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Osaka J. Math.
20 (1983), 123-136

BP OPERATIONS AND HOMOLOGICAL PROPERTIES OF BP_*BP-COMODULES

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(Received April 22, 1981)

BP is the Brown-Peterson spectrum for a fixed prime p and BP_*X is the Brown-Peterson homology of the CW-spectrum X. The left BP_*-module BP_*X is an associative comodule over the coalgebra BP_*BP. In [2] we have studied some torsion properties of (associative) BP_*BP-comodules, by paying attension to the behaviors of BP operations. It seems that the following result is fundamental.

Theorem 0.1. *Let M be a BP_*BP-comodule. If an element $x \in M$ is v_n-torsion, then it is v_{n-1}-torsion.* ([2, Theorem 0.1]).

After a little while Landweber [8] has obtained several results about torsion properties of associative BP_*BP-comodules in an awfully algebraic manner, as new applications of commutative algebra to the Brown-Peterson homology. In this note we will give directly new proofs of Landweber's principal results [8, Theorems 1 and 2], by making use of two basic tools (Lemmas 1.1 and 1.2) looked upon as generalizations of Johnson-Wilson results [1, Lemmas 1.7 and 1.9] handling BP operations:

Theorem 0.2. *Let M be a BP_*BP-comodule and $x \neq 0$ be an element of M. Then the radical of the annihilator ideal of x*

$$\sqrt{\mathrm{Ann}(x)} = \{\lambda \in BP_*;\ \lambda^k x = 0 \text{ for some } k>0\}$$

is one of the invariant prime ideals $I_n=(p, v_1, \cdots, v_{n-1})$ in BP_*, $1 \leqq n \leqq \infty$. (Theorem 1.3).

Theorem 0.3. *Let M be an associative BP_*BP-comodule and $1 \leqq n < \infty$. If M contains an element x satisfying $\sqrt{\mathrm{Ann}(x)} = I_n$, then there is a primitive element y in M such that the annihilator ideal of y*

$$\mathrm{Ann}(y) = \{\lambda \in BP_*;\ \lambda y = 0\}$$

is just I_n. (Theorem 2.2).

As an immediate consequence Theorem 0.2 implies Landweber's invariant prime ideal theorem [4] that the invariant prime ideals in BP_* are I_n for $1 \leqq n \leqq \infty$ (Corollary 1.4). Our technique adopted in the proof of Theorem 0.3 allows us to give a new proof of Landweber's prime filtration theorem [5] (Theorem 2.3).

We prove Theorem 0.2 and hence Invariant prime ideal theorem in §1 and Theorem 0.3 and Prime filtration theorem in §2, although Landweber has shown Theorems 0.2 and 0.3 after having known Invariant prime ideal theorem and Prime filtration theorem.

Let $\mathscr{BP}$ be the category of all associative BP_*BP-comodules and comodule maps. An associative BP_*BP-comodule has a BP_*-projective resolution in $\mathscr{BP}$. In [3] we introduced the concept of $\mathscr{BP}$-injective weaker slightly than that of BP_*-injective. In §3 we prove

Theorem 0.4. *Let M be an associative BP_*BP-comodule with* w $\dim_{BP_*} M < \infty$. *Then M has a $\mathscr{BP}$-injective resolution in $\mathscr{BP}$* (Corollary 3.12).

Let J be an invariant regular ideal in BP_* of finite length. There is a left BP-module spectrum BPJ whose homotopy is BP_*/J. When J is trivial, BPJ is just BP. we do prove our results for (associative) BPJ_*BPJ-comodules. A reader who is interested only in associative BP_*BP-comodules may neglect the "J" in the BPJ notation.

1. The radicals of annihilator ideals

Let us fix an invariant regular ideal $J=(\alpha_0, \cdots, \alpha_{q-1})$ in $BP_* \cong Z_{(p)}[v_1, v_2, \cdots]$. There is an associative left BP-module spectrum whose coefficient is $BPJ_* \cong BP_*/(\alpha_0, \cdots, \alpha_{q-1})$. BPJ becomes a quasi-associative ring spectrum [2].

Let $E=(e_1, e_2, \cdots)$ be a finitely non-zero sequence of non-negative integers and $A=(a_0, \cdots, a_{q-1})$ be a q-tuple consisting of zeros and ones. We put $|E| = \sum_i 2(p^i-1)e_i$ and $|A| = \sum_j (|\alpha_j|+1)a_j$ where $|\alpha_j|$ represents dimension of $\alpha_j \in BP_*$. BPJ^*BPJ is the free left BPJ_*-module whose free basis is formed by elements $z^{E,A}$ with dimension $|E|+|A|$. When BPJ_*BPJ is viewed as a right BPJ^*-module, its free basis is given by the elements $c(z^{E,A})$ where c denotes the canonical conjugation of BPJ_*BPJ.

BPJ^*BPJ is the direct product of copies of BPJ_* indexed by all BPJ operations $S_{E,A}\colon BPJ \to \Sigma^{|E|+|A|}BPJ$. When J is trivial, operations $S_{E,0}$ coincide with the BP operations r_E. BPJ operations $S_{E,A}$ satisfy the Cartan formula, i.e., for the BP-module structure map $\phi\colon BP \wedge BPJ \to BPJ$ we have

$$(1.1) \qquad S_{E,A}\phi = \sum_{F+G=E} \phi(r_F \wedge S_{G,A})\colon BP \wedge BPJ \to \Sigma^{|E|+|A|}BPJ\,.$$

The operation $S_{0,0}\colon BPJ \to BPJ$ is a homotopy equivalence, which is uniquely written in the form of

(1.2) $$S_{0,0} = 1 + \sum_{A\neq 0} q_A S_{0,A}$$

with certain coefficients $q_A \in BPJ_*$. The composition $S_{E,A}S_{F,B}$ has a unique representation as a formal sum

(1.3) $$S_{E,A}S_{F,B} = \sum_{(G,C)} q_{G,C} S_{G,C}$$

for certain coefficients $q_{G,C} = q_{G,C}(E, A; F, B) \in BPJ_*$.

