ can be estimated by the second term of the left hand side, the proof is completed from the following proposition and usual finite covering argument over $K\times S^{n-1}$.

Proposition 2.2. *Let* $h(\xi)\in C_0^\infty(R^n)$ *be* 1 *in a neighborhood of* 0 *and let* ξ_0 *belong to* S^{n-1}. *Then one can find some* $\psi_j(\xi)\in\bar{S}^0$ $(j=1,2)$ *such that*

(2.5) $\psi_j(\xi_0)\neq 0$, *supp* $\psi_j\subset$ *some conic neighborhood of* ξ_0 *and for any* N *we have for some constant* $C>0$

$$\begin{aligned}&C^{-1}\|\psi_1(D_x)u\|^2 \\ &\leqq\int_0^1\|h(\lambda^2D_x-\xi_0)u\|^2/\lambda\,d\lambda+\|u\|_{-N} \\ &\leqq C(\|\psi_2(D_x)u\|^2+\|u\|_{-N})\,,\qquad u\in\mathcal{S}_x\end{aligned}$$

Proof. Put $r=|\xi|$, $\theta=\xi/|\xi|$. Then

$$\begin{aligned}&\int_0^1\|h(\lambda^2D_x-\xi_0)u\|^2/\lambda\,d\lambda \\ &=\int d\theta\int_0^1 d\lambda\int_0^\infty h(\lambda^2r\theta-\xi_0)^2|\hat{u}(r\theta)|^2/\lambda\,dr\,.\end{aligned}$$

It is easy to see that supp $h(\lambda^2r\theta-\xi_0)$ is evaluated from above and below by

$$\{(\theta, r, \lambda);\ \theta\in\operatorname{supp}\psi_j\cap S^{n-1}\ \text{and}\ C_j^{-1}\leqq r\lambda^2\leqq C_j\}$$

for some $\psi_j\in\bar{S}^0$ satisfying (2.5) and some $C_j>0$ $(j=1,2)$. Therefore the integral is bounded by constant times

$$\int\psi_j^2(\theta)d\theta\int_{1/C}^{\infty}|\hat{u}(r\theta)|^2dr\int_{(Cr)^{-1/2}}^{(C/r)^{1/2}}d\lambda\,,$$

where we used $(r/C)^{1/2}<1/\lambda<(Cr)^{1/2}$ and $C=C_j$. This gives the desired estimate.

REMARK. The content of this proposition is briefly stated in [4, p. 143].

Since (2.3) is valid for $\gamma\notin\Gamma$, in view of Lemma 2.1 we now fix a $\gamma\in\Gamma$. Let a function $f_\lambda(y, \eta)\in\mathscr{B}^\infty(R_y^n\times R_\eta^n)$ with a parameter $0<\lambda\leqq 1$ satisfy

$$|\partial_y^\alpha\partial_\eta^\beta f_\lambda(y, \eta)|\leqq C_{\alpha\beta}\lambda^{|\alpha+\beta|} \tag{2.6}$$

for any α, β, where $C_{\alpha\beta}$ is a constant independent of λ. Define a pseudodifferential operator $F_\lambda(y, D_y)$ by

$$F_\lambda v=\int e^{iy\eta}f_\lambda(y, \eta)\hat{v}(\eta)d\eta\,,\qquad v\in\mathcal{S}\,. \tag{2.7}$$

If $f_\lambda(y, \eta)$ equals $f(\lambda y, \lambda\eta)$ for some $f(x, \xi)\in\mathscr{B}^\infty(R_x^n\times R_\xi^n)$, then we say that the operator $F_\lambda(y, D_y)$ has an original symbol $f(x, \xi)$. Under this notation, (2.3) for a fixed γ becomes

$$\begin{aligned}&\lambda^{-2(l-2+\sigma')}\|H_{\varepsilon_1,\lambda}v\|+\lambda^{-2(l-1+\sigma)}\|P'_{\varepsilon_1,\lambda}v\|\\&\leqq C(\|(\lambda^{-2l}P_{\varepsilon_2,\lambda}+\lambda^{-2(l-1)}A_{\varepsilon_2,\lambda})v\|\\&\quad+\lambda^{-2(l-1)}\|P'_{\varepsilon_3,\lambda}v\|+\lambda^{-2(l-2+\sigma)}\|v\|)\,,\qquad v\in\mathcal{S}\,,\end{aligned} \tag{2.8}$$

where the original symbols of $H_{\varepsilon,\lambda}$, $P_{\varepsilon,\lambda}$, $P'_{\varepsilon,\lambda}$ and $A_{\varepsilon,\lambda}$ are $h_\varepsilon(x, \xi)=h_\varepsilon(x)h_\varepsilon(\xi)$, $h_\varepsilon(p_{1,\gamma})^l$, $h_\varepsilon(p_{1,\gamma})^{l-1}$ and $h_\varepsilon a_\gamma$, respectively. Here $p_{1,\gamma}(x, \xi)=p_1(x+x_0, \xi+\xi_0)$, $a_\gamma(x, \xi)=a(x+x_0, \xi+\xi_0)$.

Lemma 2.3. *If* (2.8) *is valid and* χ *is a* C^∞ *canonical transformation keeping* 0 *fixed which is defined near* 0, *then* (2.8) *remains valid with some other* ε_j *and* C *if* $P_{\varepsilon,\lambda}$, $A_{\varepsilon,\lambda}$ *and* $P'_{\varepsilon,\lambda}$ *are replaced by* $\tilde{P}_{\varepsilon,\lambda}$, $\tilde{A}_{\varepsilon,\lambda}$ *and* $\tilde{P}_{\varepsilon,\lambda}$, *respecitvely, whose original symbols are* $h_\varepsilon(p_{1,\gamma}\circ\chi)^l$, $h_\varepsilon(a_\gamma\circ\chi)$ *and* $h_\varepsilon(p_{1,\gamma}\circ\chi)^{l-1}$, *respectively.*

As pointed out in [4, the proof of Lemma 3.2] it suffices to prove the lemma when χ has a generating function $S(x, \xi)$, that is, $\chi;(x, d_xS(x, \xi))\mapsto(d_\xi S(x, \xi), \xi)$. The proof is based on several propositions on Fourier integral operators with phase function $S_\lambda(y, \eta)=\lambda^{-2}S(\lambda y, \lambda\eta)$.

