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THE BERRY-ESSEEN BOUND FOR MAXIMUM LIKELIHOOD ESTIMATES OF TRANSLATION PARAMETER OF TRUNCATED DISTRIBUTION

TADAYUKI MATSUDA

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1. Introduction. Let $X_1, \cdots, X_n$ be independent random variables with common density $f(x-\theta)$, $-\infty<x$, $\theta<\infty$, where θ is an unknown translation parameter. We shall consider here the case that $f(x)$ is a uniformly continuous density which vanishes on the interval $(-\infty, 0)$ and is positive on the interval $(0, \infty)$ and particularly

$$f(x)\sim\alpha x \qquad \text{as} \quad x\to+0$$

with $0<\alpha<\infty$. Let $\hat{\theta}_n=\hat{\theta}_n(X_1, \cdots, X_n)$ be a m.l.e. (maximum likelihood estimate) of θ for the sample size n. Woodroofe [1] showed that $(\frac{1}{2}\alpha n \log n)^{1/2} \times(\hat{\theta}_n-\theta)$ has an asymptotic standard normal distribution. The purpose of the present paper is to estimate the speed of convergence of $a_n(\hat{\theta}_n-\theta)$ to the standard normal distribution. Here $2a_n^2=\alpha n(\log n+\log\log n)$. Similar results for minimum contrast estimates in the regular case were given by Michel and Pfanzagl [2] and Pfanzagl [3]. More precisely, Pfanzagl [3] showed that for every compact K there exists a constant c_K such that for all $\theta\in K$, $n\geqq 1$ and $t\in R$

$$\left|P_\theta\left\{\frac{n^{1/2}(\theta_n^*-\theta)}{\beta(\theta)}<t\right\}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^t \exp\left(-\frac{r^2}{2}\right)dr\right|\leqq c_K n^{-1/2},$$

where θ_n^* denotes a minimum contrast estimate for the sample size n.

2. Conditions and the main result. We shall impose the following regularity conditions on $f(x)$. These conditions are stronger than those made by Woodroofe [1].

CONDITIONS

(i) $f(x)$ is a uniformly continuous density which vanishes on $(-\infty, 0)$ and is positive on $(0, \infty)$.

(ii) $f(x)$ is continuously differentiable on $(0, \infty)$ with derivative $f'(x)$ and $f'(x)$ is absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $f''(x)$.

(iii) For some α and r, $0<\alpha,\ r<\infty$

$$f'(x)=\alpha+O(x^r) \text{ and } f''(x)=O(x^{r-1}) \qquad \text{as} \quad x\to+0\,.$$

Let $g(x)=\log f(x)$ for $x>0$. Then $g(x)$ will be continuously differentiable on $(0,\ \infty)$ with derivative $g'=f'/f$ and $g'(x)$ will be absolutely continuous on every compact subinterval of $(0,\ \infty)$ with derivative $g''=(ff''-f'^2)/f^2$.

(iv) For every $t\geqq 0$

$$\int_0^\infty \{g(x+t)\}^2 f(x)\,dx<\infty\,.$$

(v) For every $a>0$, there is a $\delta>0$, for which

$$\int_a^\infty \sup_{|u|\leqq\delta}|g'(x+u)|^3 f(x)\,dx<\infty\,.$$

(vi) For every $a>0$, there is a $\delta>0$, for which

$$\int_a^\infty \sup_{|u|\leqq\delta}\{g''(x+u)\}^2 f(x)\,dx<\infty\,.$$

REMARK. Under conditions (i) and (ii), condition (iii) is equivalent to the following condition (iii)'.

(iii)' For some α and r, $0<\alpha,\ r<\infty$

$$f(x)=\alpha x+O(x^{1+r}),\ g'(x)=x^{-1}+O(x^{r-1}) \text{ and } g''(x)=-x^{-2}+O(x^{r-2}) \qquad \text{as} \quad x\to+0\,.$$

EXAMPLES ([1]). Let

$$f(x)=r\left[\Gamma\left(\frac{2}{r}\right)\right]^{-1} x\exp(-x^r), \quad x>0, \quad \text{for some } r>0\,,$$

$$\text{or } f(x)=\frac{1}{d(1+d)}\frac{x}{(1+x)^{2+d}}, \qquad x>0, \quad \text{for some } d>0\,,$$

then conditions (i)—(vi) are all satisfied.