A left BPJ_*-module M is called a *BPJ_*BPJ-comodule* [2] if it admits a coaction map ψ_M: $M \to BPJ_*BPJ \underset{BPJ_*}{\otimes} M$ represented as

$$\psi_M(x) = \sum_{(E,A)} c(z^{E,A}) \otimes s_{E,A}(x),$$

which satisfies two conditions:

(i) ψ_M is a left BPJ_*-module map, i.e.,

"Cartan formula" $$s_{E,A}(\lambda x) = \sum_{F+G=E} r_F(\lambda) s_{G,A}(x)$$

for each $\lambda \in BPJ_*$ and $x \in M$.

(ii) $$s_{0,0}(x) = x + \sum_{A\neq 0} q_A s_{0,A}(x)$$

for each $x \in M$, where the coefficients $q_A \in BPJ_*$ are those given in (1.2).

Note that ψ_M is a split monomorphism of left BPJ_*-modules when M is a BPJ_*BPJ-comodule.

A BPJ_*BPJ-comodule M is said to be *associative* if it satisfies an additional condition:

(iii) ψ_M is associative, i.e.,

$$s_{E,A}(s_{F,B}(x)) = \sum_{(G,C)} q_{G,C} s_{G,C}(x)$$

for each $x \in M$, where the coefficients $q_{G,C} \in BPJ_*$ are those given in (1.3).

Let M be a left BPJ_*-module which admits a structure of (associative) BP_*BP-comodule. Taking $s_{E,0}(x) = r_E(x)$ and $s_{E,A}(x) = 0$ if $A \neq 0$, we can regard M as an (associative) BPJ_*BPJ-comodule.

Recall that for $1 \leqq m \leqq \infty$, $I_m = (p, v_1, \cdots, v_{m-1})$ are invariant prime ideals in BP_*. Johnson-Wilson have observed nice behaviors of BP operations r_E modulo I_m [1, Lemmas 1.7 and 1.9]. We first give two useful lemmas, which descend directly from the so-called "Ballentine Lemma". The first lemma has already appeared with a short proof in [2].

Lemma 1.1. *Let E be an exponent sequence with $|E| \geqq 2kp^s(p^n - p^m)$, $n \geqq m \geqq 1$, $s \geqq 0$ and $k \geqq 1$. Then*

$$r_E(v_n^{kp^s}) \equiv \begin{cases} v_m^{kp^s} \text{ modulo } I_m^{s+1} \text{ if } E = kp^{s+m}\Delta_{n-m} \\ 0 \quad \text{modulo } I_m^{s+1} \text{ if otherwise} \end{cases}$$

where $\Delta_{n-m}=(0, \cdots, 0, 1, 0, \cdots)$ *with the single "1" with $(n-m)$-th position.* (Cf., [1, Lemma 1.7]).

Proof. Using the Cartan formula and the fact that $p\in I_m$ we can easily see that

$$r_E(v_n^{kp^s})\equiv\sum r_{E_1}(v_n)\cdots r_{E_{kp^s}}(v_n) \text{ modulo } I_m^{s+1}$$

where the summation $\sum$ runs over all kp^s-tuples $(E_1, \cdots, E_{kp^s})$ of exponent sequences such that $E=E_1+\cdots+E_{kp^s}$ and $r_{E_i}(v_n)\not\equiv 0$ modulo I_m for all i, $1\leqq i\leqq kp^s$. The result now follows immediately from [1, Lemma 1.7].

Define an ordering on exponent sequences as follows: $E=(e_1, e_2, \cdots)<F=(f_1, f_2, \cdots)$ if $|E|<|F|$ or if $|E|=|F|$ and $e_1=f_1, \cdots, e_{i-1}=f_{i-1}$ but $e_i>f_i$.

Lemma 1.2. *Let $m\geqq 1$, $s\geqq 0$ and $\lambda\in BP_*$. If λ is not contained in I_m, then there is an exponent sequence E and a unit $u\in Z_{(p)}$ such that*

$$r_F(\lambda^{p^s})\equiv\begin{cases} uv_m^{kp^s} & \text{modulo } I_m^{s+1} \text{ if } F=p^s\sigma_m E \\ 0 & \text{modulo } I_m^{s+1} \text{ if } F>p^s\sigma_m E\end{cases}$$

where $\sigma_m E=(p^m e_{m+1}, p^m e_{m+2}, \cdots)$ and $k=e_m+e_{m+1}+\cdots$ for $E=(e_1, \cdots, e_m, \cdots)$. (Cf., [1, Lemma 1.9]).

Proof. Put $\lambda=\sum_G a_G v^G\notin I_m$, $a_G\in Z_{(p)}$, by defining $v^G=v_1^{g_1}\cdots v_n^{g_n}$ for $G=(g_1, \cdots, g_n, 0, \cdots)$. We may assume that $G=(0, \cdots, 0, g_m, g_{m+1}, \cdots)$ and a_G is a unit of $Z_{(p)}$. Pick up the exponent sequence E so that $\sigma_m E$ is maximal among $\sigma_m G$. By [1, Corollary 1.8] we have

$$r_H(\lambda)=\sum_G a_G r_H(v^G)\equiv\begin{cases} a_E v_m^{k(E)} & \text{modulo } I_m \text{ if } H=\sigma_m E \\ 0 & \text{modulo } I_m \text{ if } H>\sigma_m E\end{cases}$$

where $k(G)=g_m+g_{m+1}+\cdots$. By a similar argument to the proof of Lemma 1.1 we can compute $r_F(\lambda^{p^s})$ modulo I_m^{s+1} to obtain the required result.

