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ANALYTICITY OF SOLUTIONS OF QUASILINEAR EVOLUTION EQUATIONS II

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0. Introduction

In this paper we establish analyticity in t of solutions to quasilinear evolution equations

$$(0.1)\quad \frac{du}{dt}+A(t, u)u=f(t, u), \qquad 0\leqq t\leqq T,$$

$$(0.2)\quad u(0)=u_0 .$$

The unknown, u, is a function of t with values in a Banach space X. For fixed t and $v\in X$, the linear operator $-A(t, v)$ is the generator of an analytic semigroup in X and $f(t, v)\in X$. In the case that the domain $D(A(t, v))$ of $A(t, v)$ does not depend on t and v, Massey [7] discussed analyticity in t for equation of the form (0.1).

In the present paper, we consider analyticity for (0.1), (0.2) under the assumptions that the domain $D(A(t, v)^h)$ of $A(t, v)^h$ is independent of t, v for some $h=1/m$ where m is a positive integer and that $A(t, A_0^{-\alpha}v)^h$ is Hölder continuous in v in the sense that $||A(t, A_0^{-\alpha}v)^h A(t, A_0^{-\alpha}w)^{-h}-I||\leqq C||v-w||^\eta$, while in the previous papers [2], [3] we discussed the same problem in the case that $A(t, A_0^{-\alpha}v)^h$ is Lipschtz continuous. In order to prove the theorems we shall make use of the linear theory of Kato [5].

In the following $L(X, Y)$ is the space of linear operators from a normed space X to another normed space Y, and $B(X, Y)$ is the space of bounded linear operators belonging to $L(X, Y)$. $L(X)=L(X, X)$ and $B(X)=B(X, X)$. $||\ ||$ will be used for the norm in both X and $B(X)$; it should be clear from the context which is intended. $\sum(\phi; T)\equiv\{t\in \boldsymbol{C};\ |\arg t|<\phi, 0<|t|<T\}\cup\{0\}$ is the sector in the complex plane.

We shall make the following assumptions:
(A-1) There exist $h=1/m$, where m is an integer, $m\geqq 2$, and $0\leqq\alpha<h/2$ such that $A_0^{-\alpha}$ is a well-defined operator $\in B(X)$ and $u_0\in D(A_0^{1+\alpha})$ where $A_0\equiv A(0, u_0)$.
(A-2) A_0^{-1} is a completely continuous operator.

(A–3) There exist $R>0$, $M>0$, $T_0>0$ and $\phi_0>0$ such that $A(t, A_0^{-\alpha}v)$ is a well-defined operator $\in L(X)$ for each $t\in\sum(\phi_0; T_0)$ and $v\in N\equiv\{v\in X; ||v-A_0^{\alpha}u_0||<R\}\cap Y\cup\{A_0^{\alpha}u_0\}$, and the domain, $D(A(t, A_0^{-\alpha}v))$, of $A(t, A_0^{-\alpha}v)$ is dense in X. Where $Y\equiv\bigcup_{t>0}\{v\in X; ||v-(A_0^{\alpha}u_0+ta)||<tM\}$ $(0<M\leqq||a||)$ and we shall define $a\in X$ in the next section.

(A–4) For any $t\in\sum(\phi_0; T_0)$, $v\in N$

$$(0.3)\quad \begin{cases}\text{the resolvent set of } A(t, A_0^{-\alpha}v) \text{ contains the left half-plane and there} \\ \text{exists } C_1 \text{ such that } ||(\lambda-A(t, A_0^{-\alpha}v))^{-1}||\leqq C_1(1+|\lambda|)^{-1},\ Re\lambda\leqq 0\,.\end{cases}$$

(A–5) The domain $D(A(t, A_0^{-\alpha}v)^h)=D$ of $A(t, A_0^{-\alpha}v)^h$ is independent of $t\in\sum(\phi_0; T_0)$, $v\in N$.

(A–6) There exist C_2, C_3, σ, $1-h+\alpha<\sigma\leqq 1$, α'', $\alpha<\alpha''<h/2$, η, $\dfrac{1-h+\alpha''}{1-\alpha}<\eta<1$ such that

$$(0.4)\quad ||A(t, A_0^{-\alpha}v)^h A(s, A_0^{-\alpha}w)^{-h}||\leqq C_2 \quad t, s\in\sum(\phi_0; T_0)\,,\quad v, w\in N\,.$$

$$(0.5)\quad ||A(t, A_0^{-\alpha}v)^h A(s, A_0^{-\alpha}w)^{-h}-I||\leqq C_3\{|t-s|^{\sigma}+||w-v||^{\eta}\} \quad t, s\in\sum(\phi_0; T_0)\,,\quad v, w\in N\,.$$

(A–7) The map Φ: $(t, v)\mapsto A(t, A_0^{-\alpha}v)^h A_0^{-h}$ is analytic from $(\sum(\phi_0; T_0)\backslash\{0\})\times(N\backslash\{A_0^{\alpha}u_0\})$ to $B(X)$.

(A–8) $f(t, A_0^{-\alpha}v)$ is defined and belongs to X for each $t\in\sum(\phi_0; T_0)$ and $v\in N$, $f(0, u_0)\in D(A_0^h)$, and there exists C_4 such that

$$(0.6)\quad ||f(t, A_0^{-\alpha})-f(s, A_0^{-\alpha}w)||\leqq C_4\{|t-s|^{\sigma}+||w-v||^{\eta}\} \quad t, s\in\sum(\phi_0; T_0)\,,\quad w, v\in N\,.$$

(A–9) The map Ψ: $(t, v)\mapsto f(t, A_0^{-\alpha}v)$ is analytic from $(\sum(\phi_0; T_0)\backslash\{0\})\times N$ into X.

These constants C_i $(i=1, 2, 3, 4)$ do not depend on t, s, v, w.

The main result of this paper is the following theorem.

Theorem 1. *Let the assumptions* (A–1)–(A–9) *hold. Then there exist* T, $0<T\leqq T_0$, ϕ, $0<\phi\leqq\phi_0$, $K>0$, k, $1-h<k<1$ *and at least one continuous function* u *mapping* $\sum(\phi; T)$ *into* X *such that* $u(0)=u_0$, $u(t)\in D(A(t, u(t)))$ *and* $||A_0^{\alpha}u(t)-A_0^{\alpha}u_0||<R$ *for* $t\in\sum(\phi; T)$, u: $\sum(\phi; T)\backslash\{0\}\to X$ *is analytic,* $du/dt+A(t, u(t))u(t)=f(t, u(t))$ *for* $t\in\sum(\phi; T)\backslash\{0\}$, *and* $||A_0^{\alpha}u(t)-A_0^{\alpha}u_0||\leqq K|t|^k$ *for* $t\in\sum(\phi; T)$.

REMARKS. (1) Under the assumption that $D(A(t, u)^h)$ is constant, Sobolevskii [10] gave the existence of solutions to (0.1) with differentiable coefficients. But, as far as the author knows, the proof of [10] (or similar results) is not published yet.

(2) From the assumptions (A–3) and (A–4), $-A(t, A_0^{-\alpha}v)$ generates an analytic semigroup in X, and the fractional powers $A(t, A_0^{-\alpha}v)^{\beta}$ are defined for $\beta \in \boldsymbol{R}$. Properties of analytic semigroups and fractional powers, see Tanabe [11] Sobolevskii [9] Krein [6] Friedman [1] etc.
(3) In the previous papers [2] [3] we proved similar results with $\eta=1$. In this case we need not the assumption (A–2).

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1. Preliminaries

We shall make the following assumptions:
I) For each $t \in [0, T]$, $A(t)$ is a densely defined, closed linear operator in X with its spectrum contained in a fixed sector $S_\theta \equiv \{z \in C;\ |\arg z| < \theta \leqq \pi/2\}$. The resolvent of $A(t)$ satisfies the inequality

(1.1) $\|[z-A(t)]^{-1}\| \leqq M_0/|z| \qquad \text{for} \quad z \notin S_\theta$

where M_0 is a constant independent of t. Furthermore, $z=0$ also belongs to the resolvent set of $A(t)$ and

(1.2) $\|A(t)^{-1}\| \leqq M_1$

M_1 being independent of t.
II) For some $h=1/m$, where m is a positive integer $\geqq 2$, $D(A(t)^h)=D$ is independent of t, and there are constants k, M_2 and M_3 such that

(1.3) $\|A(t)^h A(s)^{-h}\| \leqq M_2\,, \qquad 0 \leqq t \leqq T\,,\quad 0 \leqq s \leqq T\,.$

(1.4) $\|A(t)^h A(s)^{-h} - I\| \leqq M_3 |t-s|^k\,, \qquad 0 \leqq t \leqq T\,,\quad 0 \leqq s \leqq T\,,\quad 1-h<k\leqq 1\,.$

REMARK. From (1.2) there exists $C>0$ such that

(1.2)′ $\|A(t)^{-h}\| \leqq C \qquad \text{for} \quad t \in [0, T]$

C being independent of t.