Let $M_n=\min(X_1,\cdots,X_n)$ and $G_n(t)=\sum_{i=1}^n g(X_i-t)$ for $t<M_n$. Condition (i) insures that m.l.e.'s exist in the interval $(-\infty,\ M_n)$. Let $\hat\theta_n$, $n\geqq 1$, be a sequence of m.l.e.'s. If conditions (i) and (ii) are satisfied, then

$$-\infty<\hat\theta_n<M_n \text{ and } G_n'(\hat\theta_n)=0$$

with probability 1.

Theorem. *Suppose that conditions* (i)—(vi) *are all satisfied. Let* $\hat\theta_n$, $n\geqq 1$, *denote a sequence of m.l.e.'s for* $\prod_{i=1}^n f(X_i-\theta)$ *and let* $2a_n^2=\alpha n(\log n+\log\log n)$. *Then there exists a constant* c_1 *such that for all* $\theta\in R$, $n\geqq 1$ *and* $t\leqq 0$

$$(2.1)\qquad |P_\theta\{a_n(\hat{\theta}_n-\theta)\leqq t\}-\Phi(t)|\leqq c_1(\log n)^{-1}.$$

Also, for every s, $0<s<1$, there exists a constant c_2 such that for all $\theta\in R$, $n\geqq 1$ and $t>0$

$$(2.2)\qquad |P_\theta\{a_n(\hat{\theta}_n-\theta)\leqq t\}-\Phi(t)|\leqq c_2(\log n)^{s-1}.$$

Here

$$\Phi(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp\left(-\frac{x^2}{2}\right)dx.$$

REMARK. (1) The assertion of (2.2) holds with $(\log n)^{-1}$ instead of $(\log n)^{s-1}$ provided t is restricted to a finite interval $(0, M]$.

(2) We used $\{\frac{1}{2}\alpha n(\log n+\log\log n)\}^{1/2}$ as the convergence order of m.l.e. to the true parameter θ. However our result is true for any a_n, $n\geqq 1$, satisfying that $\alpha n a_n^{-2}\log a_n=1+O((\log n)^{-1})$. Obviously, this condition includes the case that $a_n=\{\frac{1}{2}\alpha n(\log n+\log\log n)\}^{1/2}$ but excludes the case that $a_n=(\frac{1}{2}\alpha n\log n)^{1/2}$.

3. Some lemmas. Since θ is a translation parameter, it will suffice to prove our result in the special case that $\theta=0$. Hereafter, suppose that $\theta=0$. The following Lemma 1 refines the result of Woodroofe [1].

Lemma 1. *Let conditions* (i)–(iii) *and* (vi) *be satisfied. Then, for sufficiently small $\varepsilon>0$, there exists $c\geqq 0$ such that*

$$(3.1)\qquad P\Big\{\sup_{-\varepsilon\leqq t<M_n}\frac{1}{n}G_n''(t)\geqq -1\Big\}\leqq cn^{-1}$$

for all $n\geqq 1$.

Proof. Let $a>0$ be so small that $g''(x)\leqq -\frac{1}{2}x^{-2}$ for $0<x\leqq 2a$. There is a sufficiently small number $0<\varepsilon<a$ such that

$$\int_0^a(x+\varepsilon)^{-2}f(x)\,dx>2\int_a^\infty\sup_{|t|\leqq\varepsilon}|g''(x+t)|\,f(x)\,dx+5$$

because the left-hand integral diverges to ∞ as $\varepsilon\to 0$. Then the event $M_n\leqq\varepsilon$ implies that