Let $J=(\alpha_0, \cdots, \alpha_{q-1})$ be an invariant regular ideal in BP_* of length q, and M be a left BPJ_*-module. Recall that the *annihilator ideal* $\mathrm{Ann}(x)$ *of* $x\in M$ in BP_* is defined by

$$\mathrm{Ann}(x)=\{\lambda\in BP_*;\ \lambda x=0\}$$

and that the *radical* $\sqrt{\mathrm{Ann}(x)}$ *of the annihilator ideal* $\mathrm{Ann}(x)$ is done by

$$\sqrt{\mathrm{Ann}(x)}=\{\lambda\in BP_*;\ \lambda^k x=0 \text{ for some } k\}\ .$$

For the element v_n of BP_* (by convention $v_0=p$) we say that an element $x\in M$

is v_n-*torsion* if $v_n^k x=0$ for some k and that $x \in M$ is v_n-*torsion free* if not so. Since the radical $\sqrt{J}$ of J is just I_q [6, Proposition 2.5], we note that

(1.4) *every left BPJ_*-module M is at least v_n-torsion for each n, $0 \leqq n < q$,* i.e., *$v_n^{-1}M=0$ for $0 \leqq n < q$.*

Making use of Lemma 1.1 we have obtained the following result in [2, Lemma 2.3 and Corollary 2.4].

(1.5) *Let M be a BPJ_*BPJ-comodule and assume that $x \in M$ is v_n-torsion. Then $x \in M$ is v_m-torsion for all m, $0 \leqq m \leqq n$. More generally, $s_{E,A}(x)$ is v_m-torsion for all m, $0 \leqq m \leqq n$ and for all elementary BPJ operations $s_{E,A}$.*

Given exponent sequences $E=(e_1, e_2, \cdots)$, $F=(f_1, f_2, \cdots)$, $A=(a_0, \cdots, a_{q-1})$ and $B=(b_0, \cdots, b_{q-1})$ we define an ordering between pairs (E, A) and (F, B) as follows: $(E, A)<(F, B)$ if i) $|E|+|A|<|F|+|B|$, or if ii) $|E|+|A|=|F|+|B|$ and $E<F$, or if iii) $E=F$, $|A|=|B|$ and $a_0=b_0, \cdots, a_{j-1}=b_{j-1}$ but $1=a_j>b_j=0$.

As a principal result in [8] Landweber has determined the radical $\sqrt{\mathrm{Ann}(x)}$ of $x \in M$ for an associative BP_*BP-comodule M. Using Lemma 1.2 we give a new proof without the restriction of associativity on M.

Theorem 1.3 (Landweber [8, Theorem 1]). *Let J be an invariant regular ideal in BP_* of length q, M be a BPJ_*BPJ-comodule and $n \geqq q$. An element $x \in M$ is v_{n-1}-torsion and v_n-torsion free if and only if $\sqrt{Ann(x)}=I_n$.*

Proof. Assume that $x \in M$ is v_{n-1}-torsion and v_n-torsion free when $n \geqq 1$. Obviously $I_n \subset \sqrt{\mathrm{Ann}(x)}$. If $0 \neq \lambda \in \sqrt{\mathrm{Ann}(x)} - I_n$, then we may choose an integer $s \geqq 0$ such that $\lambda^{p^s}x=0$ and $I_n^{s+1}s_{E,A}(x)=0$ for all (E, A). By Lemma 1.2 there is an exponent sequence F so that

$$r_H(\lambda^{p^s}) \equiv \begin{cases} uv_n^{kp^s} \text{ modulo } I_n^{s+1} \text{ if } H=p^s\sigma_m F \\ 0 \quad\;\; \text{ modulo } I_n^{s+1} \text{ if } H>p^s\sigma_m F \end{cases}$$

for some $k>0$ and some unit $u \in Z_{(p)}$. There exists a pair (G', B') such that $s_{G',B'}(x)$ is v_n-torsion free because $x \in M$ is so. Pick up the maximal (G, B) of such pairs, and choose an integer $t \geqq 0$ such that $v_n^t s_{E,A}(x)=0$ whenever $(E, A)>(G, B)$. Using the Cartan formula we compute

$$\begin{aligned} 0 &= v_n^t s_{G+p^s\sigma_m F,B}(\lambda^{p^s}x) = v_n^t r_{p^s\sigma_m F}(\lambda^{p^s})s_{G,B}(x) \\ &= uv_n^{t+hp^s}s_{G,B}(x) \,. \end{aligned}$$

Thus $s_{G,B}(x)$ is v_n-torsion. This is a contradiction. The "if" part is evident.

In the $n=0$ case the above proof works well if we apply [1, Lemma 1.9 (b)] in place of Lemma 1.2.

Corollary 1.4. *If I is an invariant ideal in BP_*, then the radical $\sqrt{I}$ of I is I_n for some n, $1 \leqq n \leqq \infty$. In particular, the invariant prime ideals in BP_* are I_n for $1 \leqq n \leqq \infty$.* (Cf., [1, Corollary 1.10] or [4]).

2. Prime filtration theorem

Let M be a BPJ_*BPJ-comodule. An element $x \in M$ is said to be *primitive* if $s_{E,A}(x)=0$ for all $(E, A) \neq (0, 0)$.

Lemma 2.1. *Let M be a BPJ_*BPJ-comodule and $q \leqq n < \infty$ where $\sqrt{J}=I_q$. If a primitive element $x \in M$ is v_{n-1}-torsion and v_n-torsion free, then there is a primitive element given in the form of $v^K x$ such that $Ann(v^K x)=I_n$, where we put $v^K=p^{k_0}v_1^{k_1}\cdots v_n^{k_n}$ for some $(n+1)$-tuple $K=(k_0, k_1, \cdots, k_n)$ of non-negative integers. In particular, we may take $k_n=0$ when M is v_n-torsion free.*

Proof. Inductively we construct a primitive element $y_m=v^{K_m}x \in M$ so that $I_m y_m=0$ and y_m is again v_n-torsion free, where $K_m=(k_0, \cdots, k_{m-1}, 0, \cdots, 0, k_{n,m})$ is a certain $(n+1)$-tuple with "0" in the positions of $(m+1)$-th through n-th. Beginning with $y_0=x$ we inductively assume the existence of $y_m=v^{K_m}x$, $m<n$. Choose an integer $k_m \geqq 0$ such that $v_m^{k_m}y_m$ is v_n-torsion free but $v_m^{k_m+1}y_m$ is v_n-torsion. Then there is an integer $s \geqq 0$ such that $I_n^{s+1}y_m=0$ and $v_m^{k_m+1}v_n^{p^s}y_m=0$. Taking $y_{m+1}=v_m^{k_m}v_n^{p^s}y_m$, it is v_n-torsion free and $v_m y_{m+1}=0$. Applying the induction hypothesis that y_m is primitive and $I_m y_m=0$, we have

$$s_{E,A}(y_{m+1}) = r_E(v_m^{k_m}v_n^{p^s})s_{0,A}(y_m) = \begin{cases} v_m^{k_m}r_E(v_n^{p^s})y_m & \text{if } A=0 \\ 0 & \text{if } A \neq 0 . \end{cases}$$

By use of Lemma 1.1 we verify that $y_{m+1}=v^{K_{m+1}}x$ is primitive, where $K_{m+1}=(k_0, \cdots, k_m, 0, \cdots, 0, k_{n,m}+p^s)$.