Under these assumptions, we get the following theorems. They are due to Kato.

Theorem A. *Let the conditions* I) *and* II) *be satisfied. Then there exists a unique evolution operator* $U(t, s) \in B(X)$ *defined for* $0 \leqq s \leqq t \leqq T$, *with the following properties.* $U(t, s)$ *is strongly continuous for* $0 \leqq s \leqq t \leqq T$ *and*

(1.5) $U(t, r) = U(t, s)U(s, r)\,, \quad r \leqq s \leqq t\,,$

(1.6) $U(t, t) = I\,.$

For $s<t$, the range of $U(t, s)$ is a subset of $D(A(t))$ and

(1.7) $A(t)U(t, s)\in B(X)$, $\|A(t)U(t, s)\|\leqq M|t-s|^{-1}$,

where M is a constant depending only on θ, h, k, T, M_0, M_1, M_2 and M_3. Furthermore, $U(t, s)$ is strongly continuously differentiable in t for $t>s$ and

(1.8) $\frac{\partial}{\partial t}U(t, s)+A(t)U(t, s)=0$.

If $u\in D$, $U(t, s)u$ is strongly continuously differentiable in s for $s<t$. If in particular $u\in D(A(s_0))$, then

(1.9) $\frac{\partial}{\partial s}U(t, s)u\,|_{s=s_0}=U(t, s_0)A(s_0)u$.

If $f(t)$ is continuous in t, any strict solution of

(1.10) $\frac{du}{dt}+A(t)u=f(t)$

must be expressible in the form

(1.11) $u(t)=U(t, 0)u(0)+\int_0^t U(t, s)f(s)ds$.

Conversely, the $u(t)$ given by (1.11) *is a strict solution of* (1.10) *if $f(t)$ is Hölder continuous on* $[0, T]$; *here $u(0)$ may be an arbitrary element of X.*

Proof. See, [5].

Theorem B. *Assume that $A(t)$ can be continued to a complex neighborhood Δ of the interval* $[0, T]$ *in such a way that the conditions* I), II) *are satisfied for $t, s\in\Delta$. Furthermore, let $A(t)^{-h}$ be holomorphic for $t\in\Delta$. Then the evolution operator $U(t, s)$ exists for $s\leqq t$, satisfies the assertions of* Theorem A *and is holomorphic in s and t for $s<t$. (Here "$s<t$" should be interpreted as meaning "$t-s\in\sum$", where $\sum$ is the sector $|\arg t|<\pi/2-\theta$ of the t-plane, and "$s\leqq t$" as "$s<t$ or $s=t$".) If $f(t)$ is holomorphic for $t\in\Delta$, $t>0$, and Hölder continuous at $t=0$, every solution of* (1.10) *has a continuation holomorphic for $t\in\Delta$, $t>0$.*

Proof. See, [5].

It follows from I) and II) that

Proposition 1.

(1.12) $\|A(t)^{\alpha}\exp(\tau A(t))\|\leqq N_6|\tau|^{-\alpha}$ $: 0\leqq\alpha\leqq 2$, $|\arg\tau|\leqq\pi/2-\theta$

(1.13) $\|A(t)^{\alpha}U(t, s)\|\leqq (h+k-\alpha)^{-1}N_{18}(t-s)^{-\alpha}$ $: 0\leqq\alpha<k+h$

(1.14) $\|A(t)^{\alpha+h}U(t, s)A(s)^{-h}\| \leqq (k-\alpha)^{-1}N_{19}(t-s)^{-\alpha} \quad : 0 \leqq \alpha < k,\ 0 \leqq s \leqq t \leqq T$.

Here the constants $N_i (i \geqq 4, i \in \boldsymbol{N})$ are determined by M_0, M_1, M_2, M_3, θ, h, k, T. The above Proposition is essentially proved in [5]. In addition to these, we need the following estimates in [3].

Proposition 2. *If* $1-h<k<1$, $0<\alpha<\alpha'<1-k$, *then for any* $0 \leqq r \leqq s \leqq t \leqq T$, *the following inequalities hold:*

(1.15) $\|A(0)^{\alpha}[U(t, 0)-U(s, 0)]A(0)^{-1}\| \leqq C(t-s)^{1-\alpha'}$

(1.16) $\|A(0)^{\alpha}[U(t, r)-U(s, r)]\| \leqq C(t-s)^{1-\alpha'}(s-r)^{-1}$,

where the constant C *is determined by* M_0, M_1, M_2, M_3, θ, h, k, α, T.

Proposition 3. *Let the function* $f(t)$ *be continuous on* $[0, T]$. *Then for any* $0 \leqq s \leqq t \leqq T$, $0<\alpha<\alpha'<\alpha''<h$, *the following inequality holds:*

$$(1.17)\quad \|A_0^{\alpha}[\int_0^t U(t, r)f(r)dr - \int_0^s U(s, r)f(r)dr]\| \leqq C_{\alpha\alpha'}|t-s|^{1-\alpha''}(|\log(t-s)|+1)\max_{0 \leqq r \leqq T}\|f(r)\|.$$

Proposition 4. *If* $0<\alpha'<\alpha''<h$, *then for any* $0 \leqq r \leqq t \leqq T$, *the following inequality holds:*

(1.18) $\|A(t)^{\alpha'}U(t, r)A(r)^{1-ph}\| \leqq C(t-r)^{ph-\alpha''-1} \qquad p = 1, 2, \cdots, m$.

Proposition 5. *Let the function* $f(t)$ *be Hölder continuous on* $[0, T]$. *Then for any* $0 \leqq r \leqq T$, *the following inequality holds:*

(1.19) $\|A(r)^{ph}\int_0^r U(r, s)f(s)ds\| \leqq Cr^{1-ph} \quad : p = 1, 2, \cdots, m$.

Now we shall define a. We shall make the following assumptions;

(a–1)=(A–1)

(a–2) There exists $T_0>0$, such that $A_{u_0}(t)=A(t, u_0)$ is a well-defined operator from X to X for each $t \in [0, T_0)$.

(a–3) For any $t \in [0, T_0)$ the resolvent of $A_{u_0}(t)$ contains the left half-plane and there exists C_1 such that $\|(\lambda - A_{u_0}(t))^{-1}\| \leqq C_1(1+|\lambda|)^{-1}$, $Re\,\lambda \leqq 0$, and the domain, $D(A_{u_0}(t))$, of $A_{u_0}(t)$ is dense in X.

(a–4) The domain $D(A_{u_0}(t)^h)=D$ of $A_{u_0}(t)^h$ is independent of $t \in [0, T_0)$ and there exist C_2, C_3, σ, $1-h+\alpha<\sigma \leqq 1$ such that

$$\|A_{u_0}(t)^h A_{u_0}(s)^{-h}\| \leqq C_2 \qquad t, s \in [0, T_0),$$
$$\|A_{u_0}(t)^h A_{u_0}(s)^{-h} - I\| \leqq C_3|t-s|^{\sigma} \qquad t, s \in [0, T_0).$$

(a–5) $f_{u_0}(t)=f(t, u_0)$ is defined and belongs to X for each $t \in [0, T_0)$ and there exists C_4 such that

$$||f_{u_0}(t)-f_{u_0}(s)||\leqq C_4|t-s|^{\sigma} \qquad t,\ s\in[0,\ T_0)\ .$$

These constants $C_i(i=1, 2, 3, 4)$ do not depend on t, s.

Then from the Theorem A, there is a unique solution of

$$(1.20)\quad \begin{cases} \dfrac{d\hat{u}}{dt}+A_{u_0}(t)\hat{u}=f_{u_0}(t) \\ \hat{u}(0)=u_0\ . \end{cases}$$

With the solution of (1.20) set

$$(1.21)\quad a=\frac{d^+}{dt}A_0^{\alpha}\hat{u}(t)|_{t=0}\ .$$

We can define a since by $u_0\in D(A_0^{1+\alpha})$, $f_{u_0}(0)\in D(A_0^h)$ and $1-h+\alpha<\sigma\leqq 1$. In fact from (1.13), (a–5) and (a–4) we have

$$\begin{aligned} &||A_0^{\alpha}\int_0^t U_{u_0}(t,\ s)f_{u_0}(s)ds|| \\ &\leqq\int_0^t ||A_0^{\alpha}U_{u_0}(t,\ s)||\cdot||f_{u_0}(s)-f_{u_0}(0)||ds \\ &\qquad +\int_0^t ||A_0^{\alpha}U_{u_0}(t,\ s)A_{u_0}(s)^{-h}||\cdot||A_{u_0}(s)^hA_0^{-h}||\cdot||A_0^hf_{u_0}(0)||ds \\ &\leqq\int_0^t (h+k-\alpha)^{-1}N_{18}(t-s)^{-\alpha}C_4s^{\sigma}ds+\int_0^t C(t-s)^{h-\alpha'}C_2||A_0^hf_{u_0}(0)||ds \\ &\leqq Ct^{1+h-\alpha'}\ . \end{aligned}$$

2. Existence of solutions on the real axis

We consider the Cauchy problem

$$(2.1)\quad du/dt+A(t,\ u)u=f(t,\ u) \qquad 0\leqq t\leqq T$$

$$(2.2)\quad u(0)=u_0\ .$$

We shall make the following assumptions:

(R–1) There exist $h=1/m$, where m is an integer, $m\geqq 2$, and $0\leqq\alpha<h/2$ such that $A_0^{-\alpha}$ is a well-defined operator$\in B(X)$ and $u_0\in D(A_0^{1+\alpha})$ where $A_0\equiv A(0,\ u_0)$.