DEFINITION 2.4. For any $f_\lambda(y, \eta)\in\mathscr{B}^\infty$ with (2.6) and for any $S(x, \xi)\in C^\infty(R_x^n\times R_\xi^n)$ satisfying

$$\det \partial_x \partial_\xi S(0, 0) \neq 0\,, \quad (d_x S(0, 0), d_\xi S(0, 0)) = (0, 0) \tag{2.9}$$

we define the Fourier integral operator $F_{S_\lambda}(y, D_y)$ with a parameter $0<\lambda\leqq 1$ by

$$F_{S_\lambda} v = \int e^{iS_\lambda(y,\eta)} k_\lambda(y, \eta) f_\lambda(y, \eta) \hat{v}(\eta) d\eta\,, \qquad v \in \mathcal{S} \tag{2.10}$$

where $S_\lambda(y, \eta)=\lambda^{-2}S(\lambda y, \lambda\eta)$ and $k_\lambda(y, \eta)=k(\lambda y, \lambda\eta)$. Here we assume that $k(x, \xi)\in C_0^\infty$ is 1 in a neighborhood of 0 and $\det \partial_x\partial_\xi S(x, \xi)\neq 0$ on supp k. We define the conjugate Fourier integral operator $F^*_{S_\lambda}$ with a parameter $0<\lambda\leqq 1$ by

$$F^*_{S_\lambda}(y, D_y) v = \iint e^{i(y\eta - S_\lambda(\tilde{y},\eta))} k_\lambda(\tilde{y}, \eta) f_\lambda(\tilde{y}, \eta) v(\tilde{y}) d\tilde{y}\, d\eta\,, \quad v \in \mathcal{S}\,, \tag{2.11}$$

We call $f_\lambda(y, \eta)$ the symbol of $F_{S_\lambda}(y, D_y)$ $(F^*_{S_\lambda}(y, D_y))$, and morover if $f_\lambda(y, \eta)= f(\lambda y, \lambda\eta)$ for some $f\in\mathscr{B}^\infty$, then we call $f(x, \xi)$ the original symbol of $F_{S_\lambda}(F^*_{S_\lambda})$. We write $F_{S_\lambda}=I_{S_\lambda}$, $F^*_{S_\lambda}=I^*_{S_\lambda}$ if $f=1$.

Put

$$\tilde{d}_\xi S(x, \tilde{\xi}, \xi) = \int_0^1 d_\xi S(x, \tilde{\xi}+\theta(\xi-\tilde{\xi}))d\theta \tag{2.12}$$

$$\tilde{d}_x S(\tilde{x}, x, \xi) = \int_0^1 d_x S(\tilde{x}+\theta(x-\tilde{x}), \xi)d\theta$$

Put $\tilde{x}=\tilde{d}_\xi S(x, \tilde{\xi}, \xi)$ and $\tilde{\xi}=\tilde{d}_x S(\tilde{x}, x, \xi)$. Then the inverses

$$x = \phi(\tilde{x}; \tilde{\xi}, \xi) \ \text{ and } \ \xi = \psi(\tilde{\xi}; \tilde{x}, x) \tag{2.13}$$

exist, respectively, in a neighborhood of 0 on account of (2.9) if supp k is sufficiently small.

Proposition 2.5. *If supp k in* (2.10) *and* (2.11) *is sufficiently small, then $I_{S_\lambda}I^*_{S_\lambda}$ and $I^*_{S_\lambda}I_{S_\lambda}$ are pseudodifferential operators whose symbols are*

$$\int e^{-i\tilde{y}\tilde{\eta}} r_\lambda(y, \eta+\tilde{\eta}, y+\tilde{y}) d\tilde{y} d\tilde{\eta} \tag{2.14}$$

and

$$\iint e^{-i\tilde{y}\tilde{\eta}} r^*_\lambda(\eta+\tilde{\eta}, y+\tilde{y}, \eta) d\tilde{y} d\tilde{\eta} \tag{2.15}$$

respectively, where $r_\lambda(y, \tilde{\eta}, \tilde{y})$ and $r^*_\lambda(\eta, \tilde{y}, \eta)$ are given by

$$\left(k(x, \xi)k(\tilde{x}, \xi)\,|\,det \int_0^1 \partial_x\partial_\xi S(\tilde{x}+\theta(x-\tilde{x}), \xi)d\theta\,|^{-1}\right)_{\xi = \psi(\tilde{\xi}; \tilde{x}, x)} \tag{2.16}$$

and

$$\left(k(x, \xi)k(x, \tilde{\xi})\,|\,det \int_0^1 \partial_x\partial_\xi S(x, \tilde{\xi}+\theta(\xi-\tilde{\xi})d\theta\,|^{-1}\right)_{x = \psi(\tilde{x}; \tilde{\xi}, \xi)}\,, \tag{2.17}$$

respectively. Here $(x, \tilde{x}, \xi, \tilde{\xi})=\lambda(y, \tilde{y}, \eta, \tilde{\eta})$.

Proof is directly calculated by means of the change of variable; $\tilde{\xi}=\tilde{d}_x S(\tilde{x}, x, \xi)$ and $\tilde{x}=\tilde{d}_\xi S(x, \tilde{\xi}, \xi)$ respectively.

Corollary 2.6. *The operators F_{S_λ} and $F^*_{S_\lambda}$ for any f_λ with* (2.6) *are L_2-bounded uniformly with respect to $0<\lambda\leqq 1$. If ε is small enough, then $I_{S_\lambda}I^*_{S_\lambda}$ and $I^*_{S_\lambda}I_{S_\lambda}$ are elliptic on supp $h_\varepsilon(\lambda y, \lambda\eta)$, that is, the estimates*

$$(2.18)\qquad \|H_{\varepsilon,\lambda}v\|\leqq C_1(\|I_{S_\lambda}I^*_{S_\lambda}v\|+\lambda^2\|v\|),$$

$$(2.19)\qquad \|H_{\varepsilon,\lambda}v\|\leqq C_2(\|I^*_{S_\lambda}I_{S_\lambda}v\|+\lambda^2\|v\|),\quad v\in\mathcal{S},$$

hold for some constants C_1 and C_2.

Proof. The symbols of $F^*_{S_\lambda}F_{S_\lambda}$ and $F_{S_\lambda}F^*_{S_\lambda}$ are given by the versions of (2.14) and (2.15), respectively, which belong to a bounded set of $S^0_{0,0}$ uniformly on account of (2.6). The boundedness of $F^*_{S_\lambda}F_{S_\lambda}$ and $F_{S_\lambda}F^*_{S_\lambda}$ show the first statement. Note that for any $p(x, \tilde{\xi}, \tilde{x}, \xi)\in\mathcal{B}^\infty(R^{4n})$

$$(2.20)\qquad O_s-\iint e^{-i\tilde{x}\tilde{\xi}}p(x, \tilde{\xi}, \tilde{x}, \xi)d\tilde{x}d\tilde{\xi} = \sum_{|\alpha|\leqq N} p_\alpha(x, 0, 0, \xi)/\alpha! + N\sum_{|\beta|=N}\int_0^1(1-\theta)^{N-1}$$
$$O_s-\iint e^{-i\tilde{x}\tilde{\xi}}p_\beta(x, \theta\tilde{\xi}, \tilde{x}, \xi)d\tilde{x}d\tilde{\xi}d\theta/\beta!$$

holds for any positive integer N, where $p_\alpha(x,\tilde{\xi},\tilde{x},\xi)=\partial^\alpha_{\tilde{\xi}}D^\alpha_{\tilde{x}}p(x,\tilde{\xi},\tilde{x},\xi)$. Here $O_s-\iint$ denotes the oscillatory integral (see [5, p. 42]). Applicatinos of (2.20) to (2.14) and (2.15) yield (2.18) and (2.19), respectively.