We next give a new proof of another principal result in [8], treated of the annihilator ideal $\mathrm{Ann}(x)$ of $x \in M$ for an associative BP_*BP-comodule M.

Theorem 2.2 (Landweber [8, Theorem 2]). *Let J be an invariant regular ideal in BP_* of length q, M be an associative (or connective) BPJ_*BPJ-comodule and $q \leqq n < \infty$. If M contains an element x which is v_{n-1}-torsion and v_n-torsion free, then there exists a primitive element y in M satisfying $Ann(y)=I_n$.*

Proof. Pick up the maximal pair (G, B) such that $s_{G,B}(x)$ is v_n-torsion free, and then choose an integer $s \geqq 0$ for which $I_n^{s+1}s_{E,A}(x)=0$ for all (E, A) and $v_n^{p^s}s_{F,C}(x)=0$ for any $(F, C)>(G, B)$. In the case when M is associative we have

$$s_{E,A}(v_n^{p^s}s_{G,B}(x)) = s_{E,A}(s_{G,B}(v_n^{p^s}x)) = \sum q_{F,C}s_{F,C}(v_n^{p^s}x) = \sum q_{F,C}v_n^{p^s}s_{F,C}(x) = 0$$

if $(E, A)\neq(0, 0)$. Hence $z=v_n^{p^s}s_{G,B}(x)$ is primitive. So we apply Lemma 2.1 to find out a desirable element y in M.

In the connective case we use induction on dimension of x to show the existence of a primitive element $z\in M$ which is v_{n-1}-torsion and v_n-torsion free.

We are now in a position to prove directly Landweber's prime filtration theorem by repeated use of Lemma 2.1.

Theorem 2.3 (Prime filtration theorem [5]). *Let J be an invariant regular ideal in BP_* of length q and M be a BPJ_*BPJ-comodule which is finitely presented as a BPJ_*-module. Then M has a finite filtration*

$$M = M_s\supset M_{s-1}\supset\cdots\supset M_1\supset M_0 = \{0\}$$

consisting of subcomodules so that for $1\leqq i\leqq s$ each subquotient M_i/M_{i-1} is stably isomorphic to BP_*/I_k for some $k\geqq q$.

Proof. Notice that a BPJ_*-module is finitely presented if and only if it is so as a BP_*-module. By virtue of [4, Lemma 3.3] we may take M to be a cyclic comodule BP_*/I where I is an invariant finitely generated ideal including J. Since I is finitely generated, we can choose an integer $l\geqq 0$ to identify $M=BP_*/I$ with $BP_*\underset{R_l}{\otimes}R_l/I'$ for some finitely generated ideal I' in the ring $R_l=Z_{(p)}[v_1, \cdots, v_l]$. Note that any extended module from R_l to BP_* is always v_{l+1}-torsion free. On the other hand, by (1.4) we remark that the BPJ_*-module M is v_{q-1}-torsion.

When the generator $g=[1]\in M=BP_*/I$ is v_{m-1}-torsion and v_m-torsion free for some m, $q\leqq m\leqq l+1$, it is sufficient to show that M has a finite filtration of comodules

$$\{0\} = M^0\subset M^1\subset\cdots\subset M^{r+1}\subset M$$

so that for each $k\leqq r+1$, M^k/M^{k-1} is stably isomorphic to BP_*/I_m and moreover that M/M^{r+1} is an extended module from R_l, whose generator $g_{r+1}=[1]\in M/M^{r+1}$ is v_m-torsion. Assume that the generator $g=[1]\in M=BP_*/I$ is v_{m-1}-torsion and v_m-torsion free, $m\leqq l+1$. By Lemma 2.1 there is a $(m+1)$-tuple $K=(k_0, \cdots, k_m)$ with $k_{l+1}=0$, for which $y=v^Kg$ is a primitive element satisfying $\mathrm{Ann}(y)=I_m$. Take $M^1=BP_*\cdot y\subset M$ so that $N^1=M/M^1$ is a cyclic comodule which is an extended module from R_l since v^K belongs to R_l. Take $K'=(k_0, \cdots, k_{i-1}, k_i-1, 0, \cdots, 0, k'_m)$ if $K=(k_0, \cdots, k_i, 0, \cdots, 0, k_m)$ with $k_i\geqq 1$. Then $v^{K'}g$ is not contained in M^1 for any $k'_m\geqq 0$, as is easily checked. By construction of an improved primitive element developed in Lemma 2.1 we gain a primitive element $y_1=v^{K_1}g_1$ in $N^1=M/M^1$ satisfying $\mathrm{Ann}(y_1)=I_m$, where $K_1=(k_0, \cdots, k_{i-1}, k_i-1, k'_{i+1}, \cdots, k'_m)$ for some $k'_j\geqq 0$, $i+1\leqq j\leqq m$. Repeating this construction we get a primitive element $y_q=v^{K'}g_q\in N^q=M/M^q$ with $\mathrm{Ann}(y_q)=I_m$ at a suitable stage $q\geqq 1$, where $K'=(k_0, \cdots, k_{i-1}, k_i-1, 0, \cdots, 0, k_{m,q})$ for some $k_{m,q}\geqq 0$. Applying a downward induction on i we lastly obtain a primitive element $y_r=v_m^k g_r \in N^r=M/M^r$ for some $k\geqq 0$ such that $\mathrm{Ann}(y_r)=I_m$. Take the subcomodule $M^{r+1}\subset M$ to be $M^{r+1}/M^r\cong BP_*\cdot y_r$, then the generator $g_{r+1}=[1]\in M/M^{r+1}$ is obviously v_m-torsion. Consequently we get a satisfactory filtration.