(R–2) A_0^{-1} is a completely continuous operator.

(R–3) There exist $R>0$ and $M>0$ such that $A(t,\ A_0^{-\alpha}v)$ is a well-defined operator $\in L(X)$ for each $t\in[0,\ T]$ and $v\in N\equiv\{v\in X;\ ||v-A_0^{\alpha}u_0||<R\}\cap Y\cup\{A_0^{\alpha}u_0\}$ where $Y\equiv\bigcup_{t>0}\{v\in X;\ ||v-(A_0^{\alpha}u_0+ta)||<tM\}$, $0<M\leqq||a||$, and the domain, $D(A(t,\ A_0^{-\alpha}v))$, of $A(t,\ A_0^{-\alpha}v)$ is dense in X.

(R–4) For any $t\in[0,\ T]$ and $v\in N$

(2.3) $\begin{cases} \text{the resolvent set of } A(t, A_0^{-\alpha}v) \text{ contains the left half-plane and there} \\ \text{exists } C_1 \text{ such that } \|(\lambda - A(t, A_0^{-\alpha}v))^{-1}\| \leqq C_1(1+|\lambda|)^{-1}, \quad Re\,\lambda \leqq 0 . \end{cases}$

(R–5) The domain $D(A(t, A_0^{-\alpha}v)^h) = D$ of $A(t, A_0^{-\alpha}v)^h$ is independent of $t \in [0, T]$ and $v \in N$.

(R–6) There exist C_2, C_3, σ, $1-h+\alpha<\sigma\leqq 1$, $\alpha<\alpha''<h/2$, $\dfrac{1-h+\alpha''}{1-\alpha}<\eta<1$ such that

(2.4) $\|A(t, A_0^{-\alpha}v)^h A(s, A_0^{-\alpha}w)^{-h}\| \leqq C_2 \qquad t, s \in [0, T], \quad v, w \in N,$

(2.5) $\|A(t, A_0^{-\alpha}v)^h A(s, A_0^{-\alpha}w)^{-h} - I\| \leqq C_3\{|t-s|^\sigma + \|v-w\|^\eta\}$
$t, s \in [0, T], \quad v, w \in N.$

(R–7) $f(t, A_0^{-\alpha}v)$ is defined and belongs to X for each $t \in [0, T]$ and $v \in N$, and there exists C_4 such that

(2.6) $\|f(t, A_0^{-\alpha}v) - f(s, A_0^{-\alpha}w)\| \leqq C_4\{|t-s|^\sigma + \|v-w\|^\eta\} \quad t, s \in [0, T], \quad w, v \in N.$

Theorem 2. *Let the assumptions* (R–1)–(R–7) *hold. Then there exists* S_0, $0<S_0\leqq T$, *such that there exists at least one continuously differentiable solution of* (2.1) *for* $0<t<S_0$ *that is continuous for* $0\leqq t<S_0$ *and satisfies* (2.2).

Proof. Let $\alpha<\alpha''<h/2$, $(1-h+\alpha'')/\eta<\zeta<1-\alpha$, $L>0$ and $0<\varepsilon<1$. We consider the set $F(S)$ of all functions $v(t)$, defined on $[0, S)$ which satisfy the following;

(2.7) $v(0) = A_0^\alpha u_0 ,$

(2.8) $\|v(t_1)-v(t_2)\| \leqq L|t_1-t_2|^\zeta \qquad \text{for any} \quad t_1, t_2 \in [0, S),$

(2.9) $\|v(t)-(A_0^\alpha u_0+ta)\| \leqq Mt(1-\varepsilon) \qquad \text{for} \quad t \in [0, S)$

Suppose $S_1 \in (0, T]$. Then for any $v \in F(S_1)$

$$\|v(t)-A_0^\alpha u_0\| = \|v(t)-v(0)\| \leqq L|t|^\zeta \qquad \text{for} \quad t \in [0, S_1).$$

So if $0<S_2<\min\{S_1, (RL^{-1})^{1/\zeta}\}$, then

(2.10) $\|v(t)-A_0^\alpha u_0\| < L(RL^{-1}) = R \qquad \text{for} \quad t \in [0, S_2).$

Therefore from (2.9) we have $v(t) \in N$ for $t \in (0, S_2)$. Hence the operator

(2.11) $A_v(t) = A(t, A_0^{-\alpha}v(t))$

is well defined for $t \in [0, S_2)$ and, by (2.3)

$$\|(\lambda - A_v(t))^{-1}\| \leqq C_1/(1+|\lambda|) \qquad \text{if} \quad Re\,\lambda \leqq 0, \quad t \in [0, S_2).$$

From (2.4) we obtain

$$||A_v(t)^h A_v(s)^{-h}|| \leqq C_2 \qquad \text{if} \quad t, s \in [0, S_2) .$$

From (2.5) and (2.8) we also get

$$||A_v(t)^h A_v(s)^{-h} - I|| \leqq C_3\{|t-s|^\sigma + ||v(t)-v(s)||^\eta\} \leqq C_3\{S_2^{\sigma-k} + L^\eta S_2^{\zeta\eta-k}\}\,|t-s|^k$$
$$t, s \in [0, S_2)$$

where $k = \min\{\sigma, \zeta\eta\}$.

Note that $1-h+\alpha < \sigma \leqq 1$ and $(1-h+\alpha'')/\eta < \zeta < 1-\alpha$ imply $1-h<k<1$.

By Theorem A, there exists a fundamental solution $U_v(t, s)$ corresponding to $A_v(t)$ and all the estimates for fundamental solutions in the previous section hold uniformly with respect to v in $F(S_2)$. In particular, from (1.15) and (1.16) we get for $0<\alpha<\alpha'<1-\zeta$, $0 \leqq r \leqq s \leqq t \leqq S_2$

(2.12) $\quad ||A_0^\alpha[U_v(t, 0) - U_v(s, 0)]A_0^{-1}|| \leqq \tilde{C}\,|t-s|^{1-\alpha'}$

(2.13) $\quad ||A_0^\alpha[U_v(t, r) - U_v(s, r)]|| \leqq \tilde{C}\,|t-s|^{1-\alpha'}\,|s-r|^{-1}$

where $\tilde{C}$ is a constant depending on θ, h, ζ, α, C_1, C_2, C_3, S_2.

Setting $f_v(t) = f(t, A_0^{-\alpha}v(t))$, it follows from (2.6) and (2.8) that

(2.14) $\quad ||f_v(t) - f_v(s)|| \leqq C_4\{|t-s|^\sigma + ||v(t)-v(s)^\eta||\} \leqq C_4\{T^{\sigma-k} + L^\eta T^{\zeta\eta-k}\}\,|t-s|^k .$

Since $f_v(0) = f(0, A_0^{-\alpha}v(0)) = f(0, u_0)$ is independent of v, (2.14) implies that

(2.15) $\quad \max\limits_{0 \leqq t < S_2} ||f_v(t)|| \leqq ||f(0, u_0)|| + C_4\{S_2^{\sigma-k} + L^\eta S_2^{\zeta\eta-k}\}\,S_2^k \leqq C_5 .$

Set $w_{v,\alpha}(t) = A_0^\alpha w_v(t)$, where w_v is the unique solution of

(2.16) $\quad dw_v/dt + A_v(t)w_v = f_v(t) \qquad t \in [0, S_2)$

(2.17) $\quad w_v(0) = u_0 .$

Then from (2.14) and Theorem A, $w_{v,\alpha}$ is given by

(2.18) $\quad w_{v,\alpha}(t) = A_0^\alpha U_v(t, 0)u_0 + A_0^\alpha \int_0^t U_v(t, s) f_v(s)\,ds .$

In view of (2.18), for any t_1, t_2 in $[0, S_2)$ we obtain

(2.19) $$||w_{v,\alpha}(t_1) - w_{v,\alpha}(t_2)|| \leqq ||A_0^\alpha[U_v(t_1, 0) - U_v(t_2, 0)]A_0^{-1}|| \cdot ||A_0 u_0|| + ||A_0^\alpha[\int_0^{t_1} U_v(t_1, s)f_v(s)\,ds - \int_0^{t_2} U_v(t_2, s)f_v(s)\,ds]|| .$$