Proposition 2.7. *Let $F_\lambda(y, D_y)$ be the pseudodifferential operator with original symbol $f(x,\xi)\in\mathcal{B}^\infty$. Let G_{S_λ} and $\tilde{G}_{S_\lambda}$ be Fourier integral operators whose original symbols are $f(x, d_x S(x,\xi))$ and $f(d_\xi S(x,\xi),\xi)$, respectively. Then $\lambda^{-2}(F_\lambda I_{S_\lambda}-G_{S_\lambda})$ and $\lambda^{-2}(I_{S_\lambda}F_\lambda-\tilde{G}_{S_\lambda})$ are L_2-bounded operators uniformly with respect to $0<\lambda\leqq 1$.*

Proof. It is easy to check that the equations

$$F_\lambda I_{S_\lambda}v=\int e^{iS_\lambda(y,\eta)}r_\lambda(y,\eta)\hat{v}(\eta)d\eta$$

and

$$I_{S_\lambda}F_\lambda v=\int e^{iS_\lambda(y,\eta)}\tilde{r}_\lambda(y,\eta)\hat{v}(\eta)d\eta,$$

for

$$(2.21)\qquad r_\lambda(y,\eta)=O_s-\iint e^{-i\tilde{y}\tilde{\eta}}f(x, \tilde{\xi}+\tilde{d}_x S(x, x+\tilde{x}, \xi)k(x+\tilde{x}, \xi)d\tilde{y}d\tilde{\eta}$$

and

$$(2.22)\qquad \tilde{r}_\lambda(y,\eta)=O_s-\iint e^{-i\tilde{y}\tilde{\eta}}f(\tilde{x}+\tilde{d}_\xi S(x, \xi+\tilde{\xi}, \xi), \xi)k(x, \xi+\tilde{\xi})d\tilde{y}d\tilde{\eta},$$

respectively, where $(x,\tilde{x},\xi,\tilde{\xi})=\lambda(y,\tilde{y},\eta,\tilde{\eta})$. Applications of (2.20) to (2.21) and (2.22) complete the proof.

Corollary 2.8. *For $f(x,\xi)\in C_0^\infty$ with support contained a sufficiently small neighborhood of* 0, *set $\tilde{f}(x,\xi)=(f\circ\chi)(x,\xi)$, where χ is defined by*

$$(x, d_xS(x,\xi)) \to (d_\xi S(x,\xi), \xi)\,.$$

Then $\lambda^{-2}(F_\lambda I_{S\lambda}-I_{S\lambda}\tilde{F}_\lambda)$ is L_2-bounded uniformly for $0<\lambda\leqq 1$. Furthermore, if $f(x,\xi)=(p_1(x,\xi))^l h_\varepsilon(x,\xi)$ then for $\tilde{f}=(p_1{}^l h_\varepsilon)\circ\chi$, the estimate

$$\|(F_\lambda I_{S\lambda}-I_{S\lambda}\tilde{F}_\lambda)v\|\leqq C(\lambda^2\|F_\lambda' I_{S\lambda}v\|+\lambda^2\|I_{S\lambda}\tilde{F}_\lambda' v\|)\,,\quad v\in\mathcal{S}\,, \tag{2.23}$$

holds for some constant C, where the original symbols of F_λ' and $\tilde{F}_\lambda'$ are $(p_1)^{l-1}h_{\varepsilon'}$ and $((p_1)^{l-1}h_{\varepsilon'})\circ\chi$ for some $\varepsilon'>\varepsilon$.

Proof. The first part follows from Proposition 2.7. The second part is obtained by checking the second terms of the expansions of (2.21) and (2.22).

REMARK. It is clear that $\lambda^{-2}(I_{S\lambda}^*F_\lambda-F_\lambda I_{S\lambda}^*)$ is L_2-bounded operator uniformly for $0<\lambda\leqq 1$. Indeed, this follows from $\|F_\lambda I_{S\lambda}-I_{S\lambda}\tilde{F}_\lambda\|=\|I_{S\lambda}^*F_\lambda^*-\tilde{F}_\lambda^* I_{S\lambda}^*\|$ and the fact that $\lambda^{-2}(F_\lambda-F_\lambda^*)$ and $\lambda^{-2}(\tilde{F}_\lambda-\tilde{F}_\lambda^*)$ are L_2-bounded.

Proof of Lemma 2.3. Taking $I_{S\lambda}v$ as v in (2.8) and noting Corollary 2.8, we obtain (2.8) for operators transformed, by using the fact that for any small $\varepsilon>0$ the eatimate

$$C^{-1}\|H_{\varepsilon',\lambda}v\|\leqq\|I_{S\lambda}H_{\varepsilon,\lambda}v\|+\lambda^2\|v\|\leqq C(\|H_{\varepsilon'',\lambda}v\|+\lambda^2\|v\|)\,,\quad v\in\mathcal{S}\,,$$

holds for some $0<\varepsilon'<\varepsilon<\varepsilon''$ and some constant C, which follows from (2.19).

Now we take a canonical transformation χ such that $p_{1,\gamma}\circ\chi=\xi_1$ and $q_{0,\gamma}\circ\chi=x_1$, where $q_{0,\gamma}$ is defined from q_0 given in (1.5) by the same way as in $p_{1,\gamma}$. Darboux theorem (see [7, Proposition 3.1)] shows that (1.6) guarantees the existence of such a χ. Application of Lemma 2.3 gives

Lemma 2.9. *The estimate* (2.8) *is valid if for some ε and ε' ($\varepsilon'>\varepsilon>0$) the estimate*

$$\begin{aligned}\lambda^{-2(l-2+\sigma')}\|h_\varepsilon(\lambda y',\lambda D_{y'})v\|+\lambda^{-2\sigma}\|D_{y_1}^{l-1}h_\varepsilon(\lambda y',\lambda D_{y'})v\|\\ \leqq C(\|(D_{y_1}^l+\lambda^{-2(l-1)}\tilde{A}_{\varepsilon',\lambda}(y,D_{y'}))v\|\\ +\|D_{y_1}^{l-1}v\|+\lambda^{-2(l-2+\sigma)}\|v\|)\,,\\ \text{if } v\in\mathcal{S} \text{ vanishes for } |y_1|>\varepsilon\,,\end{aligned} \tag{2.24}$$

holds for some C, where the symbol of $\tilde{A}_{\varepsilon,\lambda}(y,D_{y'})$ is $\tilde{a}(y_1,\lambda y',\lambda\eta')h_\varepsilon(\lambda y',\lambda\eta')$. Here $\tilde{a}(x,\xi')=(a_\gamma\circ\chi)(x,0,\xi')$.

Proof. Condition (1.7) leads us to the

(2.25) $$a_\gamma\circ\chi(x, \xi) = a_\gamma\circ\chi(x, 0, \xi')+b(x, \xi)\xi_1^{l-1}$$

for some b near origin since H_{q_0} is transformed to ∂_{ξ_1}. Therefore, substituting $h_\varepsilon(y_1, \lambda^2 D_{y_1})h_{\varepsilon''}(\lambda y', \lambda D_{y'})v$ $(\varepsilon<\varepsilon''<\varepsilon')$ for v of (2.24) with replaced $\tilde{a}$ by $a_\gamma\circ\chi$ (x,ξ) and changing variable (y_1, η_1) into $(\lambda y_1, \lambda^{-1}\eta_1)$ we get the estimate (2.8) canonically transformed.