$$\sup_{-\varepsilon\leqq t<M_n}\frac{1}{n}G_n''(t)\leqq\frac{-1}{2n}\sum\nolimits_0^a(X_i+\varepsilon)^{-2}+\frac{1}{n}\sum\nolimits_a^\infty\sup_{|t|\leqq\varepsilon}|g''(X_i+t)|$$

where $\sum_u^v$ denotes summation over $i\leqq n$ for which $u\leqq X_i<v$. Hence the relations $M_n\leqq\varepsilon$ and $\sup\limits_{-\varepsilon\leqq t<M_n}\frac{1}{n}G_n''(t)\geqq -1$ imply

$$\left|\frac{1}{n}\sum_0^a (X_i+\varepsilon)^{-2}-\int_0^a (x+\varepsilon)^{-2}f(x)\,dx\right|\geqq 1$$

or

$$\left|\frac{1}{n}\sum_a^\infty \sup_{|t|\leqq\varepsilon}|g''(X_i+t)|-\int_a^\infty \sup_{|t|\leqq\varepsilon}|g''(x+t)|\,f(x)dx\right|\geqq 1\,.$$

Hence we have

$$\begin{aligned}&P\Big\{\sup_{-\varepsilon\leqq t<M_n}\frac{1}{n}G_n''(t)\geqq -1\Big\}\\&\leqq P\{M_n>\varepsilon\}+P\Big\{\Big|\frac{1}{n}\sum_0^a (X_i+\varepsilon)^{-2}-\int_0^a (x+\varepsilon)^{-2}f(x)\,dx\Big|\geqq 1\Big\}\\&\quad+P\Big\{\Big|\frac{1}{n}\sum_a^\infty \sup_{|t|\leqq\varepsilon}|g''(X_i+t)|-\int_a^\infty \sup_{|t|\leqq\varepsilon}|g''(x+t)|\,f(x)\,dx\Big|\geqq 1\Big\}\,.\end{aligned}$$

Since $P\{M_n>\varepsilon\}=o(n^{-1})$, Lemma 1 follows from condition (vi) and Chebyshev's inequality.

Woodroofe [1] mentioned that condition (i) and

$$\int_0^\infty -g(x)f(x)\,dx<\infty\,,$$

which is a weaker condition than (iv), imply all assumptions of Wald [4]. Thus we can make use of his results.

Lemma 2. *Let $\hat\theta_n$, $n\geqq 1$, be a sequence of m.l.e.'s. Suppose that conditions* (i)–(iii) *and* (iv) *hold. Then for every $\varepsilon>0$ there exists $c\geqq 0$ such that*

$$P\{|\hat\theta_n|\geqq\varepsilon\}\leqq cn^{-1}$$

for all $n\geqq 1$.

Proof. Let M be a positive number chosen such that

$$E\{\log\sup_{t<-M} f(X-t)\}<E\{\log f(X)\}\,.$$

For every $t\in[-M,-\varepsilon]$ there exists an open neighborhood U_t of t such that

$$E\{\log\sup_{u\in U_t} f(X-u)\}<E\{\log f(X)\}\,.$$

The existence of such a positive number M and that of such a U_t follow from Wald [4]. As $\{U_t\colon t\in[-M,-\varepsilon]\}$ covers the compact set $[-M,-\varepsilon]$, there exists a finite subcover of this set $[-M,-\varepsilon]$ determined by $t_j\in[-M,-\varepsilon], j=1,\cdots,m$. For notational convenience, let $U_0=(-\infty,-M)$ and $U_j=U_{t_j}$, $j=1,\cdots,m$. If $|\hat\theta_n|\geqq\varepsilon$ and $M_n<\varepsilon$, then $-\infty<\hat\theta_n\leqq-\varepsilon$ and therefore $\hat\theta_n\in U_j$ for some $j\in\{0,1,\cdots,m\}$, that is to say,

$$n^{-1}\sum_{i=1}^{n}\log\sup_{t\in U_j}f(X_i-t)\geqq n^{-1}\sum_{i=1}^{n}\log f(X_i)$$