Making use of (2.14), (2.15) and (1.17), we find that

(2.20) $$||A_0^\alpha[\int_0^{t_1} U_v(t_1, s)f_v(s)\,ds - \int_0^{t_2} U_v(t_2, s)f_v(s)\,ds]|| \leqq \tilde{C}\,|t_1-t_2|^{1-\tilde{\alpha}}(|\log(t_1-t_2)|+1) \qquad \text{where} \quad \zeta < 1-\tilde{\alpha} < 1-\alpha .$$

Therefore from (2.19), (2.12) and (2.20) it follows that

$$||w_{v,\alpha}(t_1)-w_{v,\alpha}(t_2)||\leqq\tilde{C}\,|t_1-t_2|^{1-\alpha'}||A_0u_0||+C\,|t_1-t_2|^{1-\tilde{\alpha}}(\,|\log(t_1-t_2)|+1)\,.$$

Hence if a positive number S_3 satisfies $\tilde{C}S_3^{1-\zeta-\alpha'}||A_0u_0||+CS_3^{1-\zeta-\tilde{\alpha}-\varepsilon}|t_1-t_2|^{\varepsilon}\times(\,|\log(t_1-t_2)|+1)\leqq L$ where $0<\varepsilon<1-\zeta-\tilde{\alpha}$ and if $S_3\leqq S_2$, the inequality

$$(2.21)\qquad ||w_{v,\alpha}(t_1)-w_{v,\alpha}(t_2)||\leqq L\,|t_1-t_2|^{\zeta}\qquad\text{for}\quad t_1,\,t_2\in[0,\,S_3)$$

holds.

We shall prove that if S_4 is sufficiently small, the following inequality holds;

$$(2.22)\qquad ||w_{v,\alpha}(t)-(A_0^{\alpha}u_0+ta)||\leqq Mt(1-\varepsilon)\qquad\text{for all}\quad t\in[0,\,S_4)\,.$$

First, if S_5, $0<S_5\leqq S_3$, is sufficiently small, for any $0\leqq t<S_5$ the following inequality holds;

$$(2.23)\qquad ||w_{v,\alpha}(t)-A_0^{\alpha}\hat{u}(t)||\leqq Mt(1-\varepsilon)/2\qquad\text{for}\quad t\in[0,\,S_5)\,.$$

1) The case of bounded $A(t,\,A_0^{\alpha}v)$.

If $A(t_1,\,A_0^{-\alpha}v_1)$ is assumed to be bounded for some $t_1\in[0,\,S_4)$ and some $v_1\in N$, in addition to the assumption (R–4) and (R–5), it follows that $A(t,\,A_0^{-\alpha}v)\in B(X)$ for all $t\in[0,\,S_4)$ and $v\in N$. In fact the boundedness of $A(t_1,A_0^{-\alpha}v_1)$ implies that of $A(t_1,\,A_0^{-\alpha}v_1)^h$ so that the constant domain $D=D(A(t_1,\,A_0^{-\alpha}v_1)^h)$ must coincide with X. Thus from closed graph theorem $A(t,\,A_0^{-\alpha}v)^h\in B(X)$ and hence $A(t,\,A_0^{-\alpha}v)\in B(X)$ for all t and v.

Let v_1, v_2 belong to $F(S_4)$ and set

$$(2.24)\qquad \begin{cases} A_i(t) = A(t,\,A_0^{-\alpha}v_i(t)) \\ U_i(t,\,s) = U_{v_i}(t,\,s) \\ f_i(t) = f(t,\,A_0^{-\alpha}v_i(t)) \\ w_i(t) = A_0^{-\alpha}w_{v_i,\alpha}(t) \qquad\qquad i=1,\,2\,. \end{cases}$$

Thus, for $i=1,\,2$,

$$(2.25)\qquad \begin{cases} dw_i/dt+A_i(t)w_i = f_i(t) \\ w_i(0) = u_0\,. \end{cases}$$

Note that $w_1(t)\in D(A_2(t))$, $w_2(t)\in D(A_1(t))$ since $A_i(t)\in B(X)$ ($i=1,\,2$), and we get

$$(2.26)\qquad \frac{d}{dt}(w_1-w_2)+A_1(t)(w_1-w_2) = [A_2(t)-A_1(t)]w_2+[f_1(t)-f_2(t)]\,.$$

Now, we shall show the following,

Lemma 1. *$[A_2(t)-A_1(t)]w_2(t)$ is Hölder continuous in t for $0\leqq t<S_4$.*

Proof of Lemma. Write

$$(2.27)\quad [A_2(t)-A_1(t)]w_2(t)-[A_2(s)-A_1(s)]w_2(s) = [A_2(t)-A_2(s)]w_2(t)+A_2(s)[w_2(t)-w_2(s)] - [A_1(t)-A_1(s)]w_2(t)-A_1(s)[w_2(t)-w_2(s)]\,.$$

First we verify the following two inequalities:

$$(2.28)\quad ||[A_i(t)-A_i(s)]w_2(t)||\leqq D_1(t-s)^k \qquad 0\leqq s\leqq t<S_4\,,\quad i=1,2\,,$$

$$(2.29)\quad ||A_i(s)[w_2(t)-w_2(s)]||\leqq D_2(t-s)^{1-h} \qquad 0\leqq s\leqq t<S_4\,,\quad i=1,2\,,$$

where the constants D_1, D_2 do not depend on v_i, s, t but depend on $||A_0^h||$.

From (2.4), (2.5), (1.13) and (2.15) we have

$$\begin{aligned} &||[A_i(t)-A_i(s)]w_2(t)|| \\ &\leqq \sum_{p=1}^{m} ||A_i(t)^{1-ph}[A_i(t)^h A_i(s)^{-h}-I]A_i(s)^{ph}\{U_2(t,0)u_0+\int_0^t U_2(t,r)f_2(r)dr\}|| \\ &\leqq \sum_{p=1}^{m} ||A_i(t)^h||^{m-p}||A_i(t)^h A_i(s)^{-h}-I||\cdot||A_i(s)^h||^p[||U_2(t,0)u_0||+\int_0^t ||U_2(t,r)f_2(r)||dr] \\ &\quad \leqq mC^m(t-s)^k[(h+k)^{-1}N_{18}||u_0||+t(h+k)^{-1}N_{18}C_5]||A_0^h||^m C_3 \\ &\quad \leqq D_1(t-s)^k\,. \end{aligned}$$

From (2.4), (2.12) and (2.20) we have

$$\begin{aligned} &||A_i(s)[w_2(t)-w_2(s)]|| \\ &\leqq ||A_i(s)A_0^{-\alpha}||\cdot||A_0^\alpha\{U_2(t,0)u_0+\int_0^t U_2(t,r)f_2(r)dr-U_2(s,0)u_0-\int_0^s U_2(s,r)f_2(r)dr\}|| \\ &\leqq ||A_i(s)A_0^{-\alpha}||\{||A_0^\alpha[U_2(t,0)-U_2(s,0)]A_0^{-1}||\cdot||A_0u_0|| \\ &\qquad\qquad +||A_0^\alpha[\int_0^t U_2(t,r)f_2(r)dr-\int_0^s U_2(s,r)f_2(r)dr]||\} \\ &\leqq C_2^m||A_0^h||^m||A_0^{-\alpha}||\{\tilde{C}(t-s)^{1-\alpha'}||A_0u_0||+C(t-s)^{1-\alpha''}(|\log(t-s)|+1)\} \\ &\leqq D_2(t-s)^{1-h}\,. \end{aligned}$$

Thus using (2.27), (2.28) and (2.29) we obtain

$$\begin{aligned} (2.30)\quad &||[A_2(t)-A_1(t)]w_2(t)-[A_2(s)-A_1(s)]w_2(s)|| \\ &\leqq 2D_1|t-s|^\sigma+2D_2|t-s|^{1-h} \\ &\leqq D_3|t-s|^{1-h} \end{aligned}$$

so that $[A_2(t)-A_1(t)]w_2(t)$ is Hölder continuous. q.e.d.