Let us denote by $\mathcal{BPJ}$ the category of all *associative* BPJ_*BPJ-comodules and comodule maps. Clearly $\mathcal{BPJ}$ is an abelian category. By employing (1.3) we can show the following result due to Landweber [7, Proposition 2.4].

(2.1) *The category $\mathcal{BPJ}$ has enough projectives. That is, for each associative BPJ_*BPJ-comodule M there is an associative BPJ_*BPJ-comodule F which is BPJ_*-free and an epimorphism $f\colon F\to M$ of comodules. F may be taken to be finitely generated if so is M.*

Using (2.1) and the exactness of direct limit we obtain

(2.2) *every associative BPJ_*BPJ-comodule is a direct limit of finitely presented associative comodules.*

Let G be a right BPJ_*-module. We define the *$\mathcal{BPJ}$-weak dimension of G*, denoted by $\mathrm{w\,dim}_{\mathcal{BPJ}}G$, to be less than n if $\mathrm{Tor}_i^{BPJ_*}(G, M)=0$ for all $i\geqq n$ and all comodules M in $\mathcal{BPJ}$. Let N be a BPJ_*BPJ-comodule. We regard N as a right BPJ_*BPJ-comodule. Since the right comodule structure map ${}_N\psi\colon N\to N\underset{BPJ_*}{\otimes} BPJ_*BPJ$ is split monic, we can easily see that $\mathrm{w\,dim}_{\mathcal{BPJ}}N$ is the same as the BPJ_*-weak dimension of N.

For the fixed invariant regular ideal J of length q we consider the invariant ideals $J_{(m)}=J+I_m$ and $J'_{(m)}=\{\lambda\in BP_*;\ \lambda v_m\in J_{(m)}\}$ for any $m\geqq 0$. Then we have an exact sequence

$$0\to BP_*/J'_{(m)}\xrightarrow{v_m} BP_*/J_{(m)}\to BP_*/J_{(m+1)}\to 0 \tag{2.3}$$

of comodules. Note that $J_{(k)}=J'_{(k)}=I_k$ for each $k\geqq q$. When J is just I_q, $J_{(i)}=I_q$ for any $i\leqq q$ and hence $J'_{(i)}=BP_*$, $i<q$.

From Theorem 2.3, (2.1) and (2.2) we can immediately derive the $\mathcal{BPJ}$-version of Landweber's exact functor theorem (as extended) [7].

Theorem 2.4. *Let J be an invariant regular ideal in BP_* of length q, G be a right BPJ_*-module and $n\geqq 0$. Then the following conditions are equivalent*:
(i) $\mathrm{w\,dim}_{\mathcal{BPJ}}G\leqq n$,
(ii) $\mathrm{Tor}_{n+1}^{BPJ_*}(G, BP_*/I_k)=0$ for all $k\geqq q$,
(iii) *multiplication by v_k is monic on* $\mathrm{Tor}_n^{BPJ_*}(G, BP_*/I_k)$ *for each $k\geqq q$ and in addition* $\mathrm{Tor}_{n+1}^{BPJ_*}(G, BP_*/I_q)=0$, *and*
(iv) *the induced multiplication* $v_m\colon \mathrm{Tor}_n^{BPJ_*}(G, BP_*/J'_{(m)})\to \mathrm{Tor}_n^{BPJ_*}(G, BP_*/J_{(m)})$ *is monic for all $m\geqq 0$.*

We call a right BPJ_*-module G *$\mathcal{BPJ}$-flat* when $\mathrm{w\,dim}_{\mathcal{BPJ}}G=0$.

Corollary 2.5. *If a right BP_*-module G is $\mathcal{BP}$-flat, then the extended module $G \underset{BP_*}{\otimes} BPJ_*$ is $\mathcal{BPJ}$-flat.*

Recall that for any l, $0 \leqq l \leqq \infty$, $BP\langle l\rangle_* \cong Z_{(p)}[v_1, \cdots, v_l]$ is viewed as a quotient of BP_*. Setting $v_m^{-1}BP\langle l, J\rangle_* = v_m^{-1}BP\langle l\rangle_* \underset{BP_*}{\otimes} BPJ_*$ with $0 \leqq m \leqq l$, it is $\mathcal{BPJ}$-flat.

Using the technique of Landweber [8, Theorem 3] by aid of Theorem 2.2 we can show

Proposition 2.6. *Let M be an associative (or connective) BPJ_*BPJ-comodule, G be a right BPJ_*-module and $m \geqq q$ where $\sqrt{J} = I_q$. Assume that G is $\mathcal{BPJ}$-flat with $Gv_m^{-1} \underset{BPJ_*}{\otimes} BP_*/I_m \neq 0$. Then M is v_m-torsion if and only if $Gv_m^{-1} \underset{BPJ_*}{\otimes} M = 0$.*

Recall that $E(m)_* = v_m^{-1}BP\langle m\rangle_*$ and $E(m, J)_* = E(m)_* \underset{BP_*}{\otimes} BPJ_*$.

Corollary 2.7 ([8, Theorem 3]). *Let M be an associative (or connective) BPJ_*BPJ-comodule. Then M is v_m-torsion if and only if $E(m)_* \underset{BP_*}{\otimes} M = 0$.*

This allows us to give a simple proof of the following result [2, Proposition 2.8].