for some $j\in\{0, 1, \cdots, m\}$. Write

$$b_j = E\{\log f(X)\}-E\{\log\sup_{t\in U_j}f(X-t)\}>0, \quad j=0, 1, \cdots, m$$

and let $2b=\min\{b_j; j=0, 1, \cdots, m\}>0$. Then

$$|n^{-1}\sum_{i=1}^{n}\log\sup_{t\in U_j}f(X_i-t)-E\{\log\sup_{t\in U_j}f(X-t)\}|<b, \quad j=0, 1, \cdots, m$$

and

$$|n^{-1}\sum_{i=1}^{n}\log f(X_i)-E\{\log f(X)\}|<b$$

imply

$$n^{-1}\sum_{i=1}^{n}\log\sup_{t\in U_j}f(X_i-t)<n^{-1}\sum_{i=1}^{n}\log f(X_i), \quad j=0, 1, \cdots, m\,.$$

Hence we have

$$\begin{aligned}P\{|\hat{\theta}_n|\geqq\varepsilon\}&\leqq P\{|\hat{\theta}_n|\geqq\varepsilon, M_n<\varepsilon\}+P\{M_n\geqq\varepsilon\}\\&\leqq\sum_{j=0}^{m}P\{|n^{-1}\sum_{i=1}^{n}\log\sup_{t\in U_j}f(X_i-t)-E\{\log\sup_{t\in U_j}f(X-t)\}|\geqq b\}\\&\quad+P\{|n^{-1}\sum_{i=1}^{n}\log f(X_i)-E\{\log f(X)\}|\geqq b\}+P\{M_n\geqq\varepsilon\}\,.\end{aligned}$$

Now, by conditions (i)–(iii) and (iv), the assertion follows from Chebyshev's inequality.

For $i=1, \cdots, n$ and $0\leqq t\leqq(\log n)^{1/2}$, let

$$Z_{ni}=Z_{ni}(X_i, t)=Y_{ni}-E\{Y_{ni}\}\,,$$

where

$$\begin{aligned}Y_{ni}=Y_{ni}(X_i, t)&=g'(X_i+a_n^{-1}t), &&\text{if } X_i>a_n^{-1}\,,\\&=0 &&\text{if } X_i\leqq a_n^{-1}\,.\end{aligned}$$

Here E denotes expectation. Moreover, let $b_n(t)=E\{Z_{n1}(X_1, t)\}^2$.

Lemma 3. *Let conditions* (i)—(iii), (v) *and* (vi) *be satisfied. Then there exists a constant c such that for all* $x\in R$, $n\geqq 1$ *and* $0\leqq t\leqq(\log n)^{1/2}$

$$|P\{(nb_n(t))^{-1/2}\sum_{i=1}^{n}Z_{ni}(X_i, t)<x\}-\Phi(x)|\leqq c(\log n)^{-1}\,.$$

Proof. We shall first show that

$$E\{Y_{n1}\}=-t\alpha a_n^{-1}\log a_n(1+t)^{-1}+O(a_n^{-1}(1+t))\,, \tag{3.2}$$

$$E\{Y_{n1}^2\}=\alpha\log a_n(1+t)^{-1}+O(1)\,, \tag{3.3}$$

$$(3.4) \qquad E\{|Y_{n1}|^3\} = O(a_n(1+t)^{-1}).$$

By condition (iii)′, choose $a>0$ and $c_0 \geqq 0$ such that

$$(3.5) \qquad |f(x)-\alpha x| \leqq c_0 x^{1+r}, \ |g'(x)-x^{-1}| \leqq c_0 x^{r-1} \text{ and } |g''(x)+x^{-2}| \leqq c_0 x^{r-2}$$

for $0<x\leqq 2a$. Next choose $\delta>0$ such that conditions (v) and (vi) hold. Then we may establish (3.2) as follows. Since

$$g'(x+a_n^{-1}t) = g'(x)+\int_0^{a_n^{-1}t} g''(x+u)\,du$$

we have

$$\begin{aligned} E\{Y_{n1}\} &= \int_{a_n^{-1}}^{\infty} g'(x)f(x)\,dx + \int_{a_n^{-1}}^{\infty} \{\int_0^{a_n^{-1}t} g''(x+u)\,du\} f(x)\,dx \\ &= I_1+I_2, \quad \text{say.} \end{aligned}$$