From (2.6) for any $0\leqq s\leqq t<S_4$ it follows that

$$(2.31)\quad ||[f_1(t)-f_2(t)]-[f_1(s)-f_2(s)]||\leqq 2C|t-s|^k\,.$$

Hence from (2.30) and (2.31) the right-hand of (2.26) is Hölder continuous. Then applying Theorem A to (2.25) and $w_1(0)-w_2(0)=0$ we can write

$$(2.32)\quad w_1(t)-w_2(t)=\int_0^t U_1(t, r)\{[A_2(r)-A_1(r)]w_2(r)+[f_1(r)-f_2(r)]\}\,dr$$

Therefore from the definition of $w_{v,\alpha}$ we get the identity

$$\begin{aligned}(2.33)\quad & w_{v_1,\alpha}(t)-w_{v_2,\alpha}(t)\\ &= A_0^\alpha w_1(t)-A_0^\alpha w_2(t)\\ &= -A_0^\alpha\int_0^t U_1(t, r)\{[A_1(r)-A_2(r)]w_2(r)+[f_2(r)-f_1(r)]\}\,dr\\ &= -A_0^\alpha\int_0^t U_1(t, r)\sum_{p=1}^m A_1(r)^{1-ph}[A_1(r)^h A_2(r)^{-h}-I]A_2(r)^{ph}w_2(r)dr\\ &\qquad +A_0^\alpha\int_0^t U_1(t, r)[f_1(r)-f_2(r)]dr\\ &= -\sum_{p=1}^m\int_0^t A_0^\alpha U_1(t,r)A_1(r)^{1-ph}[A_1(r)^h A_2(r)^{-h}-I]A_2(r)^{ph}w_2(r)dr\\ &\qquad +\int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr\,.\end{aligned}$$

In the following the constants $E_1, E_2, \cdots$ do not depend on s, t, v_i, $||A_0^h||$. So, put $v_1=v$ and $v_2=A_0^\alpha\hat{u}$;

$$\begin{aligned}(2.34)\quad & ||w_{v,\alpha}(t)-w_{A_0^\alpha\hat{u},\alpha}(t)||\\ &= -\sum_{p=1}^m\int_0^t A_0^\alpha U_1(t, r)A_1(r)^{1-ph}[A_1(r)^h A_2(r)^{-h}-I]A_2(r)^{ph}w_2(r)dr\\ &\qquad +\int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr\,.\end{aligned}$$

From (2.8), (2.7), (2.18), (1.17) and (1.15) we get

$$\begin{aligned}(2.35)\quad & ||v(r)-A_0^\alpha\hat{u}(r)||\\ &\leqq ||v(r)-v(0)||+||A_0^\alpha\hat{u}(r)-A_0^\alpha u_0||\\ &\leqq Lr^\zeta+||A_0^\alpha\int_0^r U_{u_0}(r, s)f_{u_0}(s)ds||+||A_0^\alpha[U_{u_0}(r, 0)-U_{u_0}(0, 0)]A_0^{-1}A_0u_0||\\ &\leqq Lr^\zeta+Cr^{1-\tilde{\alpha}}[\,|\log r|+1]\max_{0\leqq t\leqq T}||f_{u_0}(t)||+Cr^{1-\tilde{\alpha}}\\ &\leqq Cr^\zeta \qquad\qquad \text{where}\quad \zeta<1-\tilde{\alpha}<1-\alpha\,.\end{aligned}$$

For any $0\leqq t<S_5$ the following inequality holds;

$$(2.36)\quad ||\int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr||\leqq Ct^{1-\alpha'+\zeta\eta}\,.$$

We see this, using (1.13), (2.6) and (2.35) for $0<\alpha<\alpha'<h/2$, as follows;

$$\begin{aligned}(2.37)\quad & ||\int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr||\\ &\leqq \int_0^t M_{\alpha\alpha'}(h+k-\alpha')^{-1}N_{18}(t-r)^{-\alpha'}C_4Cr^{\zeta\eta}dr\\ &\leqq Ct^{1-\alpha'+\zeta\eta}\,.\end{aligned}$$

We cite (1.18) for $A=A_1$, $U=U_1$;

$$(2.38)\quad \|A_1(t)^{\alpha'}U_1(t, r)A_1(r)^{1-ph}\| \leqq E_2(t-r)^{ph-\alpha''-1}.$$

Note that

$$(2.39)\quad A_2(r)^{ph}w_2(r) = A_2(r)^{ph}U_2(r, 0)u_0 + A_2(r)^{ph}\int_0^r U_2(r, s)f_2(s)ds$$

$$(2.40)\quad \begin{aligned}\|A_2(r)^{ph}U_2(r, 0)u_0\| &\leqq \|A_2(r)^{ph}U_2(r, 0)A_0^{-h}\|\cdot\|A_0^h u_0\| \\ &\leqq (k-ph+h)^{-1}N_{19}r^{h-ph}\|A_0^h u_0\| \\ &\leqq E_3 r^{h-ph}\end{aligned}$$

by (1.14).

From (1.19) we find that

$$(2.41)\quad \|A_2(r)^{ph}\int_0^r U_2(r, s)f_2(s)ds\| \leqq E_4 r^{1-ph}.$$

Hence using (2.39), (2.40) and (2.41) we have

$$(2.42)\quad \begin{aligned}\|A_2(r)^{ph}w_2(r)\| &\leqq E_3 r^{h-ph} + E_4 r^{1-ph} \\ &\leqq E_5 r^{h-ph}.\end{aligned}$$

Therefore from (2.38), (2.5), (2.42) and (2.35) it follows that

$$(2.43)\quad \begin{aligned}&\|\int_0^t A_0^{\alpha}U_1(t, r)A_1(r)^{1-ph}[A_1(r)^h A_2(r)^{-h}-I]A_2(r)^{ph}w_2(r)dr\| \\ &\leqq \int_0^t E_2(t-r)^{ph-\alpha''-1}\|v(r)-A_0^{\alpha}\hat{u}(r)\|^{\eta}E_5 r^{h-ph}dr \\ &\leqq \int_0^t E_2(t-r)^{ph-\alpha''-1}Cr^{\zeta\eta}E_5 r^{h-ph}dr \\ &\leqq Ct^{h-\alpha''+\zeta\eta}.\end{aligned}$$

Then from (2.34), (2.43) and (2.36) we have

$$(2.44)\quad \begin{aligned}\|w_{v,\alpha}(t) - w_{A_0^{\alpha}\hat{u},\alpha}(t)\| &\leqq mCt^{1-\alpha'+\zeta\eta} + Ct^{h-\alpha''+\zeta\eta} \\ &\leqq Ct^{h-\alpha''+\zeta\eta}.\end{aligned}$$

Put $v_1=A_0^{\alpha}u_0$ and $v_2=A_0^{\alpha}\hat{u}(t)$, from (2.18) and (1.15) it follows that

$$(2.45)\quad \begin{aligned}&\|\hat{u}(r)-u_0\| \\ &\leqq \|[U_{u_0}(r, 0)-U_{u_0}(0, 0)]A_0^{-1}A_0u_0\| + \int_0^r \|U_{u_0}(r, s)f_{u_0}(s)\|ds \\ &\leqq Cr^{1-\tilde{\varepsilon}} \qquad \text{where} \quad 0<\tilde{\varepsilon}<\alpha.\end{aligned}$$

Then as we get (2.44), we have

$$(2.46)\quad \|w_{A_0^{\alpha}\hat{u},\alpha}(t) - A_0^{\alpha}\hat{u}(t)\| \leqq Ct^{h-\alpha''+\eta(1-\tilde{\varepsilon})}.$$

Note that $(1-h+\alpha'')/\eta<\zeta<1-\alpha$ implies $h-\alpha''+\eta(1-\tilde{\varepsilon})>h-\alpha''+\zeta\eta>1$. Therefore from (2.44) and (2.46)

$$(*)\quad \|w_{v,\alpha}(t)-A_0^{\alpha}\hat{u}(t)\|\leqq Ct^{h-\alpha''+\zeta\eta-1}\times t$$
$$\leqq CS_5^{h-\alpha''+\zeta\eta-1}\times t \qquad \text{for any} \quad t\in[0, S_5)\,.$$

So if $0<S_5\leqq \min(S_3,\ \{M(1-\varepsilon)/2C\}^{1+\alpha''-h-\zeta\eta})$, then

$$\|w_{v,\alpha}(t)-A_0^{\alpha}\hat{u}(t)\|\leqq M(1-\varepsilon)t/2\,.$$

Thus (2.23) is obtained.

2) The general case.

We now turn to general case in which $A(t, A_0^{-\alpha}v)$ is not necessarily bounded. We first construct a sequence of bounded operators $A_n(t, A_0^{-\alpha}v)$ that approximate $A(t, A_0^{-\alpha}v)$ in a certain sense. We set

$$(2.47)\quad \begin{cases} A_n(t, A_0^{-\alpha}v) = A(t, A_0^{-\alpha}v)J_n(t, A_0^{-\alpha}v) \\ J_n(t, A_0^{-\alpha}v) = [1+n^{-1}A(t, A_0^{-\alpha}v)^h]^{-m} \qquad n=1, 2, \cdots\,. \end{cases}$$

Obviously $A_n(t, A_0^{-\alpha}v)$ belongs to $B(X)$ and satisfy the assumptions I), II). Therefore, all the estimates deduced in the preceding section are valid, whose constants do not depend on n. Hence from I) there exists a fundamental solution $U_{i,n}(t, s)$ corresponding to $A_n(t, A_0^{-\alpha}v_i(t))$ and a solution $w_{i,n}$ of

$$\begin{cases} \dfrac{dw_{i,n}}{dt}+A_n(t, A_0^{-\alpha}v_i(t))w_{i,n}=f_i(t) \\ w_{i,n}(0)=u_0 \qquad\qquad v_i\in F(S_4)\,,\quad i=1, 2\,. \end{cases}$$

Then we get by (*)

$$(2.48)\quad \|A_n(0, u_0)^{\alpha}[w_{1,n}(t)-w_{2,n}(t)]\|\leqq CS_5^{h-\alpha''+\zeta\eta-1}\times t\,.$$

Due to Kato [5], we obtained that $A_n^{\alpha}(0, u_0)U_{i,n}(t, 0)\rightarrow A_0^{\alpha}U_i(t, 0)$ as $n\rightarrow\infty$. Thus (2.23) is obtained.