Proposition 2.10. *Let* $g_\lambda(y, \eta)\in C^\infty$ *with a parameter* $0<\lambda\leqq 1$ *satisfy for any* α, β

(2.26) $$|\partial_\eta^\alpha\partial_y^\beta g_\lambda|\leqq C_{\alpha\beta}(|\eta_1|^2+\lambda^{-4\delta})^{(m-|\alpha_1|)/2}\lambda^{|\alpha'+\beta'|}$$

where $0<\delta\leqq 1$ *and* m *integer. Assume that, for some* $h_\lambda(y', \eta')=h(\lambda y', \lambda\eta')$, g_λ *satifies for some* $c_0>0$

(2.27) $$|g_\lambda|\geqq c_0(|\eta_1|^2+\lambda^{-4\delta})^{m/2} \quad on \quad \{|y_1|<1\}\times supp\ h_\lambda\,.$$

Then we get

(2.28) $$\begin{aligned}&||D_{y_1}^m h_\lambda(y', D_{y'})v||+\lambda^{-2\delta m}||h_\lambda(y', D_{y'})v||\\ &\leqq C(||G_\lambda(y, D_y)v||+||v||)\,, \qquad if\ v\in\mathcal{S}\ vanishes\ |y_1|>1\,.\end{aligned}$$

Proof is omitted. (for example, see [5, p. 77]).

Applying this proposition with $m=l$ and $\delta=(l-1)/l$ to $\eta_1^l+\lambda^{-2(l-1)}\tilde{a}(y_1, \lambda y', \lambda\eta')h_\varepsilon(\lambda y', \lambda\eta')$, we obtain (2.24) if $\mathrm{Im}\,\tilde{a}(0)\neq 0$. From now on we assume $\mathrm{Im}\,\tilde{a}(0)=0$. Let $\omega(x,\xi')$ be a l power root of $(\tilde{a}h_\varepsilon)(x, \xi')$ such that $\omega(0)$ is real (since $\mathrm{Re}\,a(0)\neq 0$ by (1.2)).

Then we obtain the factorization

$$\xi_1^l+\tilde{a}h_\varepsilon = (\xi_1+\omega)\sum_{j=1}^{l-1}(-\omega)^{j-1}\xi_1^{l-1}$$

Set $\omega_\lambda(y, \eta')=\omega(y_1, \lambda y', \lambda\eta')$ and set $l_{2,\lambda}(y, \eta)=\sum_{j=1}^{k-1}(-\lambda^{-2\delta}\omega_\lambda)^{j-1}\eta_1^{l-j}$ $(\delta=(l-1)/l)$. Since $l_{2,\lambda}(y, \eta)$ satisfies (2.27) with $m=l-1$ and $\delta=(l-1)/l$, we get (2.24) if we show for some C

(2.29) $$\begin{aligned}&\lambda^{-2\sigma}||h_{\varepsilon,\lambda}(y', D_{y'})v||\leqq C(||(D_{y_1}+\lambda^{-2\delta}\omega_\lambda(y, D_{y'}))v||+||v||)\,,\\ &\qquad if\ v\in\mathcal{S}\ \text{vanishes}\ |y_1|>\varepsilon\,.\end{aligned}$$

For brevity we denote $\tilde{a}(x, \xi')h_{\varepsilon'}(x', \xi')$ by $\tilde{a}(x, \xi')$ in what follows. Note that $\mathrm{Re}\,\tilde{a}$ is independent of x_1 on account of (1.3) because H_{p_1} and Γ were transformed to ∂_{x_1} and $\xi_1=0$, respectively, by the χ. Using the expansion $(1+z)^{1/l}=1+z/l+0(z)^2$, we obtain

$$(2.30)\qquad \omega(x,\xi') = (\mathrm{Re}\,\tilde{a})^{1/l}(1+\mathrm{Im}\,\tilde{a}/\,\mathrm{Re}\,\tilde{a})^{1/l} = r(x',\xi')+iq(x,\xi')+\mathcal{O}(|x|+|\xi'|)q(x,\xi')$$

if we set $r=(\mathrm{Re}\,\tilde{a})^{1/l}$, $q=\mathrm{Im}\,\tilde{a}/l(\mathrm{Re}\,\tilde{a})^{(l-1)/l}$. Hence (2.29) follows if we show that for some C

$$(2.31)\qquad \lambda^{-2\sigma}\|h_{\varepsilon,\lambda}(y',D_{y'})v\|+\lambda^{-2\delta}\|q_\lambda(y,D_{y'})h_{\varepsilon,\lambda}(y',D_{y'})v\| \leqq C(\|D_{y_1}+\lambda^{-2\delta}(r_\lambda(y',D_{y'})+iq_\lambda(y,D_{y'}))v\|+\|v\|),$$
$$\text{if } v\in\mathcal{S} \text{ vanishes for } |y_1|>\varepsilon,$$

where $q_\lambda(y,\eta')=q(y_1,\lambda y',\lambda\eta')$ and $r_\lambda(y',\eta')=r(\lambda y',\lambda\eta')$. Indeed, if we take $h_{\varepsilon''}(x',\xi')$ such that $h_\varepsilon(x',\xi')=1$ on supp $h_{\varepsilon''}$ and substitute $h_{\varepsilon'',\lambda}v$ into (2.31), then we get (2.29) with replaced v by $h_{\varepsilon'',\lambda}v$ since the part corresponding to third term of the left hand side of (2.30) can be estimated by the second term of the right hand side of (2.31) when ε is small enough. (Note that $\varepsilon''<\varepsilon$).

Let $\Phi(x,\xi')$ be the solution to

$$(2.32)\qquad \partial_{x_1}\Phi+r(x',d_{x'}\Phi)=0\,,\qquad \Phi(0,x',\xi')=x'\xi'$$

Without loss of generality we assume that the Φ exists on $\{|x_1|<\varepsilon\}\times\mathrm{supp}\,h_\varepsilon(x',\xi')$. Put $\Phi_\lambda(y,\eta')=\lambda^{-2}\Phi(\lambda^{2/l}y_1,y',\lambda\eta')$. Then Φ_λ of course exists on $\{|y_1|<\varepsilon\}\times\mathrm{supp}\,h_{\varepsilon,\lambda}(y',\eta')$. If we regard y_1 as a parameter, in the same way as in (2.10) and (2.11), we can define the Fourier integral operator and the conjugate Fourier integral operator with phase function $\Phi_\lambda(y,\eta')$ and the symbol $f_\lambda(y,\eta')$ satisfying

$$(2.6')\qquad |\partial_y^\alpha\partial_{\eta'}^{\beta'}f_\lambda|\leqq C_{\alpha\beta'}\lambda^{|\alpha'+\beta'|}$$

by

$$(2.33)\qquad F_{\Phi_\lambda}v(y)=\int e^{i\Phi_\lambda(y,\eta')}k_\lambda(y',\eta')f_\lambda(y,\eta')\check{v}(y_1,\eta')d\eta'$$

and

$$(2.34)\qquad F_{\Phi_\lambda}^*v(y)=\iint e^{i(y'\eta'-\Phi_\lambda(y_1,\tilde{y}',\eta'))}k_\lambda(\tilde{y}',\eta') f_\lambda(y_1,\tilde{y}',\eta')v(y_1,\tilde{y}')d\tilde{y}'d\eta',$$

when $v\in\mathcal{S}$ vanishes for $|y_1|>\varepsilon$. Here $\check{v}(y_1,\eta')$ denotes the Fourier transform of $v(y)$ with respect to y'. Set