It is easily seen that

$$I_1 = -\int_0^{a_n^{-1}} g'(x) f(x)\,dx,$$

so that $I_1=O(a_n^{-1})$ by (3.5). Next we put

$$\begin{aligned} I_2 &= \int_{a_n^{-1}}^{a} \{\int_0^{a_n^{-1}t} g''(x+u)\,du\} f(x)\,dx + \int_a^{\infty} \{\int_0^{a_n^{-1}t} g''(x+u)\,du\} f(x)\,dx \\ &= I_{21}+I_{22}, \quad \text{say.} \end{aligned}$$

By condition (vi), we have $I_{22}=O(a_n^{-1}t)$. Moreover let

$$I_{21} = I_{211}+I_{212}+I_{213},$$

where

$$\begin{aligned} I_{211} &= -\int_{a_n^{-1}}^{a} \{\int_0^{a_n^{-1}t} (x+u)^{-2}du\} \alpha x\,dx, \\ I_{212} &= -\int_{a_n^{-1}}^{a} \{\int_0^{a_n^{-1}t} (x+u)^{-2}du\} (f(x)-\alpha x)\,dx, \\ I_{213} &= \int_{a_n^{-1}}^{a} \{\int_0^{a_n^{-1}t} [g''(x+u)+(x+u)^{-2}]\,du\} f(x)\,dx. \end{aligned}$$

By easy computation, (3.5) implies that

$$\begin{aligned} I_{211} &= -t\alpha a_n^{-1} \log a_n(1+t)^{-1} + O(a_n^{-1}t), \\ I_{212} &= O(a_n^{-1}t) \end{aligned}$$

and

$$I_{213} = O(a_n^{-1}t),$$

so that (3.2) is established.

To establish (3.3), let

$$\begin{aligned} E\{Y_{n1}^2\} &= \int_{a_n^{-1}}^{a} \{g'(x+a_n^{-1}t)\}^2 f(x)\,dx + \int_a^\infty \{g'(x+a_n^{-1}t)\}^2 f(x)\,dx \\ &= J_1 + J_2\,, \quad \text{say.} \end{aligned}$$

Condition (v) implies that $J_2=O(1)$. Divide J_1 into J_{11}, J_{12}, J_{13} and J_{14} as follows:

$$\begin{aligned} J_{11} &= \int_{a_n^{-1}}^{a} (x+a_n^{-1}t)^{-2}\alpha x\,dx\,, \\ J_{12} &= \int_{a_n^{-1}}^{a} (x+a_n^{-1}t)^{-2}(f(x)-\alpha x)\,dx\,, \\ J_{13} &= \int_{a_n^{-1}}^{a} 2(x+a_n^{-1}t)^{-1}\{g'(x+a_n^{-1}t)-(x+a_n^{-1}t)^{-1}\}\,f(x)\,dx\,, \\ J_{14} &= \int_{a_n^{-1}}^{a} \{g'(x+a_n^{-1}t)-(x+a_n^{-1}t)^{-1}\}^2 f(x)\,dx\,. \end{aligned}$$

Then, by (3.5), we have

$$\begin{aligned} J_{11} &= \alpha \log a_n(1+t)^{-1}+O(1)\,, \\ J_{12} &= O(1)\,, \\ J_{13} &= O(1) \\ \text{and} \qquad J_{14} &= O(1)\,, \end{aligned}$$

so that (3.3) is established.