(2.4) *An associative BPJ_*BPJ-comodule M is v_m-torsion if M is v_{m+1}-divisible,* i.e., *if multiplication by v_{m+1} is epic on M.*

3. $\mathcal{BPJ}$-injective

Let $\mathcal{BPJ}_0$ be the full subcategory of $\mathcal{BPJ}$ consisting of all finitely presented associative comodules. For a left BPJ_*-module G we define the *$\mathcal{BPJ}$-injective dimension of G*, denoted by $\mathrm{inj\,dim}_{\mathcal{BPJ}}\, G$, to be less than n if $\mathrm{Ext}^i_{BPJ_*}(M, G)=0$ for all $i \geqq n$ and all comodules M in $\mathcal{BPJ}$. The *$\mathcal{BPJ}_0$-injective dimension of G* is similarly defined.

As a dual of Theorem 2.4 we have

Lemma 3.1. *Let G be a left BPJ_*-module, $\sqrt{J}=I_q$ and $n \geqq 0$. Then the following conditions are equivalent:*

(i) $\mathrm{inj\,dim}_{\mathcal{BPJ}_0} G \leqq n$,

(ii) $\mathrm{Ext}^{n+1}_{BPJ_*}(BP_*/I_k, G)=0$ *for all $k \geqq q$,*

(iii) *multiplication by v_k is epic on* $\mathrm{Ext}^n_{BPJ_*}(BP_*/I_k, G)$ *for each $k \geqq q$ and in addition* $\mathrm{Ext}^{n+1}_{BPJ_*}(BP_*/I_q, G)=0$, *and*

(iv) *the induced multiplication* v_m: $\mathrm{Ext}^n_{BPJ_*}(BP_*/J_{(m)}, G) \to \mathrm{Ext}^n_{BPJ_*}(BP_*/J'_{(m)}, G)$ *is epic for all $m \geqq 0$.*

Proposition 3.2. *Let G be a left BPJ_*-module and $n \geqq 0$. Then*

$\operatorname{inj}\dim_{\mathcal{BPJ}} G\leqq n$ if and only if $\operatorname{inj}\dim_{\mathcal{BPJ}_0} G\leqq n$ and moreover $\operatorname{Ext}^{n+1}_{BPJ_*}(BP_*/I, G)=0$ for any invariant ideal I including J with the radical $\sqrt{I}=I_\infty$.

Proof. The $n=0$ case is shown by using Theorems 1.3 and 2.2 and a Zorn's lemma argument (see [3, Lemma 3.13]) similar to the abelian group case. A general n case is done by induction.

As an immediate consequence we have

(3.1) *for any m, $\operatorname{inj}\dim_{\mathcal{BPJ}} v_m^{-1}G\leqq n$ if $\operatorname{inj}\dim_{\mathcal{BPJ}_0} G\leqq n$.*

We call a left BPJ_*-module G *$\mathcal{BPJ}$-injective* when $\operatorname{inj}\dim_{\mathcal{BPJ}}G=0$. Similarly for *$\mathcal{BPJ}_0$-injective.*

Corollary 3.3. *If a left BP_*-module G is $\mathcal{BP}$-injective, then the coextended BPJ_*-module $\operatorname{Hom}_{BP_*}(BPJ_*, G)$ is $\mathcal{BPJ}$-injective.*

Consider the BP_*-modules $N^m_{\langle l\rangle}$ and $M^m_{\langle l\rangle}$ for every $m\geqq 0$ defined inductively by setting that $N^0_{\langle l\rangle}=BP\langle l\rangle_*$, $M^m_{\langle l\rangle}=v_m^{-1}N^m_{\langle l\rangle}$ and $N^{m+1}_{\langle l\rangle}$ is the cokernel of the localization homomorphism $N^m_{\langle l\rangle}\to M^m_{\langle l\rangle}$. The sequence $0\to N^m_{\langle l\rangle}\to M^m_{\langle l\rangle}\to N^{m+1}_{\langle l\rangle}\to 0$ is exact for each $m\leqq l$, and $N^n_{\langle l\rangle}=0$ for any $n\geqq l+2$.

We can easily verify that

(3.2) *$M^m_{\langle l\rangle}$ is $\mathcal{BP}$-injective and hence $\operatorname{Hom}_{BP_*}(BPJ_*, M^m_{\langle l\rangle})$ is $\mathcal{BPJ}$-injective.*

As a dual of Proposition 2.6 we have

Proposition 3.4. *Let M be an associative (or connective) BPJ_*BPJ-comodule, G be a left BPJ_*-module and $m\geqq q$ where $\sqrt{J}=I_q$. Assume that G is $\mathcal{BPJ}$-injective with $\operatorname{Hom}_{BPJ_*}(BP_*/I_m, v_m^{-1}G)\neq 0$. Then M is v_m-torsion if and only if $\operatorname{Hom}_{BPJ_*}(M, v_m^{-1}G)=0$.*

Putting $M(m)=M^m_{\langle m\rangle}$ and $M(m, J)=\operatorname{Hom}_{BP_*}(BPJ_*, M(m))$ we obtain

Corollary 3.5. *Let M be an associative (or connective) BPJ_*BPJ-comodule. Then M is v_m-torsion if and only if $\operatorname{Hom}_{BP_*}(M, M(m))=0$.*

For the invariant regular ideal $J=(\alpha_0, \cdots, \alpha_{q-1})$ we put $J_k=(\alpha_0, \cdots, \alpha_{k-1})$ for each $k\leqq q$. The exact sequence $0\to BP_*/J_k\xrightarrow{\alpha_k} BP_*/J_k\to BP_*/J_{k+1}\to 0$ induces isomorphisms $\operatorname{Ext}^k_{BP_*}(BP_*/J_k, BP_*/J)\cong\operatorname{Ext}^{k+1}_{BP_*}(BP_*/J_{k+1}, BP_*/J)$ and $\operatorname{Ext}^q_{BP_*}(BP_*/J, BP_*/J_k)\cong\operatorname{Ext}^q_{BP_*}(BP_*/J, BP_*/J_{k+1})$. So we observe that $BP_*/J\cong \operatorname{Hom}_{BP_*}(BP_*, BP_*/J)\xrightarrow{\cong}\operatorname{Ext}^q_{BP_*}(BP_*/J, BP_*/J)\xleftarrow{\cong}\operatorname{Ext}^q_{BP_*}(BP_*/J, BP_*)$.