$$(2.35)\qquad \Psi_\lambda(y,\eta)=y_1\eta_1+\Phi_\lambda(y,\eta')\,.$$

Let χ_λ denote a canonical transformation with generating function Ψ_λ, that is, $\chi_\lambda;\,(y,d_y\Psi_\lambda(y,\eta))\mapsto(d_\eta\Psi_\lambda(y,\eta),\eta)$. Note that χ_λ and χ_λ^{-1} are defined for $\{|y_1|<\varepsilon\}\times\mathrm{supp}\,h_{\varepsilon,\lambda}(y',\eta')$ if ε is small enough. Set $\tilde{q}_\lambda(y,\eta')=q_\lambda\circ\chi_\lambda(y,\eta')$. It is easy to check that

$$(2.36)\qquad \tilde{q}_\lambda(y,\eta')=q(y_1,\psi_\lambda(y,\eta'),d_{x'}\Phi(\lambda^{2/l}y_1,\psi_\lambda(y,\eta'),\lambda\eta'))$$

where $\psi_\lambda(y, \eta')=\psi(\lambda y'; \lambda^{2/l}y_1, \lambda\eta')$ and $\psi(x'; x_1, \xi')$ is defined as the inverse of $x'=d_{\xi'}\Phi(x_1, \cdot, \xi')$.

Lemma 2.11. *Assume that for some $\varepsilon_1>0$ and some constant C_1 the estimate*

$$\begin{aligned}(2.37)\qquad &\lambda^{-2\sigma}\|h_{\varepsilon_1,\lambda}(y', D_{y'})v\|+\lambda^{-2\delta}\|\tilde{q}_\lambda(y, D_{y'})h_{\varepsilon_1,\lambda}(y', D_{y'})v\| \\ &\leqq C_1(\|(D_{y_1}+i\lambda^{-2\delta}\tilde{q}_\lambda(y, D_{y'}))v\|+\|v\|)\end{aligned}$$

holds if $v\in\mathcal{S}$ vanishes for $|y_1|>\varepsilon_1$. Then (2.3) *holds for some $\varepsilon>0$ and C.*

Proof. Fix y_1 as a parameter and let $\Phi_\lambda(y_1, y', \eta')$ correspodn to $S_\lambda(y, \eta)$ in Definition 2.4. By the remark after Corollary 2.8, we see that the $\lambda^{-2}(I^*_{\Phi_\lambda}q_\lambda-\tilde{q}_\lambda I^*_{\Phi_\lambda})$ as an operator on $L_2(R^{n-1}_{y'})$ has a uniform bound with respect to $|y_1|<\varepsilon$ and $0<\lambda\leqq 1$, therefore it has a uniform bound as an operator on $L_2([-\varepsilon, \varepsilon]\times R^{n-1}_{y'})$ by integrating with respect to y_1.
Note that

$$\begin{aligned}D_{y_1}I^*_{\Phi_\lambda}v &= I^*_{\Phi_\lambda}D_{y_1}v-(\partial_{y_1}\Phi_\lambda)^*_{\Phi_\lambda}v \\ &= I^*_{\Phi_\lambda}D_{y_1}v+\lambda^{-2\delta}T^*_{\Phi_\lambda}v\,, \\ &\quad \text{if } v\in\mathcal{S} \text{ vanishes for } |y_1|>\varepsilon\,,\end{aligned}$$

where $t_\lambda(y, \eta')=r_\lambda(y', d_{y'}\Phi_\lambda(y, \eta'))$. In fact this follows from $\partial_{y_1}\Phi_\lambda(y, \eta')=-\lambda^{-2\delta}r_\lambda(y', d_{y'}\Phi_\lambda(y, \eta'))$. The adjoint form of Proposition 2.7 shows that the second term equals $I^*_{\Phi_\lambda}\lambda^{-2\delta}R_\lambda$ modulo L_2-bounded operator. Hence, substitution $I^*_{\Phi_\lambda}v$ for v of (2.37) gives (2.31) because for $\tilde{h}_{\varepsilon,\lambda}(y_1, y', \eta')$ defined from $h_\varepsilon(x', \xi')$ in the same way as (2.36), there exists some ε' such that $\tilde{h}_{\varepsilon,\lambda}=1$ on supp $h_{\varepsilon'}(\lambda y', \lambda\eta')\times\{|y_1|<\varepsilon\}$, provided that $0<\lambda\leqq\lambda_0$ for some sufficiently small λ_0.

In the rest of this section we investigate the properties of $\tilde{q}_\lambda(y, \eta')$ derived from the assumptions. It follows from (2.36) that for any α, β' and some $C_{\alpha\beta'}$ independent of λ

$$\begin{aligned}(2.38)\qquad &|\partial^\alpha_y\partial^{\beta'}_{\eta'}\tilde{q}_\lambda(y, \eta')|\leqq C_{\alpha\beta'}\lambda^{|\alpha'+\beta'|} \\ &\text{on } \Omega_\varepsilon=\{|y_1|<\varepsilon,\quad |y'|+|\eta'|<\varepsilon\lambda^{-1}\}\,.\end{aligned}$$

The second property is that

(2.39) $\tilde{q}_\lambda(y, \eta')$ does not change sign from $+$ to $-$ for y_1 increasing if λ is small enough.

This follows from (1.10). In fact, since it follows from (2.25) that

$$d_{x,\xi}(\text{Re } a_\gamma\circ\chi)^{1/l})(x, 0, \xi')=d_{x,\xi}(\text{Re } a_\gamma\circ\chi)^{1/l})(x, 0, \xi')$$

we obtain the property (2.39) because the modified-null-bicharacteristic curve is invariant under canonical transformations χ and χ_λ. The invariance of

Poisson brackets for canonical transformations and (2.25) give the following;

$$(2.40)\qquad \begin{cases} \text{for any } (y, \eta')\in\Omega_\varepsilon=\{|y_1|<\varepsilon,\ |y'|+|\eta'|<\varepsilon\lambda^{-1}\} \\ \text{there exists a } I\in\mathcal{J}_\mu \text{ such that } \tilde{p}_{I,\lambda}(y, \eta')\neq 0 \text{ if} \\ \lambda \text{ is small enough.} \end{cases}$$

Here $\tilde{p}_{I,\lambda}(y, \eta')$ is defined in the same way as $p_I(x, \xi)$ of (1.11) with $p_1=\eta_1$ and $p_2=\lambda^{-2\delta}\tilde{q}_\lambda(y, \eta')$. Indeed, note that (1.12) is invariant under canonical transformations and in changing $p_i(i=1,2)$ to f_ip_i for non-vanising functions $f_i(i=1,2)$. By means of (2.25) and the definition of $\mathcal{J}$, we see that for any $(y, \eta')\in\Omega_\varepsilon$ there exist a $I\in\mathcal{J}_\mu$ and $c_I>0$ such that

$$|p^0_{I,\lambda}(y, \eta')|\geqq c_I\lambda^{-2+2b(I)/l},$$

where $p^0_{I,\lambda}$ is defined by setting $p_1=\eta_1$ and $p_2=\lambda^{-2\delta}q_\lambda(y, \eta')$. If $p_{I,\lambda}$ denotes $p^0_{I,\lambda}$ with replaced η_1 by $\eta_1+\lambda^{-2\delta}r_\lambda(y', \eta')$, we obtain

$$|p_{I,\lambda}-p^0_{I,\lambda}|\leqq C_I\lambda^{-2\delta+2b(I)/l}$$

for some constant C_I determined by the derivatives of r and q. Consequently, the invariance of Poisson brackets under χ_λ gives (2.40).