Finally, we shall establish (3.4). Let

$$\begin{aligned} E\{|Y_{n1}|^3\} &= \int_{a_n^{-1}}^{a} |g'(x+a_n^{-1}t)|^3 f(x)\,dx + \int_a^\infty |g'(x+a_n^{-1}t)|^3 f(x)\,dx \\ &= K_1+K_2\,, \quad \text{say.} \end{aligned}$$

By condition (v), we have $K_2=O(1)$. Also by (3.5) we have

$$\begin{aligned} K_1 &\leqq \int_{a_n^{-1}}^{a} \{(1+(2a)^r c_0)\,(x+a_n^{-1}t)^{-1}\}^3 f(x)d(x) \\ &= O(a_n(1+t)^{-1})\,. \end{aligned}$$

This implies (3.4).

From (3.2), (3.3) and (3.4), we have

$$\begin{aligned} E\{Z_{n1}^2\} &= \alpha \log a_n(1+t)^{-1}+O(1)\,, \\ E\{|Z_{n1}|^3\} &= O(a_n(1+t)^{-1})\,. \end{aligned} \tag{3.6}$$

Now, the assertion of Lemma 3 follows from the Berry-Esseen theorem ([5], Theorem 12.4).

In the rest of this section, we shall study the conditional distribution of

$a_n^{-1}\sum_{i=1}^n g'(X_i-a_n^{-1}t)$ given $M_n>a_n^{-1}t$ for $0<t\leqq(\log n)^{1/2}$. The conditional ditribution of $X_1, \cdots, X_n$, given $M_n>a_n^{-1}t$, is that of independent random variables with common density

$$\begin{aligned} f_n^*(x) &= c_n f(x)\,, & & x>a_n^{-1}t \\ &= 0 \quad , & & \text{otherwise} \end{aligned}$$

where

$$c_n = [\int_{a_n^{-1}t}^{\infty} f(x)\,dx]^{-1}\,.$$

For $i=1, \cdots, n$ and $0<t\leqq(\log n)^{1/2}$ let

$$Z_{ni}^* = Z_{ni}^*(X_i, t) = Y_{ni}^* - E^*\{Y_{ni}^*\}$$

where

$$\begin{aligned} Y_{ni}^* = Y_{ni}^*(X_i, t) &= g'(X_i-a_n^{-1}t)\,, & & \text{if } X_i>a_n^{-1}(1+2t)\,, \\ &= 0 \quad , & & \text{if } X_i\leqq a_n^{-1}(1+2t)\,. \end{aligned}$$

Here E^* denotes conditional expectation given $M_n>a_n^{-1}t$. It is easily seen that $c_n=1+O(n^{-1})$ for $0<t\leqq(\log n)^{1/2}$. Thus, in a similar way to Lemma 3, we obtain

$$\begin{aligned} E^*\{Y_{n1}^*\} &= t\alpha\, a_n^{-1}\log a_n(1+t)^{-1}+O(a_n^{-1}(1+t))\,, \\ E^*\{Z_{n1}^{*2}\} &= \alpha \log a_n(1+t)^{-1}+O(1) \end{aligned}$$

and

$$E^*\{|Z_{n1}^*|^3\} = O(a_n(1+t)^{-1})\,,$$

which lead to the following lemma.

Lemma 4. *Let conditions* (i)—(iii), (v) *and* (vi) *be satisfied. Then there exists a constant c such that for all* $x\in R$, $n\geqq 1$ *and* $0<t\leqq(\log n)^{1/2}$

$$|P\{(nb_n^*(t))^{-1/2}\sum_{i=1}^n Z_{ni}^*(X_i, t)<x\,|\,M_n>a_n^{-1}t\}-\Phi(x)|\leqq c(\log n)^{-1}\,,$$

where $b_n^*(t)=E^*\{Z_{n1}^*(X_1, t)\}^2$.

4. Proof of Theorem. As the left sides of (2.1) and (2.2) are uniformly bounded for $\theta\in R$ and $t\in R$, it sufficies to prove the assertion for all sufficiently large n. To simplify our notations we shall use n_0 as a generic constant instead of the phrase "for all sufficiently large n". In the same manner we shall use c as a generic constant to denote factors occurring in the bounds.