Next, from (1.21) for any $\delta>0$ there is a $t_0>0$ such that

$$\Big\|\frac{1}{t}[A_0^{\alpha}\hat{u}(t)-A_0^{\alpha}u_0]-a\Big\|<\delta \qquad \text{for any} \quad t\in(0, t_0]\,.$$

Then choose $\delta=M(1-\varepsilon)/2$ there is a $t_0>0$ such that

$$(2.49)\quad \|A_0^{\alpha}\hat{u}(t)-[A_0^{\alpha}u_0+ta]\|$$
$$=\Big\|\frac{1}{t}[A_0^{\alpha}\hat{u}(t)-A_0^{\alpha}u_0]-a\Big\|t$$
$$\leqq M(1-\varepsilon)t/2 \qquad\qquad \text{for} \quad t\in(0, t_0]$$

Hence if $0<S_4\leqq \min\{S_5, t_0\}$, then from (2.23) and (2.49)

$$(2.50)\quad \begin{aligned} &\|w_{v,\alpha}(t)-[A_0^\alpha u_0+ta]\| \\ &\leqq \|w_{v,\alpha}(t)-A_0^\alpha \hat{u}(t)\|+\|A_0^\alpha \hat{u}(t)-[A_0^\alpha u_0+ta]\| \\ &\leqq Mt(1-\varepsilon) \qquad\qquad \text{for any} \quad t\in[0, S_4) \end{aligned}$$

holds. Thus (2.22) is proved.

Since (2.17) implies

$$(2.51)\quad w_{v,\alpha}(0) = A_0^\alpha w_v(0) = A_0^\alpha u_0 ,$$

we get $w_{v,\alpha}\in F(S_4)$.

We defined a transformation $T: v\mapsto w_{v,\alpha}$ for $v\in F(S_4)$. Then from (2.51) (2.21) and (2.50) we have

$$\begin{aligned} &(Tv)(0) = w_{v,\alpha}(0) = A_0^\alpha u_0 , \\ &\|(Tv)(t_1)-(Tv)(t_2)\|\leqq L|t_1-t_2|^\zeta \qquad \text{for} \quad t_1, t_2\in[0, S_4) \\ &\|Tv(t)-(A_0^\alpha u_0+ta)\|\leqq Mt(1-\varepsilon) \qquad \text{for} \quad t\in(0, S_4) \end{aligned}$$

that is, T maps $F(S_4)$ into itself.

We now consider $F(S_4)$ as a subset of the Banach space $\tilde{Y}\equiv C([0, S_4); X)$ consisting of all the continuous functions $v(t)$ from $[0, S_4)$ into X with norm

$$|||v||| = \sup_{0\leqq t<S_4} \|v(t)\| .$$

We shall prove that T is a continuous mapping in $F(S_4)$ (with the topology induced by $\tilde{Y}$).

1) The case of bounded $A(t, A_0^{-\alpha}v)$.

Let v_1 and v_2 belong to $F(S_4)$. From (2.33)

$$(2.52)\quad \begin{aligned} &w_{v_1,\alpha}(t)-w_{v_2,\alpha}(t) \\ &= -\sum_{p=1}^{m}\int_0^t A_0^\alpha U_1(t, r)A_1(r)^{1-ph}[A_1(r)^h A_2(r)^{-h}-I]A_2(r)^{ph} w_2(r)dr \\ &\quad + \int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr . \end{aligned}$$

For any $0\leqq t<S_4$, the following inequality holds:

$$(2.53)\quad \left\|\int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr\right\|\leqq E_1 t^{1-h}|||v_1-v_2|||^\eta .$$

We see this, using (1.13) and (2.6) for $0<\alpha<\alpha'<h$, as follows;

$$\begin{aligned} &\left\|\int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr\right\| \\ &\leqq \int_0^t \|A_0^\alpha A_1(t)^{-\alpha'}\|\cdot\|A_1(t)^{\alpha'}U_1(t, r)\|\cdot\|f_1(r)-f_2(r)\|dr \\ &\leqq \int_0^t M_{\alpha\alpha'}(h+k-\alpha')^{-1}N_{18}(t-r)^{-\alpha'}C_4\|v_1(r)-v_2(r)\|^\eta dr \\ &\leqq E_1 t^{1-h}|||v_1-v_2|||^\eta . \end{aligned}$$

Therefore from (2.33), (2.38), (2.5), (2.42) and (2.53) it follows that

$$(2.54)\quad \begin{aligned}&||w_{v_1,\alpha}(t)-w_{v_2,\alpha}(t)||\\ &\leqq \sum_{p=1}^{m}\int_0^t ||A_0^\alpha U_1(t, r)A_1(r)^{1-ph}||\cdot||A_1(r)^hA_2(r)^{-h}-I||\cdot||A_2(r)^{ph}w_2(r)||dr\\ &\quad +||\int_0^t A_0^\alpha U_1(t, r)[f_1(r)-f_2(r)]dr||\\ &\leqq \sum_{p=1}^{m}\int_0^t E_2\,(t-r)^{ph-\alpha''-1}||v_1(r)-v_2(r)||^\eta E_5r^{h-ph}dr+E_1t^{1-h}|||v_1-v_2|||^\eta\\ &\leqq E_4(t^{h-\alpha''}+t^{1-h})|||v_1-v_2|||^\eta\\ &\leqq E_2t^{h-\alpha''}|||v_1-v_2|||^\eta .\end{aligned}$$

Hence

$$(2.55)\quad \begin{aligned}|||Tv_1-Tv_2||| &= \sup_{0\leqq t<S_4}||w_{v_1,\alpha}(t)-w_{v_2,\alpha}(t)||\\ &\leqq E_7S_4^{h-\alpha''}|||v_1-v_2|||^\eta \qquad v_1,\, v_2\in F(S_4)\,.\end{aligned}$$

This means that T is a continuous operator.

2) The general case.

we get by (2.54)

$$(2.56)\quad ||A_n(0, u_0)^\alpha[w_{1,n}(t)-w_{2,n}(t)]||\leqq E_8S_4^{h-\alpha''}|||v_1-v_2|||^\eta \qquad n\in \boldsymbol{N}_+\,.$$

Due to Kato [5], we obtain that $A_n(0, u_0)^\alpha U_{i,n}(t, 0)\to A_0^\alpha U_i(t, 0)$ as $n\to\infty$. Thus T is a continuous operator.

We now claim that the set $TF(S_4)$ is contained in a compact subset of Y. Indeed, the functions $v(t)$ of $F(S_4)$ are uniformly bounded (by (2.10)) and equicontinuous (by (2.8)). If we can show that for each t the set $\{w_{v,\alpha}(t); v\in F(S_4)\}$ is contained in a compact subset of X, then by applying Ascoli's Theorem we can prove that $TF(S_4)$ is contained in a compact set of Y.

We can write, for each $t\in[0, S_4)$, $w_{v,\alpha}(t)=A_0^{-\gamma}A_0^\gamma w_{v,\alpha}(t)$ where$0<\gamma<h-\alpha$.

From (2.12) and (2.41), we have

$$\begin{aligned}||A_0^\gamma w_{v,\alpha}(t)|| &= ||A_0^\gamma[A_0^\alpha U_v(t, 0)u_0+A_0^\alpha \int_0^t U_v(t, s)f_v(s)ds]||\\ &\leqq ||A_0^{\gamma+\alpha}[U_v(t, 0)-U_v(0, 0)]A_0^{-1}A_0u_0||+||A_0^{\gamma+\alpha}u_0||\\ &\quad +||A_0^{\gamma+\alpha}A_v(t)^{-h}||\cdot||A_v(t)^h\int_0^t U_v(t, s)f_v(s)ds||\\ &\leqq \bar{C}t^{1-\alpha-\gamma-\varepsilon}||A_0u_0||+||A_0^{\alpha+\gamma}u_0||+ME_4t^{1-h}\\ &\leqq E_9\,.\end{aligned}$$

Thus $\{A_0^\gamma w_{v,\alpha}(t); v\in F(S_4)\}$ is a bounded subset of X. And by assumption (A–2), $A_0^{-\gamma}$ is completely continuous. Therefore $\{w_{v,\alpha}(t); v\in F(S_4)\}$ is indeed contained in a compact subset of X.