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, as observed in the preceding section, it suffices to show (2.37). For the sake of simplicity, we denote $\lambda^{-2\delta}\tilde{q}_\lambda(y, \eta')$ by $q(x, \xi')$. Suppose that λ is small enough. By means of a constant scale change in the variables, we may assume from (2.38)–(2.40) that $q(x, \xi')$ satisfies the following conditions: For any α, β and some constant $C_{\alpha\beta}$ (independent of λ)

$$(3.1)\qquad |D^\alpha_\xi D^\beta_x q(x, \xi')|\leqq C_{\alpha\beta}\lambda^{|\alpha'+\beta'|-2\delta} \quad \text{in } \Omega=\{|x_1|<1,\ |(x', \xi')|<\lambda^{-1}\}$$

(3.2) $\qquad q(x, \xi')$ does not change sign from $+$ to $-$ for x_1 increasing.

(3.3) For any $(x, \xi')\in\Omega$ there exist some $\mu\in Q_0$ and some $c_0>0$ such that

$$\sum_{I\in\mathcal{J}_\mu}\lambda^{2-2b(I)/l}|p_I(x, \xi')|\geqq c_0>0$$

provided that $p_1=\xi_1, p_2=q(x, \xi')$. Then (2.37) is stated as follows; for some C

$$(3.4)\qquad \begin{aligned} &\lambda^{-2\sigma}||h(\lambda x', \lambda D')u||+||D_{x_1}h(\lambda x', \lambda D')u|| \\ &\leqq C(||(D_{x_1}+iq(x, D'))u||+||u||), \\ &\qquad\text{if } u\in\mathcal{S} \text{ vanishes for } |x_1|>1/2, \end{aligned}$$

where h is C_0^∞ with support in a ball of radius 1/2.

The proof of (3.4) for $q(x, \xi)$ with (3.1)–(3.3) is the same as in showing [4,(6.1) and (6.30)] except the difference of "weight". (See [4, (4.1) and (4.3)].) To prove [4,(6.1) and (6.30)] it was important to obtain inequalities for q in [4, Section 4]. So we sketch the argument corresponding to [4, Section 4]. Put

$$M(x, \xi') = \max_{I \in \mathcal{J}_\mu} |p_I(x, \xi')/\rho|^{1/|I|}. \tag{3.5}$$

Here ρ is a large parameter, whose role is the same as in [4, Section 4] (see [4, p. 149]). By (3.1) and (3.3) we have

$$C_1 \lambda^{-2\delta/\mu} \rho^{-(l-1)/\mu} \leqq M(x, \xi') \leqq C_2 \lambda^{-2\delta}/\rho. \tag{3.6}$$

Here and in what follows the constants are independent of λ and ρ.

The definition of $M=M(0)$ means in particular that

$$|D_{x_1}^j q(0)| < \rho M^{j+1}, \qquad j \leqq \mu - 1, \tag{3.7}$$

and where (3.1) is valid we have by (3.6)

$$|D_{x_1}^j q| \leqq \mathcal{O}(\lambda^c) \rho M^{[\mu]+1} \tag{3.8}$$

since $\rho^{l-2}/M^{[\mu+1]-\mu} \ll 1$ if λ is small enough, where c is some positive. If we set

$$F(t, y', \eta') = q(t/M, y'(\rho M)^{1/2\delta}, \eta'(\rho M)^{1/2\delta})/\rho M,$$

then the application of [4, Lemma 7.1] to $F(t, y', \eta')$ shows that

$$\begin{aligned} &|D_\xi^\alpha D_x^\beta (\tilde{q}(x, \xi') - \xi_2 \partial \tilde{q}(x_1, 0)/\partial \xi_2)| \leqq C_{\alpha\beta} \rho M^{\beta_1+1} (\rho M)^{-|\alpha'+\beta'|/2\delta} \\ &\qquad \text{if } |x_1 M| < 1,\ |(x', \xi')| < C(\rho M)^{1/2\delta}, \end{aligned} \tag{3.9}$$

where $\tilde{q}$ is determined from q by a sympletic orthogonal transformation.

Let $\varepsilon = \lambda^\kappa$ with $0 < \kappa < 1/\mu$ (which is different from ε in the preceding section). Then

$$\varepsilon^2 (\rho M)^{1/\delta} \gg 1. \tag{3.10}$$

As in [4], using this ε we consider the following two cases.

Case I. Assume that

$$|d_{x'\xi'} D_{x_1}^j q(0)| \leqq \varepsilon \rho M^{j+1}, \qquad j < \mu'. \tag{3.11}$$

where $\mu' = (l-2)(\mu-1)/2(l-1)$,. In view of (3.1), (3.6) and (3.10) we get for some $c > 0$

$$|d_{x'\xi'} D_{x_1}^j q(x, \xi')| \leqq \mathcal{O}(\lambda^c) \varepsilon \rho M^{[\mu']+1}, \qquad j > \mu',\ (x, \xi') \in \Omega. \tag{3.11$'$}$$

Then it is easy to check that the argument corresponding to Case I in [4, Section

4] follows with k and $k/2$ replaced by $[\mu-1]$ and μ'.

Case II. Assume now that (3.11) is not fulfilled. Choose $s<\mu'$ so that with $q^{(j)}=\partial^j q/\partial x_1^j$

$$d_{x'\xi'}q^{(s)}(0)=a \tag{3.12}$$

$$d_{x'\xi'}q^{(j)}(0)\leqq aM^{j-s} \qquad \text{for } j<\mu' \tag{3.13}$$

Then (3.1) and the fact that (3.11) is not valid give

$$\varepsilon\rho M^{s+1}<a\leqq C\lambda^{1-2\delta}. \tag{3.14}$$

In view of (3.11)′ we have then for every j

$$|D_{x_1}^j\tilde{q}(x_1,0)/\partial\xi_2|\leqq C_j aM^{j-s}=C_j\rho M^{j+1}A_2, \qquad |x_1M|<1, \tag{3.13$'$}$$

where $A_2=a/\rho M^{s+1}$. The equality of (3.13)′ holds when $j=s$. From (3.9) we can therefore obtain an estimate of the form [4, (4.10)] with $B_2=A_3=B_3=\cdots=(\rho M)^{-1/2\delta}$ and $K=M$. However, $A_2B_2=a/M^{s+1}(\rho M)^{1/2\delta}$ so [4, Lemma 4.1] is not applicable if $a>M^{s+1}(\rho M)^{1/2\delta}$. In that case we shall replace the orthogonal sympletic transformation which led from q to $\tilde{q}$ by a non-linear canonical transformation.