Setting $N^s_J=\operatorname{Hom}_{BP_*}(BPJ_*, N^{q+s}_{\langle\infty\rangle})$ and $M^s_J=\operatorname{Hom}_{BP_*}(BPJ_*, M^{q+s}_{\langle\infty\rangle})$ we have

Lemma 3.6. *N^s_J and M^s_J are associative BPJ_*BPJ-comodules such that $N^0_J\cong BPJ_*$, $M^s_J\cong v_{q+s}^{-1}N^s_J$ and the sequence $0\to N^s_J\to M^s_J\to N^{s+1}_J\to 0$ is an exact* *sequence of comodules.*

Proof. Since $M^m_{\langle\infty\rangle}$ is $\mathcal{BP}$-injective and $\mathrm{Hom}_{BP_*}(BP_*/J, M^{q-1}_{\langle\infty\rangle})=0$ by (1.4) we see that $\mathrm{Hom}_{BP_*}(BP_*/J, N^q_{\langle\infty\rangle})\cong\mathrm{Ext}^q_{BP_*}(BP_*/J, BP_*)$ and $\mathrm{Ext}^1_{BP_*}(BP_*/J, N^r_{\langle\infty\rangle})\cong\mathrm{Ext}^{r+1}_{BP_*}(BP_*/J, BP_*)=0$ for any $r\geqq q$. Hence $N^0_J\cong BP_*/J$ and the sequence $0\to N^s_J\to M^s_J\to N^{s+1}_J\to 0$ is exact. Obviously $M^s_J\cong v^{-1}_{q+s}N^s_J$ and it is an associative comodule by [2, Proposition 2.9] (or see [9, Lemma 3.2]).

For a left BPJ_*-module G we write $\mathrm{w\,dim}_{\mathcal{NG}}G\leqq n$ if $\mathrm{Tor}^{BPJ_*}_i(N^s_J, G)=0$ for all $i\geqq n+1$ and all $s\geqq 0$. When we regard a left BPJ_*-module as a right one by mere necessity, it is evident that

$$\text{(3.3)}\qquad \mathrm{w\,dim}_{\mathcal{NG}}\, G\leqq n \qquad \textit{if}\ \ \mathrm{w\,dim}_{\mathcal{BPG}}\, G\leqq n\,.$$

Putting $N^s_m=\mathrm{Hom}_{BP_*}(BP_*/I_m, N^{m+s}_{\langle\infty\rangle})$ we have a short exact sequence $0\to N^{s-1}_{m+1}\to N^s_m\xrightarrow{v_m}N^s_m\to 0$ of comodules for any $s\geqq 1$. Using this exact sequence and Theorem 2.4 we can show that the converse of (3.3) is valid when J is just I_q. By induction on $s\geqq 0$ we can see that there is an isomorphism

$$\text{(3.4)}\qquad N^s_J\underset{BPJ_*}{\otimes}BP_*/I_q\cong N^s_q$$

where $\sqrt{J}=I_q$. This implies that $\mathrm{Tor}^{BPJ_*}_i(N^s_J, BP_*/I_q)=0$ for all $i\geqq 1$ and $s\geqq 0$, i.e.,

$$\text{(3.5)}\qquad \mathrm{w\,dim}_{\mathcal{NG}}\, BP_*/I_q=0\,.$$

Moreover we notice that

$$\text{(3.6)}\qquad \mathrm{w\,dim}_{\mathcal{NG}}\, v^{-1}_{q+n}G\leqq n \quad \textit{and} \quad \mathrm{w\,dim}_{\mathcal{NG}}\, BPJ_*BPJ\underset{BPJ_*}{\otimes}v^{-1}_{q+n}G\leqq n\,,$$

since $\mathrm{w\,dim}_{BPJ_*}N^s_J\leqq s$ and the right BPJ_*-modules N^s_J and $N^s_J\underset{BPJ_*}{\otimes}BPJ_*BPJ$ are v_{q+s-1}-torsion.

Lemma 3.7. *Let G be a left BPJ_*-module with* $\mathrm{w\,dim}_{\mathcal{NG}}\, G<\infty$. *Assume that M is a left BPJ_*-module which is v_n-torsion for every $n\geqq 0$. Then* $\mathrm{Ext}^k_{BPJ_*}(M, G)=0$ *for all $k\geqq 0$.* (Cf., [3, Corollary 2.4]).

Proof. It is sufficient to prove the case that $\mathrm{w\,dim}_{\mathcal{NG}}\, G=0$. The sequence $0\to N^s_J\underset{BPJ_*}{\otimes}G\to M^s_J\underset{BPJ_*}{\otimes}G\to N^{s+1}_J\underset{BPJ_*}{\otimes}G\to 0$ is exact. Using this exact sequence we get immediately that $\mathrm{Ext}^k_{BPJ_*}(M, G)\cong\mathrm{Hom}_{BPJ_*}(M, N^k_J\underset{BPJ_*}{\otimes}G)=0$ for any $k\geqq 0$ since $\mathrm{Ext}^i_{BPJ_*}(M, M^s_J\underset{BPJ_*}{\otimes}G)=0$ for all $i\geqq 0$ under our assumption on M.

Combining Proposition 3.2 with Lemma 3.7 we obtain

Theorem 3.8. *Let G be a left BPJ_*-module with* w $\dim_{\mathcal{NG}} G<\infty$ *and* $n\geqq 0$. *Then* inj $\dim_{\mathcal{BPG}} G\leqq n$ *if and only if* inj $\dim_{\mathcal{BPG}_0} G\leqq n$.

Lemma 3.9. *Let M be an associative* (*or connective*) *BPJ_*BPJ-comodule and $\sqrt{J}=I_q$. If* w $\dim_{\mathcal{NG}} M\leqq n$, *then M is v_{q+n}-torsion free.* (Cf., [8, Lemma 3.4]).