We shall use ideas related to Woodroofe [1]. It follows from Lemma 1 and Lemma 2 that

$$P\{a_n\hat{\theta}_n\leqq -t\} = P\{a_n^{-1}\sum_{i=1}^n g'(X_i+a_n^{-1}t)\geqq 0\}+O(n^{-1})\,, \tag{4.1}$$

where $O(n^{-1})$ is uniform in $t\in[0, a_n\varepsilon)$. Here $\varepsilon>0$ is chosen sufficiently small so that (3.1) of Lemma 1 holds. Similarly, it follows from Lemma 1 and Lemma 2 that

$$
\begin{aligned}
(4.2)\qquad P\{a_n\hat{\theta}_n>t\} &= P\{a_n^{-1}\sum_{i=1}^n g'(X_i-a_n^{-1}t)<0,\ M_n>a_n^{-1}t\}+O(n^{-1})\\
&= P\{a_n^{-1}\sum_{i=1}^n g'(X_i-a_n^{-1}t)<0\,|\,M_n>a_n^{-1}t\}P\{M_n>a_n^{-1}t\}\\
&\quad +O(n^{-1}),
\end{aligned}
$$

where $O(n^{-1})$ is uniform in $t>0$.

We shall first show the validity of (2.1). By condition (iii)′

$$
\begin{aligned}
(4.3)\qquad |P\{a_n^{-1}\sum_{i=1}^n g'(X_i+a_n^{-1}t)\geqq 0\}-P\{a_n^{-1}\sum_{i=1}^n Y_{ni}\geqq 0\}\,|\\
\leqq \sum_{i=1}^n P\{X_i\leqq a_n^{-1}\}\\
\leqq c(\log n)^{-1}
\end{aligned}
$$

for all $n\geqq n_0$ and $0\leqq t\leqq(\log n)^{1/2}$. Since

$$
P\{a_n^{-1}\sum_{i=1}^n Y_{ni}\geqq 0\} = P\{(nb_n(t))^{-1/2}\sum_{i=1}^n Z_{ni}\geqq x_n(t)\}\ ,
$$

where

$$
x_n(t) = -n^{1/2}(b_n(t))^{-1/2}E\{Y_{n1}\}\ ,
$$

it follows from Lemma 3 that

$$
(4.4)\qquad |P\{a_n^{-1}\sum_{i=1}^n Y_{ni}\geqq 0\}-\Phi(-x_n(t))\,|\leqq c(\log n)^{-1}
$$

for all $n\geqq 1$ and $0\leqq t\leqq(\log n)^{1/2}$. According to (3.2) and (3.6)

$$
\begin{aligned}
-x_n(t) &= (na_n^{-1}E\{Y_{n1}\})\,(na_n^{-2}b_n(t))^{-1/2}\\
&= \{-t+2t\log(1+t)(\log n)^{-1}+O((1+t)(\log n)^{-1})\}\\
&\qquad\times\{1-2\log(1+t)(\log n)^{-1}+O((\log n)^{-1})\}^{-1/2}\\
&= -t+t\log(1+t)(\log n)^{-1}+O((1+t)(\log n)^{-1})\,.
\end{aligned}
$$

Hence, for $n\geqq n_0$ and $0\leqq t\leqq(\log n)^{1/2}$

$$
\begin{aligned}
(4.5)\qquad |\Phi(-x_n(t))-\Phi(-t)\,| &\leqq \frac{1}{\sqrt{2\pi}}\,|t-x_n(t)|\max\left\{\exp\left(-\frac{t^2}{2}\right),\ \exp\left(-\frac{x_n(t)^2}{2}\right)\right\}\\
&\leqq c(\log n)^{-1}\,.
\end{aligned}
$$

From (4.1), (4.3), (4.4) and (4.5), there exists a constant c such that

$$
(4.6)\qquad |P\{a_n\hat{\theta}_n\leqq -t\}-\Phi(-t)\,|\leqq c(\log n)^{-1}
$$

for all $n\geqq n_0$ and $0\leqq t\leqq(\log n)^{1/2}$. For $t>(\log n)^{1/2}$ we have

$$|P\{a_n\hat{\theta}_n \leqq -t\} - \Phi(-t)| \leqq P\{a_n\hat{\theta}_n \leqq -(\log n)^{1/2}\} + \Phi(-(\log n)^{1/2}) .$$

Using (4.6) and Feller ([6], p. 166, Lemma 2), we obtain

$$(4.7) \qquad |P\{a_n\hat{\theta}_n \leqq -t\} - \Phi(-t)| \leqq c(\log n)^{-1}$$

for $n \geqq n_0$ and $t > (\log n)^{1/2}$. Hence (4.6) and (4.7) imply (2.1).