Thus assume for the moment that (3.12), (3.13) and

$$M^{s+1}(\rho M)^{1/2\delta}<a\leqq C\lambda^{1-2\delta} \tag{3.14$'$}$$

are fulfilled. Let $b=a\lambda^{2(\delta-1)}$. Then the function

$$Q(x',\xi')=(q^{(s)}(0,bx',b\xi')-q^{(s)}(0,0,0))/ab$$

is in a bounded subset of $C^\infty(U)$ ($U=\{|(x',\xi')|<C^{-1}\}$) since $|(bx',b\xi')|<\lambda^{-1}$. Hence there exists some canonical transformation χ belonging to a bounded set in C^∞ for $0<\lambda\leqq 1$ such that

$$Q\circ\chi(x',\xi')=\xi_2$$

in a neighborhood of 0. If we put

$$\chi_b(x',\xi')=b\chi(b^{-1}x',b^{-1}\xi')$$
$$\tilde{q}(x,\xi')=q(x_1,\chi_b(x',\xi')),$$

then we obtain

$$\tilde{q}^{(s)}(x,\xi')=a\xi_2+\tilde{q}^{(s)}(0,0) \qquad \text{when } x_1=0,\ |(x',\xi')|<cb. \tag{3.15}$$

By the same way as in [4] we get

(3.16) $$|a^j(\partial/\partial x_2)^j(\partial/\partial x_1)^i\tilde{q}(0)|\leqq C_{ij}M^{j(s+1)}\rho M^{i+1}$$

for any i, j satisfying

$$j\leqq l-2 \text{ and } (i+1+j(s+1))(l-1)/(l-j-1)\leqq\mu .$$

If we introduce

(3.17) $$B_2 = M^{s+1}/a ,$$

noting incidentally that $\rho A_2B_2=1$ as required in [4, Lemma 4.1], this means that we have bounds for the derivatives of $\tilde{q}(x_1/M, x_2/B_2, 0)/\rho M$ at 0.

As stated in [4, p. 156], when we deiive (3.16), we can replace the canonical transformation $\chi_b(x', \xi')$ by another $\tilde{\chi}_b(x', \xi')$ which is linear in all variarles except x_2, provided that the integral curve of the Hamilton field of $\tilde{q}^{(s)}(0, x', \xi')$ b) new $\tilde{\chi}_b$ is also the x_2 axis $x_2=at$, $x_3=\cdots=\xi_n=0$. We denote $\tilde{\chi}_b$ by χ_b in what follows.

By the analogous calculation as in showing [4, (4.23] and (4.23)′] we obtain when $|x_1M|<1$, $|(x', \xi')|<cb$

(3.18) $$|D_\xi^\alpha D_x^\beta\tilde{q}(x, \xi')|\leqq C_{\alpha\beta}aM^{\beta_1}b^{1-|\alpha'+\beta'|} \quad \text{if } |\alpha'+\beta'|\neq 0$$

(3.18)′ $$|D_\xi^\alpha D_x^\beta\tilde{q}(x, \xi')|\leqq C'_{\alpha\beta}\lambda^{-2\delta}b^{-|\alpha'+\beta'|} \quad \text{for any } \alpha', \beta' .$$

In particular (3.18)′ is a much better estimate than (3.16) if $j\geqq l-1$ or $(i+1+j(s+1))(l-1)/(l-j-1)>\mu$, and it is not only valid at 0. Hence (3.16) leads us to uniform bounds for $\tilde{q}(x_1/M, x_2/B_2, 0)/\rho M$ and all of its derivatives when $|x_1|<1$ and $|x_2|<1$.

Since $b\gg(\rho M)^{1/2\delta}$ we can apply [4, Lemma 7.1] to

$$F(t, x_2, y) = (M\rho)^{-1}\tilde{q}(t/M, x_2/B_2, x''(\rho M)^{1/2\delta}, \xi'(\rho M)^{1/2\delta})$$

where $x''=(x_3, \cdots, x_n)$ and $y=(x'', \xi')$. Therefore, by the same way as in getting [4, (4.25)] we obtain

(3.19) $$|D_\xi^\alpha D_x^\beta(\tilde{q}(x, \xi')-\xi_2\partial\tilde{q}(x_1, x_2, 0)/\partial\xi_2)$$
$$\leqq CM^{\beta_1+1}\rho B_2^{\beta_2}(\rho M)^{-|\alpha'+\beta'|/2\delta}$$

when $$|x_1|M<1,\ |(x'', \xi')|<(\rho M)^{1/2\delta},\ |x_2|<1/B_2 .$$

As in [4], we obtain [4, (4.26)] and [4, (4.27)] with replaced the right hand side by $Cb^{-\beta_2}\lambda^{2/l}$. Hence it follows from (3.12) that $(B_2/M)\partial\tilde{q}(x_1/M, 0)/\partial\xi_2$ is essentially a normallized polinomial in x_1 of degree $[\mu']$ and $\partial\tilde{q}(x_1/M, x_2, 0)/\partial\xi_2$ is almost independent of x_2.

From (3.19) and these inequalities we obtain with $B_1=M$, $A_j=B_j=(\rho M)^{-1/2\delta}$ when $j>2$

$$(3.20)\qquad |D_\xi^\alpha D_x^\beta \tilde{q}(x, \xi')| \leqq C_{\alpha\beta}\rho M A^\alpha B^\beta\,, \quad \text{if } |x_1 M|<1,\ |x_2 B_2|<1\,,\ |\xi_2 A_2|<N,\ |(x'', \xi'')|<(\rho M)^{1/2\delta}\,,$$

where N is a fixed but arbitrary constant. If we write $\tilde{M}(x, \xi')=M(x_1, \chi_b(x', \xi'))$, it follows in view of [4, Lemma 4.1], where we take $A_1=1/M$ and $K=M$, that

$$(3.21)\qquad \tilde{M}(x, \xi')\leqq C_N M \quad \text{if } |x_1 M|<1,\ |x_2 B_2|<1\,,\ |\xi_2 A_2|<N,\ |(x'', \xi'')|<(\rho M)^{1/2\delta}\,.$$

when (3.14) is fulfilled but not (3.14)′ we get the same conclusion with B_2 replaced by $(\rho M)^{-1/2\delta}$ and χ_b equal to orthogonal sympletic transformation such that $\tilde{q}(x, \xi')=q(x_1, \chi_b(x', \xi'))$.

The argument corresponding to the rest of [4, Section 4] can be done by the same way if we let (3.1)–(3.3), (3.16), (3.18), (3.18)′, (3.19), (3.20), (3.21), $[\mu-1]$, μ', χ_b and $(\rho M)^{1/2\delta}$ correspond to [4, (4.1)–(4.3), (4.21), (4.23), (4.23)′, (4.25), (4.28), (4.29), k, $k/2$, χ_a and $\sqrt{\rho M}$] respectively.

Because we have got the result corresponding to [4, Section 4] we can easily prove (3.4) by the same way as in [4, Section 6], if we take the above correspondence. The detail is left to the reader.