Proof. Assume that M has a v_{q+n}-torsion element $x\neq 0$. If $x\in M$ is v_m-torsion for all $m\geqq 0$, then we can find a primitive element $y\neq 0$ in M which is also v_m-torsion for all $m\geqq 0$ (cf., Theorem 2.2). Taking $L=BP_*\cdot y\subset M$, it is a non-zero subcomodule of M. However Lemma 3.7 shows that $\mathrm{Hom}_{BPJ_*}(L, M)=0$. This is a contradiction. So we may assume that $x\in M$ is v_{k-1}-torsion and v_k-torsion free for some $k>q+n$. Then, by Theorem 2.2 there is a primitive element $y\in M$ satisfying $\mathrm{Ann}(y)=I_k$. Hence we have an exact sequence $0\to BP_*/I_k\to M\to N\to 0$ of comodules. Applying this exact sequence we observe that $\mathrm{Tor}_{k-q}^{BPJ_*}(N_J^{k-q}, BP_*/I_k)=0$ under the assumption that w $\dim_{\mathcal{NG}} M\leqq n$. This implies that multiplication by v_{k-1} is monic on $\mathrm{Tor}_{k-q-1}^{BPJ_*}(N_J^{k-q}, BP_*/I_{k-1})$ and hence that $\mathrm{Tor}_{k-q-1}^{BPJ_*}(N_J^{k-q}, BP_*/I_{k-1})=0$ since N_J^{k-q} is v_{k-1}-torsion. Repeating this argument we get that $N_J^{k-q}\underset{BPJ_*}{\otimes} BP_*/I_q=0$, which is not true by (3.4).

Let I be an invariant ideal in BP_* including J and G be a left BPJ_*-module. Take a BPJ_*-homomorphism $f\colon BP_*/I\to BPJ_*BPJ\underset{BPJ_*}{\otimes} G$, which is represented as $f(\lambda)=\sum_{(E,A)} c(z^{E,A})\otimes f_{E,A}(\lambda)$. $f_{E,A}$ satisfies the Cartan formula, i.e., $f_{E,A}(\lambda)=\sum_{F+G=E} r_F(\lambda)f_{G,A}(1)$. Moreover we observe that $I\cdot f_{E,A}(1)=0$ and so $f_{E,A}(1)\in \mathrm{Hom}_{BPJ_*}(BP_*/I, G)$. Consider the group homomorphism

$$T\colon \mathrm{Hom}_{BPJ_*}(BP_*/I, BPJ_*BPJ\underset{BPJ_*}{\otimes} G)\to \bigoplus_{(E,A)}\mathrm{Hom}_{BPJ_*}(BP_*/I, G)$$

defined to be $T(f)=\bigoplus_{(E,A)} f_{E,A}(1)$. As is easily checked, T is an isomorphism.

Lemma 3.10. *Let G be a left BPJ_*-module and $n\geqq 0$. Then* inj $\dim_{\mathcal{BPG}_0} G\leqq n$ *if and only if* inj $\dim_{\mathcal{BPG}_0} BPJ_*BPJ\underset{BPJ_*}{\otimes} G\leqq n$.

Proof. It is sufficient to prove the $n=0$ case. Consider the exact sequence $0\to BP_*/J'_{(m)}\xrightarrow{v_m} BP_*/J_{(m)}\to BP_*/J_{(m+1)}\to 0$ of comodules given in (2.3). We have the following commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_{BPJ_*}(BP_*/J_{(m)}, BPJ_*BPJ\underset{BPJ_*}{\otimes} G) & \xrightarrow{T} & \oplus\mathrm{Hom}_{BPJ_*}(BP_*/J_{(m)}, G)\\
\downarrow & & \downarrow\\
\mathrm{Hom}_{BPJ_*}(BP_*/J'_{(m)}, BPJ_*BPJ\underset{BPJ_*}{\otimes} G) & \xrightarrow{T} & \oplus\mathrm{Hom}_{BPJ_*}(BP_*/J'_{(m)}, G)
\end{array}$$

because v_m is primitive in $BP_*/J_{(m)}$. Since T is an group isomorphism, Lemma

3.1 shows that G is $\mathcal{BPJ}_0$-injective if and only if so is $BPJ_*BPJ \underset{BPJ_*}{\otimes} G$.

Let $\mathcal{BPJ}_w$ be the full subcategory of $\mathcal{BPJ}$ consisting of all associative comodules M with w $\dim_{\mathcal{NJ}} M < \infty$. Finally we show that the category $\mathcal{BPJ}_w$ has enough $\mathcal{BPJ}$-injectives.

Theorem 3.11. *Let J be an invariant regular ideal in BP_* of length q and M be an associative BPJ_*BPJ-comodule with* w $\dim_{\mathcal{NJ}} M < \infty$. *Then there is an associative BPJ_*BPJ-comodule Q with* w $\dim_{\mathcal{NJ}} Q < \infty$ *which is $\mathcal{BPJ}$-injective, and a monomorphism $g: M \to Q$ of comodules.*

Proof. Assume that w $\dim_{\mathcal{NJ}} M \leqq n$ for some $n \geqq 0$. By Lemma 3.9 the localization homomorphism $M \to v_{q+n}^{-1} M$ is monic. Choose an injective left BPJ_*-module D such that M is a submodule of D. Consider the composition map

$$g: M \overset{\psi_M}{\to} BPJ_*BPJ \underset{BPJ_*}{\otimes} M \to BPJ_*BPJ \underset{BPJ_*}{\otimes} D \to BPJ_*BPJ \underset{BPJ_*}{\otimes} v_{q+n}^{-1} D$$

involving the comodule structure map ψ_M of M. Obviously g is a comodule map and it is monic. Putting $Q = BPJ_*BPJ \underset{BPJ_*}{\otimes} v_{q+n}^{-1} D$, the extended comodule Q is $\mathcal{BPJ}_0$-injective by Lemma 3.10 and w $\dim_{\mathcal{NJ}} Q \leqq n$ by (3.6). From Theorem 3.8 it follows that Q is in fact $\mathcal{BPJ}$-injective.

Corollary 3.12. *Let M be an associative BPJ_*BPJ-comodule with* w $\dim_{\mathcal{NJ}} M < \infty$. *Then M has a $\mathcal{BPJ}$-injective resolution*

$$0 \to M \to Q_0 \to Q_1 \to \cdots$$

of comodules.

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