We next show the validity of (2.2). By condition (iii)$'$

$$\begin{aligned} &|P\{a_n^{-1}\sum_{i=1}^n g'(X_i - a_n^{-1}t) < 0 \mid M_n > a_n^{-1}t\} - P\{a_n^{-1}\sum_{i=1}^n Y_{ni}^* < 0 \mid M_n > a_n^{-1}t\}| \\ &\qquad \leqq \sum_{i=1}^n P\{X_i \leqq a_n^{-1}(1+2t) \mid M_n > a_n^{-1}t\} \\ &\qquad \leqq cn\, a_n^{-2}(3t^2+4t+1) . \end{aligned}$$

Hence, for every s, $0 < s < 1$, there exists a constant c such that for $n \geqq n_0$ and $0 < t \leqq (\log n)^{s/2}$

$$\begin{aligned} (4.8) \quad &|P\{a_n^{-1}\sum_{i=1}^n g'(X_i - a_n^{-1}t) < 0 \mid M_n > a_n^{-1}t\} - P\{a_n^{-1}\sum_{i=1}^n Y_{ni}^* < 0 \mid M_n > a_n^{-1}t\}| \\ &\qquad \leqq c(\log n)^{s-1} . \end{aligned}$$

Applying arguments similar to those used in (4.4) and (4.5), Lemma 4 implies

$$(4.9) \qquad |P\{a_n^{-1}\sum_{i=1}^n Y_{ni}^* < 0 \mid M_n > a_n^{-1}t\} - \{1-\Phi(t)\}| \leqq c(\log n)^{-1}$$

for $n \geqq n_0$ and $0 < t \leqq (\log n)^{1/2}$. By (4.8) and (4.9) we have

$$\begin{aligned} &|P\{a_n^{-1}\sum_{i=1}^n g'(X_i - a_n^{-1}t) < 0 \mid M_n > a_n^{-1}t\} P\{M_n > a_n^{-1}t\} - \{1-\Phi(t)\}| \\ &\quad \leqq |P\{a_n^{-1}\sum_{i=1}^n g'(X_i - a_n^{-1}t) < 0 \mid M_n > a_n^{-1}t\} - \{1-\Phi(t)\}| P\{M_n > a_n^{-1}t\} \\ &\qquad + \{1-\Phi(t)\} P\{M_n \leqq a_n^{-1}t\} \\ &\quad \leqq \{1-\Phi(t)\} P\{M_n \leqq a_n^{-1}t\} + c(\log n)^{s-1} \end{aligned}$$

for $n \geqq n_0$ and $0 < t \leqq (\log n)^{s/2}$. Using Feller ([6], p. 166, Lemma 2), we obtain

$$\begin{aligned} \{1-\Phi(t)\} P\{M_n \leqq a_n^{-1}t\} &\leqq ct \exp\left(-\frac{t^2}{2}\right)(\log n)^{-1} \\ &\leqq c(\log n)^{-1} \end{aligned}$$

for $n \geqq n_0$ and $t > 0$. Hence, from this and (4.2) it follows that for $n \geqq n_0$ and $0 < t \leqq (\log n)^{s/2}$

$$|P\{a_n\hat{\theta}_n > t\} - \{1-\Phi(t)\}| \leqq c(\log n)^{s-1} ,$$

from which (2.2) is shown by a similar argument used in (4.7). Thus we complete the proof of the theorem.

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Faculty of Economics
Wakayama University
Nishitakamatsu,
Wakayama 641, Japan