4. Proof of Proposition 1.2

The method of the proof is a version of [2] (, see also [15]). Suppose that $\tilde{L}$ is hypoelliptic at the origin. Then, as well-known, there exist a positive integer s, a constant C and some neighborhood U of 0 such that

$$(4.1)\qquad ||u||\leqq C||\Lambda^s\tilde{L}u||\,, \qquad u\in C_0^\infty(U)\,,$$

where the symbol of Λ is $(|\xi_1|^{10}+|\xi'|^8+1)$. Hence for any large N there exists a C_N such that

$$(4.1)'\qquad ||u||\leqq C_N(||\Lambda^s\tilde{L}u||+|||x|^N u||+|||x|^N\Lambda^s\tilde{L}u||)\,, \qquad u\in\mathcal{S}\,.$$

If we take the canonical change of variables (x, ξ) into

$$(4.2)\qquad (\lambda^2 x_1, \lambda^9 x', \lambda^{-2\tau-2}(\xi_{01}+\lambda^{2\tau}\xi_1), \lambda^{-2\tau-9}(\xi_0'+\lambda^{2\tau}\xi'))\,,\quad (0<\lambda\leqq 1,\ \tau=13)\,,$$

with a fixed $\xi_0=(\xi_{01}, 0, \xi_{03})$ satisfying

$$\xi_{01}^5+2\xi_{03}^4=0 \ \text{ and } \ \xi_{03}>0\,,$$

then it follows from (4.1)′ that

$$(4.3)\qquad ||u||\leqq C_N'(\lambda^{-M}||L_\lambda u||+\lambda^{2N-M}||R_0(x, \lambda^{2\tau}D_x)u||)\,, \qquad u\in\mathcal{S}\,,$$

where

$$L_\lambda = L_1(\lambda^{2\tau}D_x)+i\lambda^\tau x_1^2 x_2 \xi_{03}^4+\sum_{j=1}^{2}\lambda^{\tau(2-j)}R_j(x, \lambda^{2\tau}D_x),$$
$$L_1(\xi) = 5\xi_{01}^4\xi_1+4\xi_{03}^3(2\xi_3-\xi_2),$$
$$R_j(x, \xi) = \sum_{\substack{j\leqq|\alpha|\\ |\alpha+\beta|\leqq N_j}} a_{\alpha\beta j}(\lambda)x^\beta\xi^\alpha \qquad (j=0, 1, 2)$$
$$(a_{\alpha\beta j}(\lambda) = \mathcal{O}(\lambda^c),\ c\geqq 0).$$

Here M and $N_j(j=0, 1, 2)$ are some integers depending on s. Since $\xi_{03}>0$ we can take $w(x)$ such that

$$(4.4)\qquad \begin{aligned} &L_1(\partial_x w)+ix_1^2x_2\xi_{03}^4 = 0\\ &|\mathrm{Im}\, w|\geqq c_0|x|^4,\quad c_0>0. \end{aligned}$$

Put $u_\lambda(x)=(\exp i\lambda^{-\tau}w(x))\sum_{j=0}^{N} v_j(x, \lambda)\lambda^{\tau j}$, where v_j will be determined later such that they are polinomials of x whose coefficients have the same property as $a_{\alpha\beta j}$. Then we have

$$L_\lambda u_\lambda = \exp i\lambda^{-\tau}w\sum_{j=2}^{N}(L_1(D_x)+A(x, \partial_x w))v_{j-2}+F_j(x))\lambda^{\tau j},$$
$$(F_2=0).$$

Here $A(x, \xi)=\sum_{|\alpha|=1} a_{\alpha\beta 1}x^\beta\xi^\alpha$ and $F_j(x)$ is a linear combination of the functions $v_0,\cdots,v_{j-3}$ and their derivatives. By means of Cauchy-Kowalewska theorem, solve the transport equations

$$(L_1(D_x)+A(x, \partial_x w))v_j+F_{j+2}(x) = \mathcal{O}(|x|^{N-j})$$

under the condition $v_0(0, \lambda)=1$ and $v_j(0, \lambda)=0\ (j\geqq 1)$, successively. Then the analytic solution obtained, which is defined in a certain neighborhood ω of the origin, is to be multiplied by a cut off function $\phi\in C_0^\infty(\omega)$ which equals 1 in another neighborhood of the origin. The multiplication does not affect (4.5) because $(1-\phi)v_j=\mathcal{O}(|x|^N)$. Substituting $u_\lambda(x)\in\mathcal{S}$ into (4.3) and changing variables x into $\lambda^{\tau/4}x$ give us the contradiction if $2N>M$ and if λ tends to 0.

References

[1] Yu. V. Egorov: *Canonical transformations and pseudodifferential operators,* Trudy Moskov. Mat. Obšč. **24** (1971), 3–28. =Trans. Moscow Math. Soc. **24** (1971), 1–28.

[2] Yu. V. Egorov: *On necessary conditions for the solvability of pseudodifferential equations of principal type,* Trudy Moskov. Mat. Obšč. **24** (1971), 29–41=Trans. Moscow Math. Soc. **24** (1971), 29–42.

[3] Yu. V. Egorov: *Subelliptic operators,* Uspehi. Mat. Nauk **30**; 2 (1975), 57–114, **30**; 3 (1975), 57–104=Russian Math. Surveys **30**: 2 (1975), 59–118, **30**: 3

(1975), 55–105.

[4] L. Hörmander: *Subelliptic operators*, Seminar on Singularities of Solution of Linear Partial Differential Equations, Princeton University Press 1979, 3–49.

[5] H. Kumano-go: Pseudo-differential operators, Iwanami Publisher, Tokyo, 1974 (in Japanese).

[6] A. Menikoff: *On hypoelliptic operators with double characteristics*, Ann. Scoula Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), 689–724.

[7] R.B. Melrose: *Equivalence of glancing surfaces*, Invent. Math. **37** (1976), 165–191.

[8] S. Mizohata: *Solutions nulles et solutions non-analytiques*, J. Math. Kyoto Univ. **1** (1961/2), 271–302.

[9] Y. Morimoto: *On the hypoellipticity for infinitely degenerate semi-elliptic operators*, J. Math. Soc. Japan **30** (1978), 327–358.

[10] L. Nirenberg and F. Treves: *On local solvability of linear partial differential equations, Part* I: *Necessary conditions*, Comm. Pure Appl. Math. **23** (1970), 1–38, *Part* II: *Sufficient conditions*, Comm. Pure Appl. Math. **23** (1970), 459–509.

[11] O.A. Oleinik and E.V. Radkevich: Second orderiequations with non-negative characteristic form, Amer. Math. Soc., Providence, Rhode Island and Plenum Press, 1973.

[12] P.R. Popivanov: *On local solvability for a class of pseudodifferential equations with double characteristics*, Journal of I.G. Petrovskii Seminar **1** (1975), 237–278.

[13] R. Rubinstein: *Local solvability of the operators* $u_{tt}+ia(t)u_x+b(t)u_t+c(t)u$, J. Differential Equations **14** (1973), 185–194.

[14] F. Treves: *A new method of proof of the subelliptic estimates*, Comm. Pure Appl. Math. **24** (1971), 71–115.

[15] P. Wenston: *A necessary conditions for the local solvability of the operator* $P_m^2(x, D)+P_{2m-1}(x, D)$, J. Differential Equations **25** (1977), 90–95.

[16] P. Wenston: *A sufficient condition for the local solvability of a linear partial differential operator with double characteristics*, J. Differential Equations **29** (1978), 374–387.

[17] P. Popivanov and G. Popov: *Microlocal properties of a class of pseudodifferential operators with multiple characteristics*, *Part* I, *Part* II, Serdica **6** (1980), 169–183, Serdica **7** (1981), 131–141. (in Russian).

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