We can now apply Schauder's fixed point theorem and deduce that T has a fixed point v in $F(S_4)$. Noting $Tv=w_{v,\alpha}$ and $w_{v,\alpha}(t)=A_0^{\alpha}w_v(t)$, we have $A_0^{\alpha}w_v(t)=v(t)$ or $w_v(t)=A_0^{-\alpha}v(t)$. Applying (2.16) we find that

$$\frac{d}{dt}A_0^{-\alpha}v(t)+A(t, A_0^{-\alpha}v(t))A_0^{-\alpha}v(t)=f(t, A_0^{-\alpha}v(t)).$$

This finishes the proof of Theorem 2 for $S=S_4$ and $u=A_0^{-\alpha}v$.

3. Proof of Theorem 1

From (0.3) there are constants C_5, $\phi_1>0$, $T_1>0$ such that for $t\in\sum(\phi_1; T_1)$, $v\in N$ and $|\theta|<\phi_1$, the resolvent set of $e^{i\theta}A(t, A_0^{-\alpha}v)$ contains the left half-plane and

(3.1) $\|(\lambda-e^{i\theta}A(t, A_0^{-\alpha}v))^{-1}\|\leqq C_5(1+|\lambda|)^{-1} \qquad Re\,\lambda\leqq 0,$

We let $\phi=\min\{\phi_0, \phi_1\}$, $(1-h+\alpha'')/\eta<\zeta<1-\alpha$, $0<\varepsilon<1$ and $L>0$.

We consider the set $E(S)$ of all functions $\tilde{v}(t)$, defined on $\sum(\phi; S)$ which satisfy the following;

(3.2) $\tilde{v}: \sum(\phi; S)\setminus\{0\}\to X$ is analytic,

(3.3) $\tilde{v}(0)=A_0^{\alpha}u_0,$

(3.4) $\|\tilde{v}(t)-\tilde{v}(0)\|\leqq L|t|^{\zeta}$ for any $t\in\sum(\phi; S)$

(3.5) $\|\tilde{v}(t_1)-\tilde{v}(t_2)\|\leqq L|t_1-t_2|^{\zeta}$ for any real $t_1, t_2\in[0, S)$,

(3.6) $\|\tilde{v}(t)-(A_0^{\alpha}u_0+ta)\|\leqq M|t|(1-\varepsilon)$ for $t\in\sum(\phi; S)$

If $0<S_1<\min\{T_0, (RL^{-1})^{1/\zeta}\}$, then

$$\|\tilde{v}(t)-A_0^{\alpha}u_0\|\leqq L|t|^{\zeta}<L(RL^{-1})=R \qquad \text{for} \quad t\in\sum(\phi; S_1).$$

Let us note that if S_1 is small enough to $\tilde{v}(t)\in N$ for $t\in(0, S_1)$ the operator

$$A_{\tilde{v}}(t)=A(t, A_0^{-\alpha}\tilde{v}(t))$$

and the function

$$f_{\tilde{v}}(t)=f(t, A_0^{-\alpha}\tilde{v}(t))$$

are well defined for $t\in\sum(\phi; S_1)$, since $\sum(\phi; S_1)\subset\sum(\phi_0; T_0)$.

We first restrict t to be real in (0.1), $t\in[0, S_1)$. Then it follows from (0.3)–(0.6) that the family $\{A_{\tilde{v}}(t); 0\leqq t<S_1\}$ and the function $f_{\tilde{v}}: [0, S_1)\to X$ satisfy the hypotheses of Theorem A. Thus there is a continuous function $\tilde{w}$: $[0, S_1)\to X$ which is the unique solution of

$$(3.7)\quad \begin{cases} d\tilde{w}_{\tilde{v}}/dt+A_{\tilde{v}}(t)\tilde{w}_{\tilde{v}}=f_{\tilde{v}}(t) \\ \tilde{w}_{\tilde{v}}(0)=u_0. \end{cases}$$

For $0<\varepsilon<S_1/2$ we consider the sector $\sum(\phi; S_1-2\varepsilon)\setminus\{0\}+\varepsilon$. Since the function $t\mapsto A_{\tilde{v}}(t)^h A(0)^{-h}$ and $t\mapsto f_{\tilde{v}}(t)$ are analytic in a neighborhood of the closure of $\sum(\phi; S_1-2\varepsilon)\setminus\{0\}+\varepsilon$ and by (0.6) $f_{\tilde{v}}(t)$ is Hölder continuous, we can apply Theorem B; $\tilde{w}_{\tilde{v}}$ has an extention to $\cup\{\sum(\phi; S_1-2\varepsilon)\setminus\{0\}+\varepsilon; \varepsilon>0\}=\sum(\phi; S_1)\setminus\{0\}$ such that $\tilde{w}_{\tilde{v}}: \sum(\phi; S_1)\setminus\{0\}\to X$ is analytic, $\tilde{w}_{\tilde{v}}(t)\in D(A_{\tilde{v}}(t))$ and $d\tilde{w}_{\tilde{v}}(t)/dt+A_{\tilde{v}}(t)\tilde{w}_{\tilde{v}}(t)=f_{\tilde{v}}(t)$ for $t\in\sum(\phi; S_1)\setminus\{0\}$.

Next we shall show that $A_0^\alpha\tilde{w}_{\tilde{v}}: \sum(\phi; S_1)\setminus\{0\}\to X$ is analytic. Actually seeing that $t\mapsto A_{\tilde{v}}(t)^h A(0)^{-h}$ is analytic, $t\mapsto A(0)^h A_{\tilde{v}}(t)^{-h}$ is analytic. By rewriting the equation as $A_{\tilde{v}}(t)\tilde{w}_{\tilde{v}}(t)=f_{\tilde{v}}(t)-\tilde{w}'_{\tilde{v}}(t)$ and using the fact that $t\mapsto\tilde{w}_{\tilde{v}}(t)$ and $t\mapsto f_{\tilde{v}}(t)$ are analytic, we have that $t\mapsto A_{\tilde{v}}(t)^h\tilde{w}_{\tilde{v}}(t)=A_{\tilde{v}}(t)^{h-1}[f_{\tilde{v}}(t)-\tilde{w}'_{\tilde{v}}(t)]$ is analytic. Then $t\mapsto A_0^\alpha\tilde{w}_{\tilde{v}}(t)=A_0^{\alpha-h}A_0^h A_{\tilde{v}}(t)^{-h}A_{\tilde{v}}(t)^h\tilde{w}_{\tilde{v}}(t)$ is analytic from $\sum(\phi; S_1)\setminus\{0\}$ to X.

Set $\tilde{w}_{\tilde{v},\alpha}(t)=A_0^\alpha\tilde{w}_{\tilde{v}}(t)$.

Let us restrict t to be real, $t\in[0, S_1)$. From assumptions (A–1)–(A–6) and (A–8), assumptions (R–1)–(R–7) hold. Therefore if $S_1>0$ is small enough, as we get (2.21), we can show that

$$||\tilde{w}_{\tilde{v},\alpha}(t_1)-\tilde{w}_{\tilde{v},\alpha}(t_2)||\leq L|t_1-t_2|^\zeta \qquad \text{for} \quad t_1, t_2\in[0, S_1).$$

We shall show that

$$(3.8)\quad \begin{cases} ||\tilde{w}_{\tilde{v},\alpha}(t)-\tilde{w}_{\tilde{v},\alpha}(0)||\leq L|t|^\zeta & \text{for} \quad t\in\sum(\phi; S_1). \\ ||\tilde{w}_{\tilde{v},\alpha}(t)-(A_0^\alpha u_0+ta)||\leq M|t|(1-\varepsilon) & \text{for} \quad t\in\sum(\phi; S_1). \end{cases}$$

In order to prove it, in (3.7) we make the change of variable $t=\tau e^{i\theta}$, $\tau\in[0, S_1)$, $|\theta|<\phi$, so equations (3.7) become

$$(3.9)\quad \begin{cases} \dfrac{\partial v}{\partial\tau}+e^{i\theta}A_{\tilde{v}}(\tau e^{i\theta})v=e^{i\theta}f_{\tilde{v}}(\tau e^{i\theta}), \\ v(0, e^{i\theta})=u_0, \end{cases}$$

where $v(\tau, e^{i\theta})=\tilde{w}_{\tilde{v}}(\tau e^{i\theta})$, $\tilde{w}_{\tilde{v}}(t)=v(|t|, t/|t|)$.

We hold $|\theta|<\phi$ fixed and let

$$B(\tau, \tilde{v}, \theta)=e^{i\theta}A(\tau e^{i\theta}, \tilde{v}), \quad g(\tau, \tilde{v}, \theta)=e^{i\theta}f(\tau e^{i\theta}, \tilde{v})$$

for $\tau\in[0, S_1)$, $A_0^\alpha\tilde{v}\in N$, $|\theta|<\phi$. We shall show that for fixed θ, $B(\tau, \tilde{v}, \theta)$ and $g(\tau, \tilde{v}, \theta)$ satisfy the assumptions (R–1)–(R–7) with constants independent of θ.

First, note that

$$B_0^{-1}=B(0, u_0, \theta)^{-1}=e^{-i\theta}A(0, u_0)^{-1}=e^{-i\theta}A_0^{-1},$$

and (R–2) is verified.

Since $A(t, A_0^{-\alpha}w)$ is well defined for any $w\in N$ and $t\in\sum(\phi; T)$, and

$$B(\tau, B_0^{-\alpha}w, \theta) \equiv B(\tau, B(0, u_0, \theta)^{-\alpha}w, \theta) = e^{i\theta}A(\tau e^{i\theta}, A_0^{-\alpha}(e^{-i\alpha\theta}w))$$

$B(\tau, B_0^{-\alpha}w, \theta)$ is well defined for $w\in N_\theta$ and $\tau\in[0, T_1)$, which verifies (R–3) where $N_\theta=e^{i\alpha\theta}N$.

(R–4) is verified since by (3.1) and $D(B(\tau, B_0^{-\alpha}w, \theta)=D(A(\tau e^{i\theta}, A_0^{-\alpha}\times(e^{-i\alpha\theta}w)))$.

For any $w\in N_\theta$ and $\tau\in[0, T_1)$ we have

$$D(B(\tau, B_0^{-\alpha}w, \theta)^h) = D(e^{i\theta h}A(\tau, A_0^{-\alpha}(e^{-i\alpha\theta}w))^h) \equiv D\,,$$

and (R–5) is verified.

From (0.4) and (0.5) it follows that

$$\begin{aligned}&||B(\tau_1, B_0^{-\alpha}w, \theta)^hB(\tau_2, B_0^{-\alpha}v, \theta)^{-h}||\\ &\quad= ||e^{ih\theta}A(\tau_1e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}w)^he^{-ih\theta}A(\tau_2e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}v)^h||\\ &\quad\leqq C_2\end{aligned}$$

and

$$\begin{aligned}&||B(\tau_1, B_0^{-\alpha}w, \theta)^hB(\tau_2, B_0^{-\alpha}v, \theta)^{-h}-I||\\ &\quad\leqq||A(\tau_1e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}w)^hA(\tau_2e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}v)^{-h}-I||\\ &\quad\leqq C_3\{|\tau_1e^{i\theta}-\tau_2e^{i\theta}|^\sigma+||e^{i\alpha\theta}w-e^{-i\alpha\theta}v||^\eta\}\\ &\quad\leqq C\{|\tau_1-\tau_2|^\sigma+||w-v||^\eta\} \qquad\qquad w, v\in N_\theta\,,\ \ \tau_1,\ \ \tau_2\in[0, T_1)\,.\end{aligned}$$

Thus (R–6) is verified.

Finally, from (0.6) we get

$$\begin{aligned}&||g(\tau_1, B_0^{-\alpha}w, \theta)-g(\tau_2, B_0^{-\alpha}v, \theta)||\\ &\quad= ||e^{i\theta}f(\tau_1e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}w)-e^{i\theta}f(\tau_2e^{i\theta}, A_0^{-\alpha}e^{-i\alpha\theta}v)||\\ &\quad\leqq C_4\{|\tau_1-\tau_2|^\sigma+||w-v||^\eta\} \qquad\qquad w, v\in N_\theta\,,\quad \tau_1, \tau_2\in[0, T_1)\,,\end{aligned}$$

which verifies (R–7).

Hence as we get (2.21), we can show that there exists a unique solution $v(\tau, e^{i\theta})$ of (3.9) defined for $\tau\in[0, S_1)$, $|\theta|<\phi$, which satifies

$$||A_0^\alpha v(\tau_1, e^{i\theta})-A_0^\alpha v(\tau_2, e^{i\theta})||\leqq L|\tau_1-\tau_2|^\zeta \qquad \text{for}\quad \tau_1, \tau_2\in[0, S_1)$$

and

$$||A_0^\alpha v(\tau, e^{i\theta})-(A_0^\alpha u_0+ta)||\leqq M|t|(1-\varepsilon) \qquad \text{for}\quad \tau\in[0, S_1)\,.$$

Therefore we obtain (3.8).

Since (3.7) implies

$$\tilde{w}_{\tilde{v},\alpha}(0) = A_0^\alpha\tilde{w}_{\tilde{v}}(0) = A_0^\alpha u_0$$

we get $\tilde{w}_{\tilde{v},\alpha}\in E(S_1)$.

We define a transformation $\tilde{T}: \tilde{v}\to\tilde{w}$ for $\tilde{v}\in E(S_1)$. Then $\tilde{T}$ maps $E(S_1)$

into itself.

Denote by $F_0(S)$ the set of the restrictions $v(t)$ of all functions $\tilde{v}(t)$ in $E(S)$ to $[0, S)$. And we define a transformation T_0 in the way $(T_0v)(t)=(\tilde{T}\tilde{v})(t)$ for $t\in[0, S_1)$. Then T_0 maps $F_0(S_1)$ into itself.

Therefore we can use the argument in §2 with $F_0(S_1)$ in stead of $F(S_4)$. And we can show that w_v is a unique solution of

$$\begin{cases} dw_v/dt+A(t, A_0^{-\alpha}v(t))w_v = f(t, A_0^{-\alpha}v(t)) \\ w_v(0) = u_0 \end{cases}$$

where $v\in F_0(S_1)$, $w_v=A_0^{-\alpha}Tv$ and T is the map which is defined in §2.

Since the functions $\tilde{v}(t)$ of $E(S_1)$ are uniformly bounded. $F_0(S_1)$ is a closed convex subset of the Banach space $\tilde{Y}\equiv C([0, S_1); X)$.

On the other hand from the definitions of T_0, T and (3.7) it follows that $A_0^{-\alpha}T_0v=A_0^{-\alpha}Tv$ by uniqueness. It follows from Theorem 2 that there is a fixed point $v\in F_0(S_1)$ such that $Tv=v$. Therefore

$$(\tilde{T}\tilde{v})(t) = (T_0v)(t) = (Tv)(t) = v(t) = \tilde{v}(t) \qquad \text{for} \quad t\in[0, S_1)\,.$$

Noting $\tilde{v}$ and $\tilde{T}\tilde{v}$ are analytic from $\sum(\phi; S_1)\setminus\{0\}$ to X, we have $\tilde{T}\tilde{v}=\tilde{v}$.

This finishes the proof of Theorem 1 for $T=S_1$ and $u=A_0^{-\alpha}\tilde{v}$.

References

[1] A. Friedman: Partial differential equations, Holt, Rinehart and Winston, New York, 1969.

[2] K. Furuya: *A note on quasilinear evolution equations,* Proc. Japan Acad. Ser. A **56** (1980), 256–258.

[3] K. Furuya: *Analyticity of solutions of quasilinear evolution equations,* Osaka J. Math. **18** (1981), 669–698.

[4] T.L. Hayden and Massey III: *Nonlinear holomorphic semigroups,* Pacific J. Math. **57** (1975), 423–439.

[5] T. Kato: *Abstract evolution equations of parabolic type in Banach and Hilbert spaces,* Nagoya Math. J. **5** (1961), 93–125.

[6] S.G. Krein: Linear differential equations in a Banach space, Izdatel'stov Nauka, Moscow (in Russian); Japanease transl., Yoshioka-shoten, Kyoto, 1972.

[7] F.J. Massay III: *Analyticity of solutions of nonlinear evolution equations,* J. Differential Equations **22** (1976), 416–427.

[8] S. Ōuchi: *On the analyticity in time of solutions of initial boundary value problems for semi-linear parabolic differential equations with monotone nonlinearity,* J. Fac. Sci. Univ. Tokyo Sect. 1A **20** (1974), 19–41.

[9] P.E. Sobolevskii: *Equations of parabolic type in a Banach space,* Trudy Moskov. Mat. Obsc. 10 (1961), 297–350 (in Russian); English transl., Amer. Math. Soc. transl. (2) 49 (1965), 1–62.

[10] P.E. Sobolevskii: *Parabolic equations in Banach space with an unbounded variable*

operator, a fractional power of which has a constant domain of definition, Dokl. Akad. Nauk SSSR **138** (1961), 59–62. (in Russian); English transl., Soviet Math. Dokl. **2** (1961), 545–548.

[11] H. Tanabe: Equations of evolution, Iwanami-shoten, Tokyo, 1975. (in Japanease); English transl., Monogr. & studies in math. vol. 6, Pitman, 1979.

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